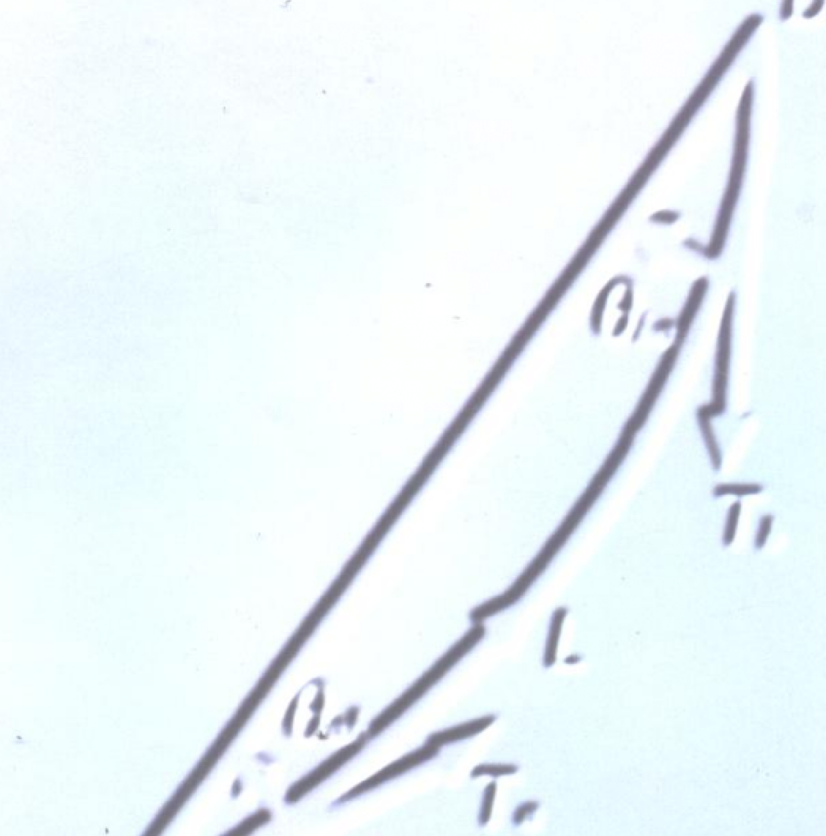
解析函数和奇异积分方程论文选集

路见可 著



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几 点 说 明

1. 本集只选入作者在解析函数边值问题和奇异积分方程及其应用这一领域中的论文(到 1996 年上半年为止),其他方面的论文概不选入;虽属上述领域中的论文,但属综述性的,也不选入。

2. 作者与他人合作的论文均不选入。

3. 同一论文用中、英文同时发表或先后转载的,只选用一种,其他的则加以注明。

4. 用英文发表的论文,多数以汉语拼音 Lu Jianke 署名,一些以英文名字 Chien-ke Lu 署名的则另加说明。

5. 本选集分三大部分:(一)解析函数边值问题;(二)奇异积分和奇异积分方程;(三)在弹性和断裂力学中的应用。每部分的论文以最初发表时间先后为序。同一论文兼及两部分内容的,则从前不从后。

6. 各论文中已发现的疏漏或误排已予改正。

7. 主题相近的论文只选刊一篇,其余的置于相应部分后的“附录论文”中仅提供篇名和出处。

目　　录

(一)解析函数边值问题

复合边值问题

摘　要

本文解决了把 Riemann 边值问题和 Hilbert 边值问题结合在一起的所谓复合边值问题. 对一个未知函数以及多个未知函数的情况都作了讨论. 所用方法是"消去法", 即先消去 Riemann 边值问题中的跳跃, 于是就把问题化为 Hilbert 边值问题.

在[1]中, 我们曾简要地介绍了所谓复合边值问题, 并叙述了用消去法求解的过程. 本文将较详细地论述这方面的一些结果. 我们仍沿用[2]中的记号, 而把 Riemann 问题和 Hilbert问题分别简称为 R 问题和 H 问题, 复合边值问题则简称为 RH 问题.

§1　单连域的 RH 问题

设一封闭的 Ляпунов 曲线 L (即其切线倾角作为弧长的函数时, 该函数满足 Hölder 条件) 围成一域 D, 而在 D 内又有一组互相外离的封闭光滑 Jordan 曲线 $\Gamma_1,\cdots,\Gamma_n$, 并记 $\Gamma=\Gamma_1+\cdots+\Gamma_n$; 取 L 的逆时针方向作为正向, Γ_j 的顺时针方向作为正向; 记 Γ_j 所围内域为 D_j^-, 而 L 与诸 Γ_j 间所围域为 D_0^+.

D 上的 RH 问题 (复合边值问题) 可表述为: 求 D 上分片全纯函数 $\Phi(z)$ (即在 D 上除 Γ 上各点外处处正则, 且分侧连续到 Γ 上), 使它在 Γ 附近, 满足边值条件:

$$\Phi^+(\tau) = G(\tau)\,\Phi^-(\tau) + g(\tau),\quad \tau\in\Gamma, \tag{1.1}$$

其中 $G(\tau), g(\tau)$ 已给在 Γ 上, 适合 Hölder 条件, 且 $G(\tau)\neq 0$; 同时又要求 $\Phi(z)$ 从 D 的内部连续到 L 上, 且满足边值条件:

$$\mathrm{Re}[\overline{\lambda(t)}\Phi(t)] = c(t),\quad t\in L, \tag{1.2}$$

其中 $\lambda(t), c(t)$ 已给在 L 上, 也适合 Hölder 条件, 且 $\lambda(t)\neq 0$ ($c(t)$ 当然是实函数). 记

$$\mathrm{Ind}_{\Gamma_j}G(\tau)=\frac{1}{2\pi}[\arg G(\tau)]_{\Gamma_j}=\kappa_j,\quad \mathrm{Ind}_\Gamma G(\tau)=\kappa=\sum_{j=1}^{n}\kappa_j,\quad \mathrm{Ind}_L\lambda(t)=k.$$

称 $K=\kappa+k$ 为所提出的 RH 问题的指标.

在每一 D_j^- 中任意取定一点 z_j，并记

$$\Pi(z)=\prod_{j=1}^{n}(z-z_j)^{\kappa_j}.$$

先求 D 上分片全纯且连续到 L 上的函数 $\Phi_1(z)$，使它满足条件(1.1)，而条件(1.2) 暂置不顾. 这问题(称作原问题的相应 R 问题) 的一般解是

$$\Phi_1(z)=X(z)[\Psi(z)+F(z)], \tag{1.3}$$

其中 $X(z)$ 为问题的典则函数：

$$X(z)=\begin{cases}X^+(z)=\Pi(z)^{-1}\mathrm{e}^{\Gamma^+(z)}, & \text{当 } z\in D_j^+,\\ X^-(z)=\mathrm{e}^{\Gamma^-(z)}, & \text{当 } z\in\bigcup_j D_j^-,\end{cases}$$

这里 $\Gamma(z)=\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{\log G(\tau)\,\Pi(\tau)}{\tau-z}\mathrm{d}\tau$ (对数可任意取定一支)；$\Psi(z)$ 可取为

$$\Psi(z)=\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{g(\tau)}{X^+(\tau)}\frac{\mathrm{d}\tau}{\tau-z},$$

而 $F(z)$ 为在 D 上全纯、在 $\overline{D}$ 上连续的任意函数. 由于原RH 问题只考虑 $\overline{D}$ 上的问题，故与这相应的R 问题一定可解，且出现了上述任意函数 $F(z)$ 用以代替通常全平面上R 问题的多项式.

注意当 $g(\tau)=0$ 时 $X(z)$ 本身也是问题(1.1) 的解：

$$X^+(\tau)=G(\tau)X^-(\tau),\quad \tau\in\Gamma. \tag{1.4}$$

用下式把所求未知函数 $\Phi(z)$ 变换为另一个未知函数 $\Phi_0(z)$，使满足

$$\Phi(z)=\Phi_1(z)+X(z)\Phi_0(z), \tag{1.5}$$

则显然 $\Phi_0(z)$ 仍在 D 上分片全纯，且连续到 L 上. 此外，当 $\tau\in\Gamma$ 时，注意 $\Phi_1(z)$ 满足(1.1)，并由(1.4) 式，得知

$$\begin{aligned}\Phi^+(\tau)&=\Phi_1^+(\tau)+X^+(\tau)\Phi_0^+(\tau)\\&=G(\tau)\Phi_1^-(\tau)+g(\tau)+G(\tau)X^-(\tau)\Phi_0^+(\tau)\\&=G(\tau)[\Phi_1^-(\tau)+X^-(\tau)\Phi_0^+(\tau)]+g(\tau).\end{aligned}$$

如果 $\Phi(z)$ 是所求解，则它也应满足(1.1)，即

$$\Phi^+(\tau)=G(\tau)\Phi^-(\tau)+g(\tau)=G(\tau)[\Phi_1^-(\tau)+X^-(\tau)\Phi_0^-(\tau)]+g(\tau).$$

比较这两式，并注意 $X^-(\tau)\neq 0$，立刻知道

$$\Phi_0^+(\tau)=\Phi_0^-(\tau),\quad \tau\in\Gamma,$$

从而 $\Phi_0(z)$ 在 D 上全纯、在 $\overline{D}$ 上连续.

反之，若 $\Phi_0(z)$ 在 D 上全纯、在 $\overline{D}$ 上连续，则也容易证明由(1.5) 所确定的分片全纯函数 $\Phi(z)$ 必满足(1.1)，且连续到 L 上.

由此可见，提出的RH 问题就转化为求在 D 上全纯、在 $\overline{D}$ 上连续的函数 $\Phi_0(z)$，使它满足由(1.2) 转化的相应条件. 将(1.5) 代入(1.2)，便立刻得到这个条件：

$$\mathrm{Re}[\overline{\lambda(t)}X(t)\Phi_0(t)]=c^*(t),\quad t\in L, \tag{1.6}$$

其中

$$c^*(t)=c(t)-\mathrm{Re}[\overline{\lambda(t)}\Phi_1(t)]. \tag{1.7}$$

这样，原问题便化为了 D 上的H 问题(1.6)，而消去了条件(1.1) 以及诸 Γ_j. 这也就是

我们称这种方法为消去法的理由.

至于(1.3)中任意函数 $F(z)$ 的选法关系不大. 因为，如果 $F(z)$ 加上一函数 $f(z)$ 时，$\Phi_1(z)$ 就增加一项 $X(z)f(z)$，而由(1.5)，只要在所求函数 $\Phi_0(z)$ 中减去 $f(z)$，就能使 $\Phi(z)$ 不变；这等于在(1.6)式左右两边各减去一项 $\mathrm{Re}(\bar{\lambda}Xf)$，因而 $\Phi_0(z)$ 也没有改变. 所以，在(1.3)中不妨取 $F(z)\equiv 0$.

对于 $\Phi_0(z)$ 所满足的 H 问题(1.6)，其指标为

$$\mathrm{Ind}_L[\lambda(t)\,\overline{X(t)}] = \kappa - \mathrm{Ind}_L X(t). \tag{1.8}$$

注意当 $t\in L$ 时，$X(t) = X^+(t) = \prod_{j=1}^{n}(t-z_j)^{-\kappa_j}e^{\Gamma^+(t)}$，故

$$\mathrm{Ind}_L X(t) = -\kappa + \mathrm{Ind}_L e^{\Gamma^+(t)}. \tag{1.9}$$

因为 $\Gamma^+(t)$ 在 L 上单值连续，故

$$\mathrm{Ind}_L e^{\Gamma^+(t)} = \frac{1}{2\pi i}[\log e^{\Gamma^+(t)}]_L = \frac{1}{2\pi i}[\Gamma^+(t)]_L = 0,$$

代入(1.8)和(1.9)，最后得知

$$\mathrm{Ind}_L[\lambda(t)\,\overline{X(t)}] = k + \kappa = K. \tag{1.10}$$

这就是说，转化后的 H 问题的指标就是原来 RH 问题的指标.

于是，运用通常对 H 问题的解的个数或可解性的理论，就完全可以对 RH 问题作类似的讨论：

1) 设 $g\equiv 0$, $c\equiv 0$（齐次问题）. 当 $K\geqslant 0$ 时，原问题有 $2K+1$ 个（对实系数而言）线性无关的解：

$$\Phi(z) = X(z)\sum_{s=1}^{2K+1}c_s\Phi_{0s}(z), \tag{1.11}$$

其中 $\Phi_{0s}(z)$ $(s=1,\cdots,2K+1)$ 为相应齐次 H 问题(1.6)（这时 $c^*\equiv 0$）的完全解系，而 c_s 为任意实常数（注意这时 $\Psi(z)\equiv 0$）.

当 $K<0$ 时，相应 H 问题只有零解 $\Phi_0(z)\equiv 0$，从而原问题也只有零解.

2) 设 $g\equiv 0$, $c\not\equiv 0$. 由于这时可以取 $\Phi_1(z)\equiv 0$，故 $\Phi(z) = X(z)\Phi_0(z)$，而 $\Phi_0(z)$ 为 D 上非齐次 H 问题

$$\mathrm{Re}[\overline{\lambda(t)}X(t)\Phi_0(t)] = c(t),\quad t\in L$$

的解.

这里，当 $K\geqslant 0$ 时，问题必有解；设 $\Phi_{00}(z)$ 为一特解，则原问题的一般解为

$$\Phi(z) = X(z)[\Phi_{00}(z) + \sum_{s=1}^{2K+1}c_s\Phi_{0s}(z)]. \tag{1.12}$$

当 $K<0$ 时，如果 $c(t)$ 满足 $-2K-1$ 个条件，则问题有且才有解，且解唯一.

3) 设 $g\not\equiv 0$，于是 $\Phi_1(z)\not\equiv 0$. 注意在 L 上，$c^*(t)$ 仍满足 Hölder 条件，故为经典的 H 问题. 这时又可分为两种情况：

① 如果 $c^*(t)\not\equiv 0$ 即 $c(t)\not\equiv \mathrm{Re}[\overline{\lambda(t)}\Phi_1(t)]$，则与前述情况 2) 相同. 当 $K\geqslant 0$ 时，一般解(1.12)右边还要添加一项 $\Phi_1(z)$；当 $K<0$ 时，当且仅当 c,λ,G,g 间满足 $-2K-1$ 个条件时才有唯一解.

② 如果 $c^*(t)\equiv 0$ 即 $c(t)\equiv \mathrm{Re}[\overline{\lambda(t)}\Phi_1(t)]$，则问题仍转化为齐次 H 问题. 当 $K\geqslant 0$ 时，

问题一般解与(1.11)相同，但右端要添加一项 $\Phi_1(z)$；当 $K<0$ 时，H 问题只有零解 $\Phi_0(z)\equiv 0$，故原问题有唯一非零解 $\Phi(z)=\Phi_1(z)$.

我们把这最后一情况称为准齐次 RH 问题，而把 2) 和 3)① 统称为真非齐次问题.

准齐次问题的条件为 $c(t)=\mathrm{Re}[\overline{\lambda(t)}\Phi_1(t)]$，其中 $\Phi_1(t)=X(t)\Psi(t)$. 如果一开始就把相应 R 问题的特解取作(1.3)，其中 $F(z)$ 为在 D 上全纯、在 $\overline{D}$ 上连续的某一函数(且设在 L 上满足 Hölder 条件)，则准齐次 RH 问题就可转化为 $\Phi_0(z)$ 的非齐次 H 问题：

$$\mathrm{Re}[\overline{\lambda(t)}X(t)\Phi_0(t)]=-\mathrm{Re}[\overline{\lambda(t)}X(t)F(t)].$$

当 $K\geqslant 0$ 时，问题的一般解为

$$\Phi(z)=\Phi_1(z)+X(z)[\Phi_{00}(z)+\sum_{s=1}^{2K+1}c_s\Phi_{0s}(z)],$$

其中 $\Phi_{00}(z)$ 为这个 H 问题的一特解. 当 $K<0$ 时，H 问题至多只可能有一个解，但 $\Phi_0(z)=-F(z)$ 显然是它的解，故 $\Phi(z)=\Phi_1(z)-F(z)=\Phi^*(z)$ 是原问题的唯一非零解. 结果与前相同.

所以，RH 问题是准齐次问题的一般条件是（除 $g\not\equiv 0$ 外）：有满足条件(1.1)的在 D 上分片全纯、连续到 L 上的函数 $\Phi_1(z)$ 存在（以(1.3)表出），使

$$\mathrm{Re}[\overline{\lambda(t)}\Phi_1(t)]=c(t),\quad t\in L. \tag{1.13}$$

以上的讨论，可归纳为

定理 1 对单连域的 RH 问题，当其指标 $K=\kappa+k\geqslant 0$ 时，问题的一般解中含有 $2K+1$ 个任意实常数；当 $K<0$ 时，1) 对齐次问题 ($c\equiv 0$, $g\equiv 0$)，只有零解，2) 对准齐次问题，有唯一非零解，3) 对真非齐次问题，当且仅当问题中各系数满足 $-2K-1$ 个条件时，才有唯一解.

当 $K<0$ 时，齐次或准齐次问题也可统一说成问题有唯一解（包括零解).

准齐次问题也可看作真非齐次问题的一特例. 当 $K\geqslant 0$ 时，已由定理 1 的前一结论看出；现设 $K<0$，由于准齐次问题有唯一解，故诸系数间也必满足非齐次情况下的 $-2K-1$ 个条件. 或者，也可这样看：我们知道（参看 [2]，§28，§29)，如果设 $p(s)$ 为 $\lambda(s)$ 的正则化因子，而令

$$\mathrm{e}^{-\omega_1(s)}=p(s)|\lambda(s)|,$$

则这 $-2K-1$ 个条件可写为

$$\int_0^{2\pi}\mathrm{e}^{\omega_1(s)}c^*(s)\mathrm{e}^{-ihs}\mathrm{d}s=0\quad (h=1,\cdots,-2K-1),$$

而在准齐次情况下，可适当选择 $\Phi_1(z)$，使它满足(1.13)，从(1.7)可知，$c^*(s)\equiv 0$，因此这些条件当然满足.

如一开始讨论的是 L 所围外域的问题(Γ_j 也在外域中)，而附带要求 $\Phi(\infty)$ 有界，则显然可得类似于定理 1 的结果. 或者，用反演法，也可变成内域问题来处理. 因而这里不再多述.

在[1]中我们曾经指出，单连域上的 RH 问题也可用 Н. И. Мусхелишвили 的方法([3]，§42) 解决，但这一方法不能应用于下一节中所述的多连域情形.

§2 多连域的 RH 问题

今设 D 为由一组封闭 Ляпунов 曲线 $L_0,L_1,\cdots,L_m$ 所围的多连域，而 L_0 围住其余各曲线

(L_0 也可不存在)，并记 $L=\sum_{p=0}^{m}L_p$[①]. 在 D 内仍有一组曲线 $\Gamma_1,\cdots,\Gamma_n$，情况如前所述，但注意，某些 Γ_j 可能围住某些 L_p. 仍取 L_0 的逆时针方向为正向，而其余各曲线均取顺时针方向为正向.

这时，D 上的RH 问题为：求在 D 上分片全纯、连续到 L 上的函数 $\Phi(z)$，使它仍满足条件(1.1) 和(1.2)，已给诸系数假设条件如前. 如没有 L_0，则还要求 $\Phi(\infty)$ 有界.

κ 和 k 定义如前，但这里

$$k=\sum_{p=0}^{m}k_p,\quad k_p=\mathrm{Ind}_{L_p}\lambda(t),$$

仍称 $K=\kappa+k$ 为RH 问题的指标.

问题的解法仍与单连域时一样. 在 Γ_j 的内部任取 z_j，当 Γ_j 围住某些 L_p 时，z_j 取在 L_p 之内、外或其上均可. 先求得满足问题的R 部分的解 (1.3)，也不妨取 $F(z)\equiv 0$，再通过(1.5)，把未知函数化为 $\Phi_0(z)$，而归结为解 D 上的H 问题(1.6). 这一问题的指标是

$$\mathrm{Ind}_L[\lambda(t)\overline{X(t)}]=\mathrm{Ind}_L\lambda(t)-\mathrm{Ind}_LX(t)=k-\sum_{p=0}^{m}\mathrm{Ind}_{L_p}X(t).$$

对于 L_0，

$$\mathrm{Ind}_{L_0}X(t)=\mathrm{Ind}_{L_0}X^+(t)=-\kappa,$$

仍如前. 对于 $L_p(p\geqslant 1)$，如没有任何 Γ_j 围住它，则有

$$\mathrm{Ind}_{L_p}X(t)=\mathrm{Ind}_{L_p}X^+(t)=\mathrm{Ind}_{L_p}[\Pi(t)e^{\Gamma^+(t)}]=0,$$

因为所有 z_j 在 L_p 之外，故 $\mathrm{Ind}_{L_p}\Pi(t)=0$，又已见 $\mathrm{Ind}_{L_p}e^{\Gamma^+(t)}=0$. 如 L_p 被某一 Γ_j 围住，则

$$\mathrm{Ind}_{L_p}X(t)=\mathrm{Ind}_{L_p}X^-(t)=\mathrm{Ind}_{L_p}e^{\Gamma^-(t)}=0.$$

所以，不论如何，转化后的H 问题的指标仍为 $K=\kappa+k$.

这样，运用多连域上H 问题的已知结果，立刻可得：

定理2 *对 $m+1$ 连域上的RH 问题，当其指标 $K>m-1$ 时，问题恒可解，且一般解中含有 $2K-m+1$ 个独立常数；当指标 $K<0$ 时，齐次问题只有零解，准齐次问题有唯一(非零) 解，而真非齐次问题当且仅当已给系数满足 $-2K+m-1$ 个条件时有唯一解.*

这里所谓准齐次问题，仍指满足条件(1.13).

对于奇异情况 $0\leqslant K\leqslant m-1$，则可有类似于 Ф. Д. Гахов [4] 和 Б. В. Боярский [5] 的结果，这里从略.

§3 开口弧段上的RH 问题

设 D 如前 (单连或多连)，而设 $\Gamma_1,\cdots,\Gamma_n$ 为一些开口的光滑 Jordan 弧段：$\Gamma_j=\widehat{a_jb_j}$ $(j=1,\cdots,n)$，并取自 a_j 至 b_j 的方向作为 Γ_j 的正向[②]. 这时RH 问题提法仍如前，(1.1) 和(1.2) 中诸系数的条件也同前，但在各 Γ_j 的端点附近，可分别要求 $\Phi(z)$有界或无界可积.

① 当 L_0 不存在时，p 自1 加起，下同.

② 若某些 Γ_j 为封闭的，某些为开口的，可同样地讨论.

在某些端点附近，将会看到，只能要求 $\Phi(z)$ 几乎有界（即对于任何 $\varepsilon>0$，$|z-c|^{\varepsilon}\Phi(z)\to 0$，当 $z\to c$ 时），它们称作 RH 问题的特异端点. 下面将要证明，它们就是相应的 R 问题 (1.1) 的特异端点 $c_{r+1},\cdots,c_{2n}$（意义见[2]，§42 等）. 在其余非特异端点处，例如，可要求 $\Phi(z)$ 在 $c_1,\cdots,c_q$ 附近有界，而在 $c_{q+1},\cdots,c_r$ 附近，要求 $\Phi(z)$ 至少无界可积. 采用[3]中记号，把这种解 $\Phi(z)$ 记作属于 $h(c_1,\cdots,c_q)$ 类.

这时前述消去法仍有效. 说明如下.

在各 Γ_j 上取 $\log G(\tau)$ 的一确定支，使

$$\alpha_j=\operatorname{Re}\gamma_j=\operatorname{Re}\left(-\frac{\log G(a_j)}{2\pi\mathrm{i}}\right)$$

满足条件

$$\begin{cases}0\leqslant\alpha_j<1, & \text{当 } a_j\in\{c_1,\cdots,c_q,c_{r+1},\cdots,c_{2n}\},\\ -1<\alpha_j<0, & \text{当 } a_j\in\{c_{q+1},\cdots,c_r\};\end{cases}$$

当 $a_j\in\{c_{r+1},\cdots,c_{2n}\}$ 时，当然 $\alpha_j=0$. 又设

$$\alpha_j'=\operatorname{Re}\gamma_j'=\operatorname{Re}\frac{\log G(b_j)}{2\pi\mathrm{i}},$$

并选取整数 κ_j，使得

$$\begin{cases}0\leqslant\alpha_j'-\kappa_j<1, & \text{当 } b_j\in\{c_1,\cdots,c_q,c_{r+1},\cdots,c_{2n}\},\\ -1<\alpha_j'-\kappa_j<0, & \text{当 } b_j\in\{c_{q+1},\cdots,c_r\};\end{cases}$$

当 $b_j\in\{c_{r+1},\cdots,c_{2n}\}$ 时，当然 $\alpha_j'-\kappa_j=0$，于是

$$\kappa=\sum_{j=1}^{n}\kappa_j$$

为相应的 R 问题属于 $h(c_1,\cdots,c_q)$ 类的指标. 仍记 $k=\operatorname{Ind}_{\Gamma}\lambda(t)$，则仍称 $K=\kappa+k$ 为 RH 问题属于 $h(c_1,\cdots,c_q)$ 类的指标.

相应 R 问题的典则函数为

$$X(z)=\prod_{j=1}^{n}(z-b_j)^{-\kappa_j}\mathrm{e}^{\Gamma(z)},\tag{3.1}$$

其中

$$\Gamma(z)=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{\log G(\tau)}{\tau-z}\mathrm{d}\tau.\tag{3.2}$$

先解 D 上相应的 R 问题，求出其一特解（任意函数 $F(z)$ 不妨取作 $\equiv 0$）：

$$\Phi_1(z)=\frac{X(z)}{2\pi\mathrm{i}}\int_{\Gamma}\frac{g(\tau)}{X^{+}(\tau)}\frac{\mathrm{d}\tau}{\tau-z}.\tag{3.3}$$

利用消去法换元，令

$$\Phi(z)=\Phi_1(z)+X(z)\Phi_0(z),\tag{3.4}$$

其中 $\Phi_0(z)$ 为新的未知函数.

和 §1 中一样，易证在各 Γ_j 弧上的内点 τ 处，$\Phi_0^{+}(\tau)=\Phi_0^{-}(\tau)$，故 $\Phi(z)$ 在 D 上单值解析，只是各端点 a_j,b_j 处可能除外. 我们来研究 $\Phi_0(z)$ 在各端点附近的性质.

首先回顾一下 $X(z)$ 的性质. 由 (3.2) 知，

$$\Gamma(z)=\begin{cases}\gamma_j\log(z-a_j)+\Gamma_j(z), & \text{当 } z \text{ 在 } a_j \text{ 附近},\\ \gamma_j'\log(z-b_j)+\Gamma_j'(z), & \text{当 } z \text{ 在 } b_j \text{ 附近},\end{cases}$$

其中$\Gamma_j(z)$,$\Gamma'_j(z)$分别在a_j,b_j附近有界①. 因之，

$$e^{\Gamma(z)}=\begin{cases}(z-a_j)^{\gamma_j}e^{\Gamma_j(z)}, & \text{当 } z \text{ 在 } a_j \text{ 附近},\\ (z-b_j)^{\gamma'_j}e^{\Gamma'_j(z)}, & \text{当 } z \text{ 在 } b_j \text{ 附近}.\end{cases}$$

于是知道，

$$X(z)=\begin{cases}(z-a_j)^{\gamma_j}e^{\Gamma_j^*(z)}, & \text{当 } z \text{ 在 } a_j \text{ 附近},\\ (z-b_j)^{\gamma'_j-\kappa_j}e^{\Gamma_j^{*\prime}(z)}, & \text{当 } z \text{ 在 } b_j \text{ 附近};\end{cases}$$

或者写作，在任何端点$z=c$附近有：

$$X(z)=(z-c)^{\gamma_c}e^{\Gamma_c(z)}, \tag{3.5}$$

其中$\Gamma_c(z)$在c附近有界，而

当$c\in\{c_1,\cdots,c_q,c_{r+1},\cdots,c_{2n}\}$时，$0\leqslant \mathrm{Re}\,\gamma_c<1$，

当$c\in\{c_{q+1},\cdots,c_r\}$时，$-1<\mathrm{Re}\,\gamma_c<0$，

且当c为特异端点时，$\mathrm{Re}\,\gamma_c=0$.

根据Cauchy型积分的熟知性质，由(3.2)知，R问题的特解$\Phi_1(z)$已属于$h(c_1,\cdots,c_q)$类. 根据对未知函数$\Phi(z)$的要求可知，$X(z)\Phi_0(z)$必须在各端点$c_1,\cdots,c_q$附近有界，在其余端点附近至少无界可积.

设c为一特异端点. 为了保证$\Phi(z)$在c附近至少无界可积，$X(z)\Phi_0(z)$必须也如此. 但因这时$\mathrm{Re}\,\gamma_c=0$，故在$z=c$附近，$X(z)$有界，且$\neq 0$，而$\Phi_0(z)$既以c点为孤立奇点，又要$X(z)\Phi_0(z)$可积，故$\Phi_0(z)$必须以c为常点，因此$X(z)\Phi_0(z)$在c附近有界，从而$\Phi(z)$在c附近几乎有界. 由此得知，在特异端点附近，$\Phi(z)$一定几乎有界.

若$c\in\{c_1,\cdots,c_q\}$，则因$0<\mathrm{Re}\,\gamma_c<1$，为了保证$X(z)\Phi_0(z)$在$c$附近有界，$\Phi_0(z)$也必以$c$为常点. 若$c\in\{c_{q+1},\cdots,c_r\}$，同样因为$-1<\mathrm{Re}\,\gamma_c<0$，为了保证$X(z)\Phi_0(z)$在$c$附近至少无界可积，$c$也只可能是$\Phi_0(z)$的常点.

总之，无论c为Γ_j的哪一端点，它都是$\Phi_0(z)$的常点，因而$\Phi_0(z)$在D上全纯. 又因为$\Phi_1(z)$,$X(z)$均连续到L上，且在L附近$X(z)\neq 0$，故$\Phi_0(z)$亦必连续到L上. 至于$\Phi_0(z)$这时所应满足的条件，仍同(1.6). 于是问题仍化为$\Phi_0(z)$的H问题. 且由(3.1)，立刻得知，

$$\mathrm{Ind}_L X(t)=-\sum_{j=1}^{n}\kappa_j=-\kappa,$$

故这个H问题的指标仍为

$$\mathrm{Ind}_L[\lambda(t)\overline{X(t)}]=k+\kappa=K.$$

因此，其余讨论均与§1或§2相同.

对于间断系数的情况，显然也可类似地讨论，现从略.

§4 多个未知函数的RH问题

记号L，Γ与§1相同. 设所求$\Phi(z)$为D上分片全纯、连续到L上的N维向量，要求

① 对数这样理解：沿Γ_j并延伸到∞点将平面割开，然后任取一支.

满足(1.1)和(1.2)，而$G(\tau)$ $(\tau\in\Gamma)$，$\lambda(t)$ $(t\in L)$均为已知N阶满秩矩阵，$g(\tau)$,$c(t)$为已知N维向量，并设它们都满足Hölder条件. 这样，我们便得到N个函数或N维向量的RH问题.

这一问题仍可用§1中所述消去法求解. 设

$$\operatorname{Ind}_\Gamma \det G(\tau)=\kappa,\quad \operatorname{Ind}_\Gamma \det \lambda(t)=k,$$

称$K=\kappa+k$为上述RH问题的总指标.

求解时，先作出相应R问题(1.1)的典则矩阵（意义及一般作法见[6]），再作出这个R问题的一特解：

$$\Phi_1(z)=\frac{1}{2\pi i}X(z)\int_\Gamma\frac{[X^+(\tau)]^{-1}g(\tau)\mathrm{d}\tau}{\tau-z},$$

然后按(1.5)把未知向量$\Phi(z)$改为$\Phi_0(z)$. 注意，在现在的情况下，(1.5)以下的推演仍成立，最后仍得$\Phi_0^+(\tau)=\Phi_0^-(\tau)$，即$\Phi_0(z)$为在$D$上全纯、在$\overline{D}$上连续的$N$维向量. 于是，问题又化为H问题(1.6)了.

这一H问题的总指标可以证明恰为K（按照И. П. Векуа的定义，总指标则为$2K$，见[7]，第179页）. 为此，我们首先证明：

$$\operatorname{Ind}_L \det X(t)=-\kappa. \tag{4.1}$$

大家已经熟知(参见[7],第41页)

$$\operatorname{Ind}_\Gamma \det X^+(\tau)=\operatorname{Ind}_\Gamma \det G(\tau)=\kappa,$$

但在Γ与L之间，典则函数$\det X(z)\equiv\det X^+(z)$全纯，且无零点，故

$$\operatorname{Ind}_L \det X(t)=\operatorname{Ind}_L \det X^+(t)=-\operatorname{Ind}_\Gamma \det X^+(\tau),$$

最后出现之负号是由于L以逆时针方向为正向，而Γ以顺时针方向为正向. 于是(4.1)成立. 由此，我们立刻得到转化后相应H问题的总指标为

$$\begin{aligned}\operatorname{Ind}_L \det[\lambda(t)\overline{X(t)}]&=\operatorname{Ind}_L[\det\lambda(t)\det\overline{X(t)}]\\&=\operatorname{Ind}_L\det\lambda(t)+\operatorname{Ind}_L\overline{\det X(t)}\\&=k+\kappa=K.\end{aligned}$$

如果能够有效地算出$X(z)$，则这个H问题的系数也就可知，于是原RH问题就能化为一个已知的、确定的H问题，而可进一步求解（见[7],第三章,I). 例如，当$G(\tau)$可写成两个因子的乘积，且它们又分别是Γ内部和Γ外部的半纯矩阵的边值时，便是如此(参看[6]，§5).

对于Γ为开口弧段或有间断系数的情况，利用И. П. Векуа的结果（[7],第二章）和上述§3中的方法，可类似地求解，不再详谈.

最后注意，如在[1]中曾指出的，本文所用的消去法可广泛应用于解决某一R问题与其他类型边值问题的复合，也可用来较简便地重新处理开口弧段或间断系数的R问题，以及复杂边界的R问题等.

消去法在另一些边值问题中的应用，将另行讨论.

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ON COMPOUND BOUNDARY PROBLEMS

Abstract

In this paper, the so-called Riemann-Hilbert compound problem, or briefly, RH problem, is considered and solved. Let D be a (simply connected or multiply connected) Liapunoff region with boundary L, and Γ be a finite set of non-intersecting and mutually exclusive smooth contours in D. By Γ^+ we mean the clockwise sense of transversal along Γ, and by L^+, the sense of transversal which preserves the region D at its left-hand. The problem here considered may be formulated as follows:

Find a sectionally holomorphic function $\Phi(z)$ in D (i. e., regular everywhere in D except on Γ, but continuous to Γ on both sides of it, and also continuous to L), such that conditions (1. 1) and (1. 2) are satisfied, where $G(\tau), g(\tau)$ are given on Γ, $G(\tau)\neq 0$; $\lambda(t), c(t)$ are given on L, $\lambda(t)\neq 0$; and all of them satisfy Hölder conditions.

This problem is solved by the method of elimination: first, solve the "Riemann part" (1. 1) of the problem and take a particular solution $\Phi_1(z)$, and then, by changing the unknown function $\Phi(z)$ to $\Phi_0(z)$, holomorphic in D, by means of (1. 5), where $X(z)$ is the characteristic function of the problem (1. 1), the problem is reduced to a Hilbert problem (1. 6). The index K of the reduced problem, called the index of the RH problem, is easily proved to be the sum of the index κ of $G(\tau)$ with respect to Γ and the index k of $\lambda(t)$ with respect to L. Thus, the solubility or the number of solutions of the RH problem can be readily analyzed.

The case in which Γ consists of open arcs is also considered, and the solutions of the RH problem may also be classified into classes h, just as the Riemann problem. It is found that the special ends of the latter serve also as the special ends of the former, the solution of which must be almost bounded near them. The index relation is just the same as before.

If $G(\tau), \lambda(t)$ are regarded as non-degenerate matrices of order N, and if $g(\tau), c(t)$, $\Phi(z)$ are N-dimensional vectors, we have the vector RH problem. This problem is solved similarly and the index relation still holds, but in this case the indices must be understood as total indices.

[原载于武汉大学学报(自然科学版),1962,5:15～24;转载于高等学校自然科学学报(数学、力学、天文学版),试刊,1964,(1):1～11;又译成英文转载:Chien-ke Lu. On compound boundary problems. Scientia Sinica, 1965, 14(11): 1545～1555.]

周期 Riemann 边值问题及其在弹性力学中的应用

绪言

周期 Riemann 边值问题的更一般形式是自守函数的边值问题. 关于有限群的自守函数 Riemann 边值问题，最初在 1954 年被 Ф. Д. Гахов 和 Л. И. Чибрикова 所研究[1]. 而无限群的研究，为后者在 1956 年所研究[2]（或参看[3]中和 [4]，§ 52 中的简单介绍），最后并有更一般的研究[5]。从实用观点看来，周期边值问题将更为重要. 在[2]中虽也曾指出了这一问题的重要性，并给出了一些在流体力学上的应用，但在单周期情况下，周期解在无穷远处(即自守函数的本性奇点)的性态没有讨论，而在双周期情况下，却没有这样的问题.

本文第一章将研究单周期的 Riemann 边值问题，特别是对解在无穷远处作各种补充的要求而讨论. 这里用的是保角变换的方法，这一方法曾被 Г. Н. Савин 类似地用来解决弹性平面中的基本周期问题（参看[6]或[12]).

本文第二章将研究单周期 Riemann 边值问题在弹性平面问题中的应用. 虽一般地这类问题可化为积分方程来处理（见[6])，但具体求解时很困难. 运用 Н. И. Мусхелишвили 的方法（见[7]，第六章）到这里来，就能把他的许多结果改换到单周期问题上来，并同样地可获得许多重要特例的解的有限形式.

双周期弹性平面的基本问题也可化为积分方程，见 W. T. Koiter [11]. 利用双周期 Riemann 边值问题的结果也可获得一系列有效的结果，但不在本文讨论之列. 因此，本文以后所称“周期”都是指单周期而言.

第一章　周期 Riemann 边值问题

§ 1　封闭曲线情况

1. 问题的提法　设 L_k ($k=0,\pm1,\pm2,\cdots$) 为无穷个封闭的光滑曲线，它们彼此形状

相同，互不相交，且以 $a\pi$ 为周期水平地排列（$a>0$），如图 1 所示[①].

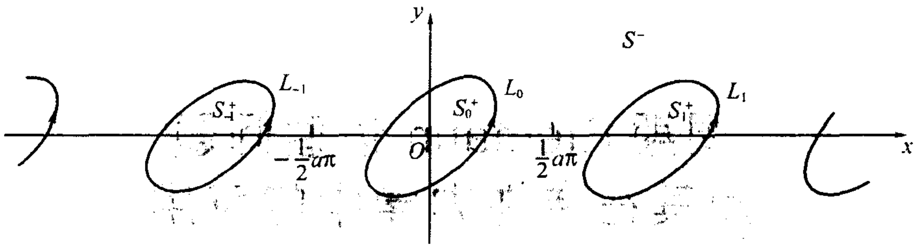

图 1

取各 L_k 的反时针方向为正向，其内域记作 S_k^+，而 $L=\sum_{k=-\infty}^{+\infty} L_k$ 的外域记作 S^-. 不妨选择原点 O 在 S_0^+ 内，且使 $\pm\frac{1}{2}a\pi$，$\pm\frac{3}{2}a\pi$，…都在 S^- 内. 这是可能的. 例如，作 L_0 的一切水平截线段(在截线段之外不再有 L_0 的点)，它们的长都小于 $a\pi$，其中点轨迹形成一 Jordan 弧，这弧的左、右侧既都有 S_0^+ 内的点，故其上也有 S_0^+ 的点；任取这样一点作为原点 O，便能达到目的.

周期的 Riemann 边值问题（简称问题 P_1）提法如下：要求一以 $a\pi$ 为周期的、在全平面中分片全纯的函数 $\Phi(z)$，使

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t),\quad t\in L, \tag{1.1}$$

其中 $G(t)$，$g(t)$ 已给在 L 上，均 $\in H$（即满足 Hölder 条件），并且 $G(t)\neq 0$；此外，它们都以 $a\pi$ 为周期：

$$G(t+a\pi)=G(t),\ g(t+a\pi)=g(t),\quad t\in L.$$

注意∞点是各曲线 L_k 的聚点，故上述问题的解（如果存在）在 $z=\infty$ 处一般不能有确定的极限. 但是当 $z=\pm\infty\mathrm{i}$（指 $z=x+\mathrm{i}y$，其中 x 任意，$y\to\pm\infty$）时，可以对 $\Phi(z)$ 提出一定的要求. 这种补充的要求可以是各种各样的，将在以后陆续给出.

当 $g(t)\equiv 0$ 时，问题称为齐次的，记作 P_1^0；否则，称为非齐次的.

在问题中，我们已要求解 $\Phi(z)$ 也以 $a\pi$ 为周期；这从解决实际问题的观点来看，是合适的. 但注意这一要求不能认为是问题的必然结果. 因为，例如，对于齐次问题 P_1^0 而言，若 $\Phi_1(z)$ 是它的一个非零周期解，则 $\Phi_1(z)I(z)$ 也是它的解，其中 $I(z)$ 为任何整函数，这个解就不一定以 $a\pi$ 为周期了. 而且也容易证明，任何非周期解一定是上述形式. 以后所称的解都是指周期解.

2. 转化为经典的 Riemann 问题 作带形 S_0：$|\mathrm{Re}\ z|<\frac{1}{2}a\pi$. 暂设 L_0 全在 S_0 中. 记 $S_0^-=S^-\cdot S_0$，并在直线 $x=\pm\frac{1}{2}a\pi$ 上取定正向使 S_0^- 在其右侧（图 2）. 如 $\Phi(z)$ 为原问题的解，其在 $S_0=S_0^++S_0^-$ 中的部分记作 $\Phi_0(z)$，则 $\Phi_0(z)$ 是 S_0 中分片全纯的函数，并连续到

① 以下为行文简便起见，只假定 L_k 是一个封闭曲线；实际上，如 L_k 是一组有限个封闭曲线时，以下的讨论只要略加修改，仍成立. 又若 L_0 是连接 $\pm\frac{1}{2}a\pi+\mathrm{i}y_0$ 的任一光滑弧段时，本节所论作适当修正后，也完全成立.

$x=\pm\frac{1}{2}a\pi$ 上，且满足条件：

$$\Phi_0^+(t)=G(t)\Phi_0^-(t)+g(t),\quad t\in L_0; \tag{1.2}$$

$$\Phi_0\left(\frac{1}{2}a\pi+\mathrm{i}y\right)=\Phi_0\left(-\frac{1}{2}a\pi+\mathrm{i}y\right),\quad |y|<+\infty. \tag{1.3}$$

反之，如 $\Phi_0(z)$ 为 S_0 中满足(1.2)和(1.3)的一个分片全纯函数，连续到 $x=\pm\frac{1}{2}a\pi$ 上者，则把它作 $a\pi$ 的周期性延拓后，便得原问题的一个解 $\Phi(z)$.

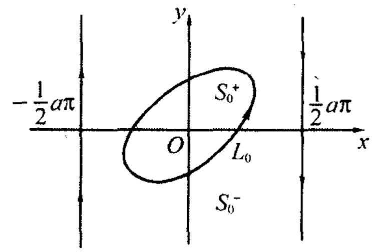

图 2

于是，问题 P_1 就转变为求 S_0 中分片全纯的函数 $\Phi_0(z)$，连续到 $x=\pm\frac{1}{2}a\pi$ 上，使满足条件(1.2)和(1.3). 把这一问题记作 R_1.

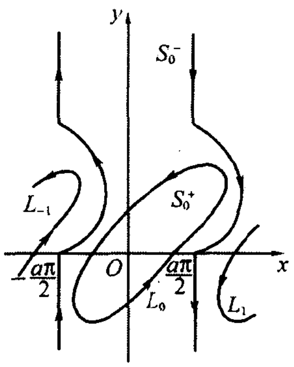

图 3

注意，可能 L_0 部分地越出带形 $|\mathrm{Re}\, z|<\frac{1}{2}a\pi$（图 3）. 这时，我们只要把它略加修改，即用两个周期合同的 Jordan 弧段代替 $x=\pm\frac{1}{2}a\pi$ 上的二直线段，使 S_0^+ 整个落在这样所得出的带形 S_0 中；不妨假定仍旧保持 $z=\pm\frac{1}{2}a\pi$ 两点在 S_0 的边界上. 仍旧记 $S_0^-=S_0-S_0^+$.

用函数

$$\zeta=\tan\frac{z}{a} \tag{1.4}$$

把带形 S_0 映射到 ζ 平面中的区域 Σ_0. 在图 2 的情况下，它是由整个 ζ 平面沿着虚轴在区间$[-\mathrm{i},\mathrm{i}]$之外剖开而成，且 $z=0,\pm\frac{1}{2}a\pi,+\infty\mathrm{i},-\infty\mathrm{i}$ 分别变为 $\zeta=0,\infty,\mathrm{i},-\mathrm{i}$，而直线 $x=\pm\frac{1}{2}a\pi$ 变成了剖线的左、右岸. 这时，L_0 变成某一光滑曲线 Γ_0，它围住原点 O，但 $\zeta=\pm\mathrm{i}$ 则在其外，且不与剖线相交（图 4）. 在图 3 的情况下，则剖线的形状有所改变，但 Γ_0 仍不穿过剖线，且也具有上述的一些其他性质（图 5）. Γ_0 的内域记作 Σ_0^+，外域记作 Σ_0^-.

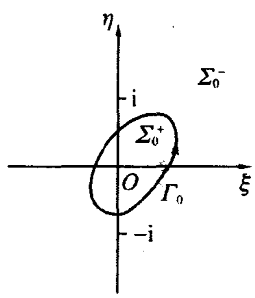

图 4

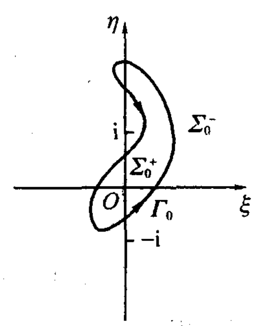

图 5

设$\Phi_0(z)$经变换后成为$\Phi_*(\zeta)$，由条件(1.3)可知，它在剖线两侧有相同极限值. 故知$\Phi_*(\zeta)$是ζ平面中的分片全纯函数(在$\zeta=\infty$处有界!)，且满足条件

$$\Phi_*^+(\tau)=G_*(\tau)\Phi_*^-(\tau)+g_*(\tau),\quad \tau\in\Gamma_0,\tag{1.5}$$

其中$G_*(\tau)$,$g_*(\tau)$分别为$G(t)$,$g(t)$变换后的结果，它们仍都$\in H$，且$G_*(\tau)\neq0$. 但要注意，$\zeta=\pm\,\mathrm{i}$现在一般是$\Phi_*(\zeta)$的孤立奇点. 于是问题化为了经典的 Riemann 问题.

我们称

$$\mathrm{Ind}_{L_0}G(t)=\frac{1}{2\pi}[\arg G(t)]_{L_0}=k\tag{1.6}$$

为原问题 P_1 的指标，显然它就是转化后 Riemann 问题的指标：

$$\mathrm{Ind}_{\Gamma_0}G_*(\tau)=\mathrm{Ind}_{L_0}G(t)=k.$$

3. 齐次问题 P_1^0 的讨论 这时$g(t)\equiv0$，于是$g_*(\tau)\equiv0$. 分几种情况讨论.

1° *要求$\Phi(\pm\infty\mathrm{i})$有界* 亦即要求$\Phi_*(\zeta)$在$\zeta=\pm\,\mathrm{i}$处也有界，即正则. 这时问题(1.5)的一般解为

$$\Phi_*(\zeta)=X_*(\zeta)P_k(\zeta),\tag{1.7}$$

其中

$$X_*(\zeta)=\begin{cases}X_*^+(\zeta)=\mathrm{e}^{\Gamma_*(\zeta)}, & \zeta\in\Sigma_0^+;\\ X_*^-(\zeta)=\zeta^{-k}\mathrm{e}^{\Gamma_*(\zeta)}, & \zeta\in\Sigma_0^-,\end{cases}\tag{1.8}$$

这里

$$\Gamma_*(\zeta)=\frac{1}{2\pi\,\mathrm{i}}\int_{\Gamma_0}\frac{\log[\tau^{-k}G_*(\tau)]}{\tau-\zeta}\mathrm{d}\tau,\tag{1.9}$$

而$P_k(\zeta)$是k次任意多项式($k<0$时，认为$P_k(\zeta)\equiv0$).

回到z平面，则有

$$\Phi_0(z)=X_0(z)P_k\left(\tan\frac{z}{a}\right),\tag{1.10}$$

这里

$$X_0(z)=\begin{cases}X_0^+(z)=\mathrm{e}^{\Gamma(z)}, & z\in S_0^+;\\ X_0^-(z)=\cot^k\dfrac{z}{a}\,\mathrm{e}^{\Gamma(z)}, & z\in S_0^-,\end{cases}\tag{1.11}$$

而

$$\begin{aligned}\Gamma(z)&=\frac{1}{2a\pi\,\mathrm{i}}\int_{L_0}\frac{\log\left[\cot^k\dfrac{t}{a}\,G(t)\right]}{\tan\dfrac{t}{a}-\tan\dfrac{z}{a}}\,\frac{\mathrm{d}t}{\cos^2\dfrac{t}{a}}\\&=\frac{1}{2a\pi\,\mathrm{i}}\int_{L_0}\log\left[\cot^k\frac{t}{a}\,G(t)\right]\cdot\left(\cot\frac{t-z}{a}+\tan\frac{t}{a}\right)\mathrm{d}t\\&=\frac{1}{2a\pi\,\mathrm{i}}\int_{L_0}\log\left[\cot^k\frac{t}{a}\,G(t)\right]\cdot\cot\frac{t-z}{a}\,\mathrm{d}t+C.\end{aligned}$$

把常数并入P_k中，因此可以认为

$$\Gamma(z)=\frac{1}{2a\pi\,\mathrm{i}}\int_{L_0}\log\left[\cot^k\frac{t}{a}\,G(t)\right]\cdot\cot\frac{t-z}{a}\,\mathrm{d}t,\tag{1.12}$$

其中对数可以任意取定一支.

如将 $\Phi_0(z)$ 作周期 $a\pi$ 的延拓，并注意到(1.10)～(1.12)诸分析式中所出现的函数均以 $a\pi$ 为周期，故不加改变，立得原问题 P_1^0 的一般解：

$$\Phi(z)=X(z)P_k\left(\tan\frac{z}{a}\right);\tag{1.10'}$$

$$X(z)=\begin{cases}X^+(z)=\mathrm{e}^{\Gamma(z)}, & z\in S^+;\\ X^-(z)=\cot^k\dfrac{z}{a}\,\mathrm{e}^{\Gamma(z)}, & z\in S^-,\end{cases}\tag{1.11'}$$

而 $\Gamma(z)$ 仍以(1.12)式给出. $X(z)$ 仍称为问题 P_1 的典则函数. 这个一般解还可写成

$$\Phi(z)=\begin{cases}\Phi^+(z)=\dfrac{1}{\cos^k\dfrac{z}{a}}\mathrm{e}^{\Gamma(z)}Q_k\left(\sin\dfrac{z}{a},\cos\dfrac{z}{a}\right), & z\in S^+;\\[2ex] \Phi^-(z)=\dfrac{1}{\sin^k\dfrac{z}{a}}\mathrm{e}^{\Gamma(z)}Q_k\left(\sin\dfrac{z}{a},\cos\dfrac{z}{a}\right), & z\in S^-,\end{cases}\tag{1.13}$$

其中 $Q_k(X,Y)$ 是 X,Y 的任意 k 次齐次多项式 ($k<0$ 时,$Q_k\equiv 0$).

因此我们得到

定理 1(Чибрикова)　*如要求 $\Phi(\pm\infty\mathrm{i})$ 有界，则齐次周期问题 P_1^0 当指标 $k\geqslant 0$ 时，问题有 $k+1$ 个线性无关解；当 $k<0$ 时，问题只有零解.*

附带指出，因为

$$\cos^{n-j}z\,\sin^j z=\sum_{\lambda=0}^{n}C_{nj}(\lambda)\mathrm{e}^{\mathrm{i}(n-2\lambda)z},$$

其中 $C_{nj}(\lambda)$ 为某些常数，故知：当 $k=2m$ 为偶数时，还可写为

$$Q_{2m}\left(\sin\frac{z}{a},\cos\frac{z}{a}\right)=a_0+\sum_{j=1}^{m}\left[a_j\cos\frac{2jz}{a}+\beta_j\sin\frac{2jz}{a}\right];\tag{1.14}$$

当 $k=2m+1$ 为奇数时，还可写为

$$Q_{2m+1}\left(\sin\frac{z}{a},\cos\frac{z}{a}\right)=\sum_{j=0}^{m}\left[a_j\cos\frac{(2j+1)z}{a}+\beta_j\sin\frac{(2j+1)z}{a}\right].\tag{1.14'}$$

因此，在一般解(1.13)中，也可把 Q_k 写成上述形式的任意三角多项式.

2°　*要求 $\Phi(+\infty\mathrm{i})=\Phi(-\infty\mathrm{i})$ (有限)*[①]　代入(1.10)′，得

$$X(+\infty\mathrm{i})P_k(+\mathrm{i})=X(-\infty\mathrm{i})P_k(-\mathrm{i}).$$

但因

$$X(+\infty\mathrm{i})=(-\mathrm{i})^k\mathrm{e}^{\Gamma(+\infty\mathrm{i})}=(-\mathrm{i})^k\exp\left\{\frac{1}{2a\pi}\int_{L_0}\log\left[\cot^k\frac{t}{a}\,G(t)\right]\mathrm{d}t\right\},$$

$$X(-\infty\mathrm{i})=\mathrm{i}^k\mathrm{e}^{\Gamma(-\infty\mathrm{i})}=\mathrm{i}^k\exp\left\{-\frac{1}{2a\pi}\int_{L_0}\log\left[\cot^k\frac{t}{a}\,G(t)\right]\mathrm{d}t\right\},$$

因而上述条件可写为

$$P_k(+\mathrm{i})=G_\infty P_k(-\mathrm{i}),\tag{1.15}$$

其中

$$G_\infty=\frac{X(-\infty\mathrm{i})}{X(+\infty\mathrm{i})}=(-1)^k\exp\left\{-\frac{1}{\pi a}\int_{L_0}\log\left[\cot^k\frac{t}{a}\,G(t)\right]\mathrm{d}t\right\}.\tag{1.16}$$

---

① 在某些实际问题中，要求 $\Phi(+\infty\mathrm{i})=-\Phi(-\infty\mathrm{i})$ 也很重要，可类似地讨论.

当$k>0$时，条件(1.15)一般限制了P_k中的一个系数. 但$k=0$时，情况有所不同. 这时因$P_0\equiv C_0$是一常数，故条件(1.15)成立当且仅当$G_\infty=1$时，亦即

$$\frac{1}{2\pi\mathrm{i}}\int_{L_0}\log G(t)\,\mathrm{d}t\equiv 0\quad(\mathrm{mod}\ a\pi).\tag{1.17}$$

因此我们得到

定理 2 如要求$\Phi(+\infty\mathrm{i})=\Phi(-\infty\mathrm{i})$(有限)，则齐次问题$\mathrm{P}_1^0$当$k\geqslant 1$时有$k$个线性无关解；当$k=0$时，当且仅当满足条件(1.17)时有一个线性无关解；当$k<0$时，问题只有零解.

3° 要求$\Phi(\pm\infty\mathrm{i})=0$ 亦即要求$\Phi_*(\pm\mathrm{i})=0$. 这时问题(1.5)的一般解为

$$\Phi_*(\zeta)=X_*(\zeta)(\zeta^2+1)P_{k-2}(\zeta).$$

回到z平面，得

$$\Phi(z)=\frac{X(z)}{\cos^2\frac{z}{a}}P_{k-2}\left(\tan\frac{z}{a}\right)=\frac{X(z)}{\cos^k\frac{z}{a}}Q_{k-2}\left(\sin\frac{z}{a},\cos\frac{z}{a}\right),\tag{1.18}$$

或

$$\Phi(z)=\begin{cases}\Phi^+(z)=\dfrac{1}{\cos^k\frac{z}{a}}\mathrm{e}^{\Gamma(z)}Q_{k-2}\left(\sin\dfrac{z}{a},\cos\dfrac{z}{a}\right), & z\in S^+;\\[2ex] \Phi^-(z)=\dfrac{1}{\sin^k\frac{z}{a}}\mathrm{e}^{\Gamma(z)}Q_{k-2}\left(\sin\dfrac{z}{a},\cos\dfrac{z}{a}\right), & z\in S^-.\end{cases}\tag{1.18$'$}$$

于是我们有

定理 3 如要求$\Phi(\pm\infty\mathrm{i})=0$，则齐次问题$\mathrm{P}_1^0$当$k\geqslant 2$时有$k-1$个线性无关解；当$k<2$时，问题只有零解.

当然也可讨论只要求$\Phi(+\infty\mathrm{i})=0$或$\Phi(-\infty\mathrm{i})=0$的解，从略.

4° 容许$\Phi(\pm\infty\mathrm{i})=\infty$(或有限) 这也就是允许$\Phi_*(\pm\mathrm{i})=\infty$，亦即允许$\Phi_*(\zeta)$在$\pm\mathrm{i}$处有任意阶极点. 注意到保持$\Phi_*(\infty)$有界，故一般解为

$$\Phi_*(\zeta)=\frac{X_*(\zeta)}{(1+\zeta^2)^m}P_{k+2m}(\zeta).$$

回到z平面，得

$$\Phi(z)=X(z)\cos^{2m}\frac{z}{a}P_{k+2m}\left(\tan\frac{z}{a}\right)=\frac{X(z)}{\cos^k\frac{z}{a}}Q_{k+2m}\left(\sin\frac{z}{a},\cos\frac{z}{a}\right),\tag{1.19}$$

或

$$\Phi(z)=\begin{cases}\Phi^+(z)=\dfrac{1}{\cos^k\frac{z}{a}}\mathrm{e}^{\Gamma(z)}Q_{k+2m}\left(\sin\dfrac{z}{a},\cos\dfrac{z}{a}\right), & z\in S^+;\\[2ex] \Phi^-(z)=\dfrac{1}{\sin^k\frac{z}{a}}\mathrm{e}^{\Gamma(z)}Q_{k+2m}\left(\sin\dfrac{z}{a},\cos\dfrac{z}{a}\right), & z\in S^-.\end{cases}\tag{1.19$'$}$$

并且因为$\Phi_*(\zeta)=O[(\zeta\pm\mathrm{i})^m]$(当$\zeta\to\pm\mathrm{i}$)，因而

$$\Phi(z)=O(\mathrm{e}^{m|y|})\quad(\text{当}\ |y|\to+\infty).\tag{1.20}$$

因之我们有

定理 4 要求 $\Phi(\pm\infty i)$ 存在(有限或无限)的齐次问题 P_1^0 的解如存在，则必有某一非负整数 m 满足(1.20)；如 m 事先指定，要求 $\Phi(z)$ 满足(1.20)，则当 $k\geqslant -2m$ 时 P_1^0 有 $k+2m+1$ 个线性无关解，当 $k<-2m$ 时，问题只有零解.

最后我们指出，运用无穷乘积，也可直接求出 $X(z)$，从而求解问题 P_1^0. 这里还要注意，下列推广的 Plemelj 公式成立：如果 $g(t)\in H$，以 $a\pi$ 为周期，而

$$\Psi(z)=\frac{1}{2a\pi i}\int_{L_0}g(t)\cot\frac{t-z}{a}dt,$$

则

$$\Psi^+(t_0)=\frac{1}{2}g(t_0)+\frac{1}{2a\pi i}\int_{L_0}g(t)\cot\frac{t-t_0}{a}dt,\quad t_0\in L,$$

$$\Psi^-(t_0)=-\frac{1}{2}g(t_0)+\frac{1}{2a\pi i}\int_{L_0}g(t)\cot\frac{t-t_0}{a}dt,\quad t_0\in L.$$

这两个公式对开口周期弧段也成立. 若要证明它们，只需把 $\cot\frac{t-z}{a}$ 展开为无穷级数. 这两个公式以后我们还要用到.

4. 非齐次问题 P_1 的讨论 这时 $g(t)\not\equiv 0$，从而 $g_*(\tau)\not\equiv 0$. 也分几种不同要求讨论.

1° 要求 $\Phi(\pm\infty i)$ 有界 当 $k\geqslant -1$ 时，问题(1.5)的一般解为

$$\Phi_*(\zeta)=X_*(\zeta)[\Psi_*(\zeta)+P_k(\zeta)],$$

其中

$$\Psi_*(\zeta)=\frac{1}{2\pi i}\int_{\Gamma_0}\frac{g_*(\tau)}{X_*^+(\tau)}\frac{d\tau}{\tau-\zeta};$$

当 $k<-1$ 时，当且仅当满足 $-k-1$ 个条件

$$\int_{\Gamma_0}\frac{g_*(\tau)}{X_*^+(\tau)}\tau^{j-1}d\tau=0\quad(j=1,\cdots,-k-1)$$

时才有唯一解.

回到 z 平面，当 $k\geqslant -1$ 时，问题 P_1 的一般解为

$$\Phi(z)=X(z)\Big[\Psi(z)+P_k\Big(\tan\frac{z}{a}\Big)\Big],\tag{1.21}$$

其中

$$\Psi(z)=\frac{1}{2a\pi i}\int_{L_0}\frac{g(t)}{X^+(t)}\Big(\cot\frac{t-z}{a}+\tan\frac{t}{a}\Big)dt.\tag{1.22}$$

当 $k\geqslant 0$ 时，可把上式右端括号中后一项略去，并入 P_k 的常数项中，而有

$$\Psi(z)=\frac{1}{2a\pi i}\int_{L_0}\frac{g(t)}{X^+(t)}\cot\frac{t-z}{a}dt;\tag{1.22$'$}$$

但当 $k=-1$ 时，此项不能略去，以保证当 $z=\pm\frac{1}{2}a\pi$ 时 $X(z)\Psi(z)$ 有界. 当 $k<-1$ 时，当且仅当满足 $-k-1$ 个条件

$$\int_{L_0}\frac{g(t)}{X^+(t)}\frac{\sin^{j-1}\frac{t}{a}}{\cos^{j+1}\frac{t}{a}}dt=0\quad(j=1,\cdots,-k-1)\tag{1.23}$$

时有唯一解(1.21)，这时 $P_k\equiv 0$，且这时 $\Psi(z)$ 必须以(1.22)给出.

这样，我们得到

定理 1′(Чибрикова) 如要求 $\Phi(\pm\infty\mathrm{i})$ 有界，则非齐次问题 P_1 当 $k\geqslant -1$ 时，问题的一般解中含 $k+1$ 个任意常数；当 $k<-1$ 时，当且仅当满足 $-k-1$ 个条件(1.23)时有唯一解.

2° 要求 $\Phi(+\infty\mathrm{i})=\Phi(-\infty\mathrm{i})$(有限) 先设 $k\geqslant 0$. 这时 1° 中一般解为

$$\Phi(z)=X(z)\left\{\frac{1}{2a\pi\mathrm{i}}\int_{L_0}\frac{g(t)}{X^+(t)}\cot\frac{t-z}{a}\,\mathrm{d}t+P_k\left(\tan\frac{z}{a}\right)\right\}. \tag{1.24}$$

令 $z=\pm\infty\mathrm{i}$ 代入，并令其相等，得

$$X(+\infty\mathrm{i})\left\{\frac{1}{2a\pi}\int_{L_0}\frac{g(t)}{X^+(t)}\mathrm{d}t+P_k(+\mathrm{i})\right\}$$
$$=X(-\infty\mathrm{i})\left\{-\frac{1}{2a\pi}\int_{L_0}\frac{g(t)}{X^+(t)}\mathrm{d}t+P_k(-\mathrm{i})\right\}.$$

由(1.16)，此条件可改写为

$$\frac{G_\infty+1}{2a\pi}\int_{L_0}\frac{g(t)}{X^+(t)}\mathrm{d}t=G_\infty P_k(-\mathrm{i})-P_k(+\mathrm{i}). \tag{1.25}$$

当 $k=0$ 时，如果 $G_\infty\neq 1$，亦即条件(1.17)不成立时，问题便有唯一解，且这时

$$P_0\equiv C_0=\frac{G_\infty+1}{G_\infty-1}\cdot\frac{1}{2a\pi}\int_{L_0}\frac{g(t)}{X^+(t)}\mathrm{d}t;$$

于是最后得

$$\Phi(z)=\frac{X(z)}{2a\pi\mathrm{i}}\int_{L_0}\frac{g(t)}{X^+(t)}\left(\cot\frac{t-z}{a}+\frac{G_\infty+1}{G_\infty-1}\mathrm{i}\right)\mathrm{d}t. \tag{1.26}$$

当 $k=0$ 时如果 $G_\infty=1$，亦即条件(1.17)成立时，则当且仅当

$$\int_{L_0}\frac{g(t)}{X^+(t)}\mathrm{d}t=0 \tag{1.27}$$

满足时，问题有一般解：

$$\Phi(z)=\frac{X(z)}{2a\pi\mathrm{i}}\left[\int_{L_0}\frac{g(t)}{X^+(t)}\cot\frac{t-z}{a}\,\mathrm{d}t+C\right], \tag{1.28}$$

其中 C 为任意常数.

当 $k\geqslant 1$ 时，问题的一般解为(1.24)，但其中 P_k 需满足条件(1.25)，亦即一般解中含有 k 个任意常数.

当 $k=-1$ 时，则 1° 中唯一解为

$$\Phi(z)=\frac{X(z)}{2a\pi\mathrm{i}}\int_{L_0}\frac{g(t)}{X^+(t)}\left(\cot\frac{t-z}{a}+\tan\frac{t}{a}\right)\mathrm{d}t. \tag{1.29}$$

要求 $\Phi(+\infty\mathrm{i})=\Phi(-\infty\mathrm{i})$，即

$$X(+\infty\mathrm{i})\int_{L_0}\frac{g(t)}{X^+(t)}\left(\mathrm{i}+\tan\frac{t}{a}\right)\mathrm{d}t=X(-\infty\mathrm{i})\int_{L_0}\frac{g(t)}{X^+(t)}\left(-\mathrm{i}+\tan\frac{t}{a}\right)\mathrm{d}t,$$

亦即

$$(G_\infty-1)\int_{L_0}\frac{g(t)}{X^+(t)}\tan\frac{t}{a}\,\mathrm{d}t=\mathrm{i}(G_\infty+1)\int_{L_0}\frac{g(t)}{X^+(t)}\mathrm{d}t. \tag{1.30}$$

这就是说，当 $k=-1$ 时，当且仅当(1.30)满足时，问题有唯一解(1.29).

当 $k<-1$ 时，则当且仅当条件(1.23)和(1.30)都满足时(共 $-k$ 个条件)，问题有唯一解(1.29)；而且，当 $G_\infty\neq 1$ 时，(1.29)仍可写成(1.26).

总之，得：

定理 2′ 如要求 $\Phi(+\infty i)=\Phi(-\infty i)$(有限)，则问题 P_1 当 $k\geqslant 1$ 时，问题的一般解中含有 k 个任意常数；当 $k=0$ 时，如(1.17)不成立，则有唯一解，而(1.17)成立时，则 $g(t)$ 要满足一个条件(1.27)时，问题就有解，且一般解中含一个任意常数；当 $k\leqslant -1$ 时，当且仅当满足 $-k$ 个条件时，问题有唯一解.

3° 要求 $\Phi(\pm\infty i)=0$ 当 $k\geqslant 0$ 时，代入(1.21)，并用(1.22)′表示 $\Psi(z)$，注意到 $X(\pm\infty i)\neq 0$，得

$$\pm\frac{1}{2a\pi}\int_{L_0}\frac{g(t)}{X^+(t)}\mathrm{d}t+P_k(\pm i)=0. \tag{1.31}$$

因此，当 $k=0$ 时，当且仅当(1.27)满足时有唯一解

$$\Phi(z)=\frac{X(z)}{2a\pi i}\int_{L_0}\frac{g(t)}{X^+(t)}\cot\frac{t-z}{a}\,\mathrm{d}t. \tag{1.32}$$

当 $k\geqslant 1$ 时，问题的一般解中含 $k-1$ 个任意常数.

当 $k=-1$ 时，要想解(1.29)满足条件 $\Phi(\pm\infty i)=0$，显然需且只需条件(1.27)以及

$$\int_{L_0}\frac{g(t)}{X^+(t)}\tan\frac{t}{a}\,\mathrm{d}t=0 \tag{1.33}$$

同时满足，且这时解成为(1.32)，且唯一.

当 $k<-1$ 时，除需满足条件(1.23)外，还需满足条件(1.27)和(1.33)，当且仅当这时(共 $-k+1$ 个条件)问题有唯一解.

故最后得

定理 3′ 如要求 $\Phi(\pm\infty i)=0$，则非齐次问题 P_1 当 $k\geqslant 1$ 时一般解中含有 $k-1$ 个任意常数；当 $k\leqslant 0$ 时，当且仅当满足 $-k+1$ 个条件时问题有唯一解.

4° 允许 $\Phi(\pm\infty i)=\infty$ 或有限 因 $X(z)\Psi(z)$ 在 $z=\pm\infty i$ 处总是有界的，故结果也与 P_1^0 问题时相同，不过要结合到可解性一同考虑. 这里不详细讨论，只把一般结果写出如下：

定理 4′ 如只要求 $\Phi(\pm\infty i)$ 存在(有限或无限)，则非齐次问题 P_1 的解如存在，则必有一非负整数 m 存在，满足(1.20)；当 m 事先指定，要求 $\Phi(z)$ 满足(1.20)时，则当 $k\geqslant -1$ 时，问题的解中含 $k+2m+1$ 个任意常数；当 $-1>k\geqslant -2m$ 时，当且仅当 $g(t)$ 满足 $-k-1$ 个条件(1.23)时，问题有解，且一般解中含 $k+2m+1$ 个任意常数；当 $k<-2m$ 时，当且仅当 $g(t)$ 满足 $-k-1$ 个条件(1.23)时，问题有唯一解，且实际上是在 $\pm\infty i$ 处有界的解.

5. 一个特例 考察 $G(t)\equiv K$ 为一常数($\neq 0$)的特殊情形，这在应用中常见.

这时 $k=0$，于是($\log K$ 任取一定值)

$$\begin{aligned}\Gamma(z)&=\frac{1}{2a\pi i}\int_{L_0}\log K\cot\frac{t-z}{a}\,\mathrm{d}t=\frac{\log K}{2\pi i}\left(\log\sin\frac{t-z}{a}\right)_L\\&=\begin{cases}\log K, & \text{当 } z\in S^+;\\ 0, & \text{当 } z\in S^-.\end{cases}\end{aligned}$$

因此$X^+(z)=K$，$X^-(z)=1$. 所以，使$\Phi(\pm\infty i)$有界的一般解是

$$\Phi(z)=\begin{cases}\dfrac{1}{2a\pi i}\displaystyle\int_{L_0}g(t)\cot\frac{t-z}{a}\,dt+C, & z\in S^+;\\ \dfrac{1}{K}\Big[\dfrac{1}{2a\pi i}\displaystyle\int_{L_0}g(t)\cot\frac{t-z}{a}\,dt+C\Big], & z\in S^-.\end{cases}\tag{1.34}$$

这直接由推广的 Plemelj 公式也可验证.

如要求$\Phi(+\infty i)=\Phi(-\infty i)$，则因现在$G_\infty=1$，故当且仅当条件(1.27)，现在是

$$\int_{L_0}g(t)dt=0\tag{1.35}$$

满足时，问题有解，且一般解就是(1.34).

如要求$\Phi(\pm\infty i)=0$，则当且仅当(1.35)满足时问题有唯一解，只要在(1.34)中令$C=0$即得.

如要求$\Phi(z)$满足条件(1.20)，则只要在(1.34)中把C改为$Q_{2m}\left(\sin\dfrac{z}{a},\cos\dfrac{z}{a}\right)$或(1.14)即可.

§2 开口弧段和间断系数情况

1. 问题的提法和转化 本节考虑L_0是由p个开口光滑弧段$\widehat{a_rb_r}(r=1,\cdots,p)$构成的情况，而$G(t),g(t)$在每一弧段上连端点在内均$\in H$，且$G(t)\neq 0$. 每一弧段设以$a_r$为起点，$b_r$为终点，而构成正向.

把a_r,b_r统一记作$c_1,\cdots,c_{2p}$，和经典情况一样(见[8]，§79)，可以把它们分成两类如下. 在每一$\widehat{a_rb_r}$上，任取$\log G(t)$的一支，设

$$\mp\frac{1}{2\pi i}\log G(c_j)=\alpha_j+i\beta_j,\quad j=1,\cdots,2p,$$

这里，当$c_j=a_r$时，左边取负号，当$c_j=b_r$时，取正号. 如α_j为一整数，则称c_j为一特殊端；如α_j不是整数，则称c_j为平常端. 记所有平常端为$c_1,\cdots,c_m$，特殊端为$c_{m+1},\cdots,c_{2p}$.

我们要求(1.1)的分片全纯周期解，使在平常端$c_1,\cdots,c_q(q\leqslant m)$附近保持有界，而在其余端点附近，至少无界可积. 把这种解仍称作属于$h_q=h(c_1,\cdots,c_q)$类. 和经典情况一样，可以证明，这样的解如存在，它在特殊端附近必几乎有界：$\lim\limits_{z\to c_j}(z-c_j)^\varepsilon\Phi(z)=0\ (j>m)$，$\varepsilon$为任意小正数.

定义问题(1.1)的指标如下：对于特殊端c_j，令$\lambda_j=-\alpha_j$；对于平常端c_j，当$j\leqslant q$时，选取整数λ_j使$0<\alpha_j+\lambda_j<1$，当$j>q$时，使$-1<\alpha_j+\lambda_j<0$. 称

$$k=-\sum_{j=1}^{2p}\lambda_j\tag{2.1}$$

为问题(1.1)关于类h_q的指标.

仍用(1.4)变换到ζ平面，问题就转化为(1.5). 因为L_0都不经过(1.4)的奇点，所以$G_*(\tau),g_*(\tau)$在Γ_0上仍保有原来的性质，且各端点的类型也不变. 因此，求原问题(1.1)在类$h(c_1,\cdots,c_q)$中的解，就变成求(1.5)在类$h(c_1^*,\cdots,c_q^*)$中的解，这里$c_j^*=\tan\dfrac{c_j}{a}$ $(j=1,$

$\cdots,2p$)，但在 $\zeta=\pm i$ 处仍保留有任意性，而 $\Phi_*(\infty)$则必须有界.

2. 齐次问题 P_1^0 的讨论 不论对 $\Phi(\pm\infty i)$作何限制，和问题(1.5)的解的性质一样，我们知道：齐次问题 P_1^0 的解 $\Phi(z)$，在特殊端附近，必定有界；在平常端附近，如有界，则必为零.

1° 要求 $\Phi(\pm\infty i)$有界 这时问题 (1.5) 在 h_q 类中的一般解为

$$\Phi_*(\zeta)=X_*(\zeta)P_k(\zeta),$$

其中 $X_*(\zeta)=\Pi_*(\zeta)e^{\Gamma_*(\zeta)}$，而

$$\Pi_*(\zeta)=\prod_{j=1}^{2p}(\zeta-c_j^*)^{\lambda_j},\quad \Gamma_*(\zeta)=\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\log G_*(\tau)}{\tau-\zeta}d\tau.$$

回到 z 平面，得(1.1) 在 h_q 类中的一般解为

$$\Phi(z)=X(z)P_k\left(\tan\frac{z}{a}\right),\tag{2.2}$$

其中典则函数

$$X(z)=\Pi(z)e^{\Gamma(z)},\tag{2.3}$$

而

$$\Pi(z)=\prod_{j=1}^{2p}\left(\tan\frac{z}{a}-\tan\frac{c_j}{a}\right)^{\lambda_j},\tag{2.4}$$

$$\Gamma(z)=\frac{1}{2a\pi i}\int_{L_0}\log G(t)\cot\frac{t-z}{a}\,dt.\tag{2.5}$$

如再把 $\Pi(z)$ 的一常数因子并入 P_k 中，则一般解(2.2) 还可写成

$$\Phi(z)=\prod_{j=1}^{2p}\sin^{\lambda_j}\frac{z-c_j}{a}\,e^{\Gamma(z)}Q_k\left(\sin\frac{z}{a},\cos\frac{z}{a}\right).\tag{2.6}$$

注意，上节定理 1 仍成立①.

2° 要求 $\Phi(+\infty i)=\Phi(-\infty i)$ (有限) 这时仍要求条件(1.15) 成立，但现在

$$G_\infty=\frac{X(-\infty i)}{X(+\infty i)}=\prod_{j=1}^{2p}\left[\frac{\tan\frac{c_j}{a}+i}{\tan\frac{c_j}{a}-i}\right]^{\lambda_j}\exp\left\{-\frac{1}{a\pi}\int_{L_0}\log G(t)\,dt\right\}$$

$$=(-1)^k\exp\left\{-\frac{2i}{a}\sum_{j=1}^{2p}\lambda_jc_j-\frac{1}{a\pi}\int_{L_0}\log G(t)\,dt\right\}.\tag{2.7}$$

注意 $k=0$ 时，$G_\infty=1$ 的条件(1.17) 现在成为

$$\frac{1}{2\pi i}\int_{L_0}\log G(t)\,dt\equiv-\sum_{j=1}^{2p}\lambda_jc_j\pmod{a\pi}.\tag{2.8}$$

当所有 c_j 为特殊端时，$\lambda_j=0$,(2.8) 便还原到(1.17).

因此，这时定理 2 仍成立，但其中条件(1.17) 一般地要改为(2.8).

3° 要求 $\Phi(\pm\infty i)=0$ 这时问题一般解为

$$\Phi(z)=\prod_{j=1}^{2p}\sin^{\lambda_j}\frac{z-c_j}{a}\cdot\frac{e^{\Gamma(z)}}{\cos^k\frac{z}{a}}Q_{k-2}\left(\sin\frac{z}{a},\cos\frac{z}{a}\right),\tag{2.9}$$

① 当然，要把解理解为 h_q 类中的解，k 理解为 h_q 类中的指标. 下同.

且定理 3 仍成立.

4° 允许 $\Phi(\pm\infty i)=\infty$（或有限） 一般解为

$$\Phi(z)=\prod_{j=1}^{2p}\sin^{\lambda_j}\frac{z-c_j}{a}\cdot\frac{e^{\Gamma(z)}}{\cos^k\frac{z}{a}}Q_{k+2m}\left(\sin\frac{z}{a},\cos\frac{z}{a}\right), \tag{2.10}$$

且条件(1.20) 和定理 4 成立.

类似地可以考虑相联周期边值问题及在相联类中的解，并进行讨论；其结果与经典情况同，从略.

3. 非齐次问题 P_1 的讨论 这时，和经典情况一样，不论对 $\Phi(\pm\infty i)$ 作何种限制，在特殊端 c_j 附近，如 $G(c_j)\neq1$，则 $\Phi(z)$ 仍有界；如 $G(c_j)=1$，则一般 $\Phi(z)$ 只能是几乎有界，除非 $g(c_j)=0$（见[8]，§81).

对 $\Phi(\pm\infty i)$ 作类似于 §1 中 1°～4°的要求时，那里的讨论这里均成立，定理 1′～4′也成立；当然，这里 $X(z)$ 要理解为(2.3)，而提到的条件(1.17)要改为(2.8).

4. 一个特例 为着以后应用需要，考察一特例：L_0 只含 x 轴上一线段 γ_0：$-l\leqslant t\leqslant l$ $(l<\frac{1}{2}a\pi)$，且 $G(t)=-K$ 是一负常数的情形. 并求在类 h_0 中的解，即在 $z=\pm l$ 处允许无界可积的解.

这里 $c_1=-l$，$c_2=+l$. 取 $\log(-K)=\ln K+\pi i$（$\ln K$ 指实值)，因此

$$\alpha_1+i\beta_1=-\frac{1}{2}+i\beta,\qquad \alpha_2+i\beta_2=\frac{1}{2}-i\beta,$$

其中已设

$$\beta=\frac{\ln K}{2\pi}. \tag{2.11}$$

因此，它们都是平常端，且在 h_0 类中，$\lambda_1=0$，$\lambda_2=-1$，即指标 $k=1$. 这时易见

$$\Gamma(z)=\frac{\log(-K)}{2a\pi i}\int_{-l}^{l}\cot\frac{t-z}{a}dt=\left(\frac{1}{2}-i\beta\right)\log\frac{\tan\frac{z}{a}-\tan\frac{l}{a}}{\tan\frac{z}{a}+\tan\frac{l}{a}},$$

从而

$$X(z)=\Pi(z)e^{\Gamma(z)}=\left(\tan\frac{z}{a}+\tan\frac{l}{a}\right)^{-\frac{1}{2}+i\beta}\left(\tan\frac{z}{a}-\tan\frac{l}{a}\right)^{-\frac{1}{2}-i\beta}, \tag{2.12}$$

这里已设 z 平面沿 γ_0 剖开，且 $X(z)$ 已任意取定一支，例如：

$$\lim_{z\to\pm\frac{1}{2}a\pi}\tan\frac{z}{a}X(z)=1. \tag{2.13}$$

因此，如补充要求 $\Phi(\pm\infty i)$ 有界，则问题的一般解为

$$\Phi(z)=\frac{X(z)}{2a\pi i}\int_{-l}^{l}\frac{g(t)}{X^+(t)}\cot\frac{t-z}{a}dt+X(z)\left(C_0\tan\frac{z}{a}+C_1\right). \tag{2.14}$$

如要求 $\Phi(+\infty i)=\Phi(-\infty i)$，则因由(2.7)，现在

$$G_\infty=-K^{\frac{a\pi}{2l}},$$

故自条件(1.25) 知，需选取 C_0,C_1 使得

$$C_0-iC_1\frac{G_\infty-1}{G_\infty+1}=\frac{1}{2a\pi i}\int_{-l}^{l}\frac{g(t)}{X^+(t)}dt,$$

所以最后有

$$\Phi(z)=\frac{X(z)}{2a\pi\mathrm{i}}\int_{-l}^{l}\frac{g(t)}{X^{+}(t)}\left(\cot\frac{t-z}{a}+1\right)\mathrm{d}t+CX(z)\left(\tan\frac{z}{a}+\mathrm{i}\frac{G_{\infty}-1}{G_{\infty}+1}\right).\quad(2.15)$$

如要求 $\Phi(\pm\infty\mathrm{i})=0$，则由(1.31)，应选取 C_0,C_1 使

$$C_0=-\frac{1}{2a\pi\mathrm{i}}\int_{-l}^{l}\frac{g(t)}{X^{+}(t)}\mathrm{d}t,\quad C_1=0,$$

因之得唯一解

$$\Phi(z)=\frac{X(z)}{2a\pi\mathrm{i}}\int_{-l}^{l}\frac{g(t)}{X^{+}(t)}\left(\cot\frac{t-z}{a}-\tan\frac{z}{a}\right)\mathrm{d}t.\quad(2.16)$$

特别地，如 $G(t)=-1$，即 $K=1$，则由(2.11)，$\beta=0$，于是

$$X(z)=\frac{1}{\mathrm{i}\sqrt{R(z)}},\quad R(z)=\tan^2\frac{l}{a}-\tan^2\frac{z}{a},\quad(2.12)'$$

其中 $\sqrt{R(z)}$ 已取定相应于条件(2.13)的一支，易见这相当于当 z 由上半平面趋于 γ_0 上的点时，$\sqrt{R(t)}$ 取正值的一支.

这时要求 $\Phi(\pm\infty\mathrm{i})$ 有界的一般解为

$$\Phi(z)=\frac{1}{2a\pi\mathrm{i}\sqrt{R(z)}}\int_{-l}^{l}g(t)\sqrt{R(t)}\cot\frac{t-z}{a}\mathrm{d}t+\frac{C_0\tan\frac{z}{a}+C_1}{\sqrt{R(z)}}.\quad(2.14)'$$

要求 $\Phi(+\infty\mathrm{i})=\Phi(-\infty\mathrm{i})$ 时的一般解为

$$\Phi(z)=\frac{1}{2a\pi\mathrm{i}\sqrt{R(z)}}\int_{-l}^{l}g(t)\sqrt{R(t)}\left(\cot\frac{t-z}{a}-\tan\frac{z}{a}\right)\mathrm{d}t+C\sqrt{\frac{\tan\frac{z}{a}-\tan\frac{l}{a}}{\tan\frac{z}{a}+\tan\frac{l}{a}}},\quad(2.15)'$$

这里后一根号应理解为：当 z 由上半平面趋于 γ_0 上的点时取正值.

要求 $\Phi(\pm\infty\mathrm{i})=0$ 时的唯一解为

$$\Phi(z)=\frac{1}{2a\pi\mathrm{i}\sqrt{R(z)}}\int_{-l}^{l}g(t)\sqrt{R(t)}\left(\cot\frac{t-z}{a}-\tan\frac{z}{a}\right)\mathrm{d}t.\quad(2.16)'$$

5. 间断系数情况 设 L_0 如 §1，而 $G(t),g(t)$ 在 L_0 上有有限个第一类间断点 $c_1,\cdots,c_p$，在每一以二相邻间断点为端点的“连续弧”上(连同端点在内)，均 $\in H$，且 $G(t)\neq 0$.

仍与通常一样，如 $G(c_j+0)/G(c_j-0)$ 不是整数，则称 c_j 为平常端，记作 $c_1,\cdots,c_m$；如它是整数，则称 c_j 为特殊端，记作 $c_{m+1},\cdots,c_p$. 要求问题(1.1)的类 $h_q=h(c_1,\cdots,c_q)$ $(q\leqslant m)$ 中的解，即在 $c_1,\cdots,c_q$ 附近有界，而在其余端点附近至少无界可积(在特殊端附近，必几乎有界).

类似地定义指标(见[8]，§85). 从某一端点开始，沿 L_0 正向环行，在第一“连续弧”上任取 $\log G(t)$ 的一支，而在每越过一间断点 c_j 时，取 $\log G(c_j+0)$ 的一支如下：设

$$-\frac{1}{2\pi\mathrm{i}}[\log G(c_j+0)-\log G(c_j-0)]=\alpha_j+\mathrm{i}\beta_j,$$

如 c_j 为特殊端，则使 $\alpha_j=0$；如 c_j 为平常端，当 $j\leqslant q$ 时，则使 $0<\alpha_j<1$，当 $m\geqslant j>q$ 时，则使 $-1<\alpha_j<0$. 这样，直到出发的端点的正侧为止. 我们称

$$k=\frac{1}{2\pi\mathrm{i}}[\log G(t)]_{L_0}=\frac{1}{2\pi}[\arg G(t)]_{L_0}$$

（包括各间断点处的跳跃）为问题(1.1)属于类 h_q 的指标. 定出指标后，§1 中的一切讨论，甚至所有公式，均可搬到这里来，这里就不再重复.

附注 1 单周期的 Hilbert 边值问题可以用这里的方法转化为经典的 Hilbert 问题，因此可以认为完全解决. Л. И. Чибрикова 曾对某种特殊域作出有效解（见 [9]，但本文作者只看到苏联数学评论中的摘要），用这里的方法，还可对更广泛的域作出有效解.

附注 2 用这里的方法，还可解决带变位的周期 Riemann 问题.

第二章 在弹性平面问题中的应用

§3 预先的说明

本章中将讨论各向同性的弹性平面或半平面中的某些周期边值问题. 这里主要是推广[7]第六章中的许多结果到周期情形. 假设所论及的边界条件都是以 $a\pi$ 为周期的. 我们永远在下述假定下求问题的解：应力、位移都以 $a\pi$ 为周期，且应力（在无穷远处）还是有界的（位移却不一定).

为了以后需要，先回忆一下 Колосов-Мусхелишвили 方程（见 [7]，第二章). 设用 $\sigma_x,\sigma_y,\tau_{xy}$ 分别表示任何点 $z=x+\mathrm{i}y$ 处的应力分量，u,v 表示位移分量，则有

$$\sigma_x+\sigma_y=2[\Phi(z)+\overline{\Phi(z)}],\tag{3.1}$$

$$\sigma_y-\sigma_x+2\mathrm{i}\tau_{xy}=2[\bar{z}\Phi'(z)+\Psi(z)],\tag{3.2}$$

$$2\mu(u+\mathrm{i}v)=\kappa\varphi(z)-z\,\overline{\varphi'(z)}-\overline{\psi(z)},\tag{3.3}$$

其中

$$\Phi(z)=\varphi'(z),\quad \Psi(z)=\psi'(z)\tag{3.4}$$

为弹性体所占域内的某两全纯函数. 一般，虽然 u,v 是单值的，但可能 $\varphi(z),\psi(z)$ 是多值解析的. 这里

$$\mu=\frac{E}{2(1+\sigma)},\tag{3.5}$$

其中 E 和 σ 分别是弹性体的杨氏模数和泊松比，又

$$\kappa=3-4\sigma\quad 或\quad \kappa=\frac{3-\sigma}{1+\sigma}\tag{3.6}$$

随平面形变或广义平面应力状态而定. 因 $E>0$, $0<\sigma<\frac{1}{2}$，故知

$$\mu>0,\quad 1<\kappa<3.\tag{3.7}$$

如弹性体是半平面，我们永远设为下半平面 S^-，则还可以只用一个全纯函数来表达应力和位移分布(见[7]，§112). 这时，当 z 在上半平面 S^+ 时，令

$$\overline{\Phi}(z)=\overline{\Phi(\bar{z})},\ \overline{\Psi}(z)=\overline{\Psi(\bar{z})},\quad z\in S^+,\tag{3.8}$$

然后再令

$$\Phi(z) = -\overline{\Phi}(z) - z\overline{\Phi}'(z) - \overline{\Psi}(z), \quad z \in S^+, \tag{3.9}$$

则(3.1) ~ (3.3) 便可改写为

$$\sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}], \tag{3.10}$$

$$\sigma_y - \sigma_x + 2\mathrm{i}\tau_{xy} = 2[(\bar{z} - z)\Phi'(z) - \Phi(z) - \overline{\Phi}(z)] \tag{3.11}$$

或

$$\sigma_y - \mathrm{i}\tau_{xy} = \Phi(z) - \Phi(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \tag{3.12}$$

$$2\mu(u' + \mathrm{i}v') = \kappa\Phi(z) + \Phi(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)}, \tag{3.13}$$

此处 $u' = \dfrac{\partial u}{\partial x}$, $v' = \dfrac{\partial v}{\partial x}$; 且

$$2\mu(u + \mathrm{i}v) = \kappa\varphi(z) + \varphi(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)} + C \tag{3.14}$$

(C 为任意常数),这里不论 z 在 S^- 或 S^+ 中,均已令 $\varphi'(z) = \Phi(z)$. 此外还要注意,在 x 轴上无载荷的部分,$\Phi(z)$ 在 S^- 和 S^+ 中互为解析延拓.

与通常弹性平面问题不完全一样,对于周期的弹性平面问题,在 $z = \pm\infty\mathrm{i}$ 处存在有应力和位移: $\sigma_x(\pm\infty\mathrm{i}),\cdots$. 对于半平面周期问题,则在 $z = -\infty\mathrm{i}$ 处有应力和位移. 它们应理解为在以上诸式中令 $z \to \pm\infty\mathrm{i}$ 时的极限. 在 $\pm\infty\mathrm{i}$ 处的外应力主矢量 $X(\pm\infty\mathrm{i}) + \mathrm{i}Y(\pm\infty\mathrm{i})$ 应理解为在一个周期线段 z 到 $z + a\pi$ 上应力主矢量的极限,即

$$X(\pm\infty\mathrm{i}) = a\pi\tau_{xy}(\pm\infty\mathrm{i}), \quad Y(\pm\infty\mathrm{i}) = a\pi\sigma_y(\pm\infty\mathrm{i}). \tag{3.15}$$

如前所述,我们恒假定它们是有限的,但一般地在 $z = +\infty\mathrm{i}$ 和 $-\infty\mathrm{i}$ 处的值并不相等.

我们有下面二引理.

引理 1 *如弹性平面中有一列以 $a\pi$ 为周期的孔,在孔边界上周期外应力主矢量为零,而在 $z = \pm\infty\mathrm{i}$ 处的外应力为 $\sigma = \sigma_y(\pm\infty\mathrm{i}), \tau = \tau_{xy}(\pm\infty\mathrm{i})$*①, *则(3.3) 中的 $\varphi(z), \psi(z)$ 均单值,且*

$$\varphi(z) = \varphi_0(z) + \beta z, \tag{3.16}$$

$$\psi(z) = \psi_0(z) - z\varphi'(z) + \kappa\bar{\beta}z = (\kappa\bar{\beta} - \beta)z - z\varphi_0'(z) + \psi_0(z), \tag{3.17}$$

其中

$$\beta = \frac{-\sigma + \mathrm{i}\tau}{1 + \kappa}, \tag{3.18}$$

而 $\varphi_0(z), \psi_0(z)$ 均以 $a\pi$ 为周期,且有界.

证 $\varphi(z), \psi(z)$ 的单值性和 $\varphi_0(z), \psi_0(z)$ 的周期性证明见[6], § 55. 我们来证明后者的有界性.

令 $\zeta = \tan\dfrac{z}{a}$, 因 $\varphi_0(z)$ 以 $a\pi$ 为周期,故知

$$\varphi_0(z) = \varphi_0^*(\zeta)$$

为 Σ_0^- 中的全纯函数 (Σ_0^- 为 ζ 平面中 S^- 的对应域). 在 $\zeta = \mathrm{i}$ 附近,设

$$\varphi_0^*(\zeta) = \sum_{n=-\infty}^{+\infty} a_n(\zeta - \mathrm{i})^n = \sum_{n=-\infty}^{+\infty} a_n r^n \mathrm{e}^{\mathrm{i}n\theta}, \quad \zeta - \mathrm{i} = r\mathrm{e}^{\mathrm{i}\theta},$$

于是

① 因在孔边上外应力主矢量为零,故在 $z = \pm\infty\mathrm{i}$ 处的外应力必相等,以保证平衡.

$$\Phi_0(z)=\varphi_0'(z)=\varphi_0^{*\prime}(\zeta)\frac{\mathrm{d}\zeta}{\mathrm{d}z}=\frac{1+\zeta^2}{a}\sum_{n=-\infty}^{+\infty}na_nr^{n-1}\mathrm{e}^{\mathrm{i}(n-1)\theta}.$$

代入(3.1)中，为要使 σ_x,σ_y 有界，易见必须

$$[(n-1)a_{n-1}+na_n\mathrm{i}]\mathrm{e}^{\mathrm{i}n\theta}+[(n-1)\overline{a}_{n-1}-n\overline{a}_n\mathrm{i}]\mathrm{e}^{-\mathrm{i}n\theta}=0,\quad n=-1,-2,\cdots.$$

由 θ 的任意性，立刻知道 $a_{-n}=0$，$n=1,2,\cdots$. 这就证明了 $\varphi_0^*(\zeta)$ 在 $\zeta=\mathrm{i}$ 附近有界，亦即 $\varphi_0(z)$ 在 $z=+\infty\mathrm{i}$ 附近有界. 同理可证它在 $z=-\infty\mathrm{i}$ 附近有界.

对于 $\psi_0(z)$，注意到(3.2)式现在可改写为

$$\sigma_y-\sigma_x+2\mathrm{i}\tau_{xy}=2[(\overline{z}-z)\varphi_0''(z)+\psi_0'(z)-\varphi_0'(z)+\beta+\kappa\overline{\beta}],\qquad(3.19)$$

又易见在 $z=\pm\infty\mathrm{i}$ 附近，$\varphi_0'(z)$ 有界，且 $(\overline{z}-z)\varphi_0''(z)\to0$，故必 $\psi_0'(z)$ 有界. 由此也立刻知道 $\psi_0(z)$ 有界. 引理证毕.

引理2 *在弹性半平面的周期问题中，(3.10)～(3.13)中的 $\Phi(z)$ 必在全平面中以 $a\pi$ 为周期，且有界*①.

证 先设 $z\in S^-$. 由(3.10)易证(见[6]，§55)

$$\Phi'(z+a\pi)=\Phi'(z),\quad z\in S^-.$$

因 S^- 是单连通的，所以立得

$$\Phi(z+a\pi)=\Phi(z)+C,\quad z\in S^-,\qquad(*)$$

其中 C 为常数. 由(3.12)，因其左边以 $a\pi$ 为周期，故

$$\Phi(\overline{z}+a\pi)=\Phi(\overline{z})+C,\quad z\in S^-.$$

亦即(*)式对 $z\in S^+$ 也成立. 再由(3.13)左边的周期性，立刻知道 $C=0$. 因而 $\Phi(z)$ 在全平面中以 $a\pi$ 为周期.

$\Phi(z)$ 在 $z=-\infty\mathrm{i}$ 附近的有界性证明与引理1相似. 在 $z=+\infty\mathrm{i}$ 处的有界性随之从(3.13)立得. 引理证毕.

§4 弹性平面中的周期焊接问题

弹性平面中一般的周期基本问题可化为积分方程处理，本节中只讨论用周期 Riemann 边值问题的方法更快地求解周期焊接问题.

1. 弹性平面和焊接物材料一致的情况 设弹性平面中有一列周期孔，其边界为 L_0，$L_{\pm1}$，…(图1)，以 $a\pi$ 为周期. 今在各孔上焊接材料相同的垫圈. 设孔边和垫圈周线间的位置差异已知：

$$(u^++\mathrm{i}v^+)-(u^-+\mathrm{i}v^-)=g(t),\quad t\in L,\qquad(4.1)$$

$g(t)$ 也以 $a\pi$ 为周期，并设 $g'(t)\in H$. 在 $z=\pm\infty\mathrm{i}$ 处外应力为 σ,τ. 求弹性体的平衡. 由上节引理1知，(3.16)，(3.17)中的 $\varphi_0(z)$，$\psi_0(z)$ 以 $a\pi$ 为周期，且有界.

由[7]，§109，问题可化为求解最简单的 Riemann 边值问题：

$$\varphi^+(t)-\varphi^-(t)=\frac{2\mu}{\kappa+1}g(t),\quad \psi^+(t)-\psi^-(t)=\frac{2\mu}{\kappa+1}h(t),$$

---

① 在 x 轴上，在载荷不为0的地方 t 处，这要理解为

$$\Phi^+(t+a\pi)=\Phi^+(t),\quad \Phi^-(t+a\pi)=\Phi^-(t).$$

其中 $t\in L$，而 $h(t)=-\overline{g(t)}-\bar{t}g'(t)$.

把 $\varphi(t),\psi(t)$ 换作 $\varphi_0(t),\psi_0(t)$，立得

$$\varphi_0^+(t)-\varphi_0^-(t)=\frac{2\mu}{\kappa+1}g(t),\tag{4.2}$$

$$\psi_0^+(t)-\psi_0^-(t)=\frac{2\mu}{\kappa+1}h_0(t),\tag{4.3}$$

其中

$$h_0(t)=-\overline{g(t)}+(t-\bar{t})g'(t).\tag{4.4}$$

显然 $h_0(t)$ 已是以 $a\pi$ 为周期的函数，且 $\in H$. 于是问题化为求解周期 Riemann 边值问题 (4.2) 和(4.3)，并要求 $\varphi_0(z),\psi_0(z)$ 在 $z=\pm\infty\mathrm{i}$ 处有界.

这两问题的指标 $k=0$，故其一般解为

$$\varphi_0(z)=\frac{\mu}{(\kappa+1)a\pi\mathrm{i}}\int_{L_0}g(t)\cot\frac{t-z}{a}\,\mathrm{d}t,\tag{4.5}$$

$$\psi_0(z)=\frac{\mu}{(\kappa+1)a\pi\mathrm{i}}\int_{L_0}h_0(t)\cot\frac{t-z}{a}\,\mathrm{d}t;\tag{4.6}$$

这里已略去了任意常数项，因为它们只会影响整个系统的刚性平移.

用(3.16),(3.17) 回到 $\varphi(z),\psi(z)$，便可得最后所需结果.

附注 1　如 L 是一排周期性裂缝，以上所论也同样地成立；但这时要求 $g(t)$ 在 L 的各端点处必须为零.

附注 2　如讨论的是有一排周期孔的半平面或无限带形，则问题可化为半平面或无限带形中的周期第一基本问题，方法同[7]，§109. 对于半平面的周期第一基本问题，还可参看下节第一段.

例 1　一排圆孔、圆垫圈的特例(材料相同). 设 L_0 是半径为 r 的圆：$|z|=r$，垫圈也是圆形，但半径略大：$r+\varepsilon$，$\varepsilon>0$. 并设在 $z=\pm\infty\mathrm{i}$ 处无外应力. 于是

$$\left.\begin{aligned}g(t)&=-\varepsilon e^{\mathrm{i}\theta}=-\frac{\varepsilon t}{r},\\ h_0(t)&=\frac{\varepsilon\bar{t}}{r}-(t-\bar{t})\frac{\varepsilon}{r}=\frac{2\varepsilon r}{t}-\frac{\varepsilon t}{r},\end{aligned}\right\}\quad t=re^{\mathrm{i}\theta}\in L_0.$$

现在 $\sigma=\tau=0$，由(4.5),(4.6) 立得

$$\varphi_0(z)=-\frac{\mu\varepsilon}{(\kappa+1)a\pi r\mathrm{i}}\int_{L_0}t\cot\frac{t-z}{a}\,\mathrm{d}t=\begin{cases}-\dfrac{2\mu\varepsilon z}{(\kappa+1)r}, & z\in S_0^+;\\ 0, & z\in S^-,\end{cases}$$

$$\begin{aligned}\psi_0(z)&=\frac{\mu\varepsilon}{(\kappa+1)a\pi\mathrm{i}}\int_{L_0}\left(\frac{2r}{t}-\frac{t}{r}\right)\cot\frac{t-z}{a}\,\mathrm{d}t\\ &=\begin{cases}\dfrac{2\mu\varepsilon}{\kappa+1}\left(\dfrac{2r}{z}-\dfrac{z}{r}-\dfrac{2r}{a}\cot\dfrac{z}{a}\right), & z\in S_0^+;\\ -\dfrac{4\mu\varepsilon r}{(\kappa+1)a}\cot\dfrac{z}{a}, & z\in S^-.\end{cases}\end{aligned}$$

回到 $\varphi(z),\psi(z)$，最后有

$$\varphi(z)=\begin{cases}-\dfrac{2\mu\varepsilon}{\kappa+1}\dfrac{z}{r}, & z\in S_0^+;\\ 0, & z\in S^-,\end{cases}$$

$$\psi(z)=\begin{cases}\dfrac{4\mu\varepsilon r}{\kappa+1}\left(\dfrac{1}{z}-\dfrac{1}{a}\cot\dfrac{z}{a}\right), & z\in S_0^+;\\ -\dfrac{4\mu\varepsilon r}{(\kappa+1)a}\cot\dfrac{z}{a}, & z\in S^-.\end{cases}$$

对于 $z\in S_j^+(j=\pm 1,\cdots)$，只要作周期延拓.

令 $a\to+\infty$ 时，还可得出无限平面一个圆孔的焊接公式：

$$\varphi(z)=\begin{cases}-\dfrac{2\mu\varepsilon}{\kappa+1}\dfrac{z}{r}, & z\in S^+;\\ 0, & z\in S^-,\end{cases}\quad \psi(z)=\begin{cases}0, & z\in S^+;\\ -\dfrac{4\mu\varepsilon r}{(\kappa+1)z}, & z\in S^-.\end{cases}$$

这和[7]，§ 109a 中的结果一致.

2. 弹性平面和焊接物剪切模数相同的情况 设弹性平面和垫圈的材料不一致，而剪切模数 μ 相同，只是杨氏模数或泊松比不同. 这时问题仍可化为周期 Riemann 边值问题而解决.

设对于垫圈 S^+ 而言，$\kappa=\kappa^+$，而对于带孔平面 S^-，$\kappa=\kappa^-$. 在 L 上应力平衡的条件为

$$\varphi^+(t)+t\,\overline{\varphi'^+(t)}+\overline{\psi^+(t)}=\varphi^-(t)+t\,\overline{\varphi'^-(t)}+\overline{\psi^-(t)},\quad t\in L. \tag{4.7}$$

仍设在 L 上位移的周期跳跃 $(u^++\mathrm{i}v^+)-(u^-+\mathrm{i}v^-)=g(t)$，且 $g'(t)\in H$，则由焊接连续性条件，得

$$\kappa^+\varphi^+(t)-t\,\overline{\varphi'^+(t)}-\overline{\psi^+(t)}=\kappa^-\varphi^-(t)+t\,\overline{\varphi'^-(t)}-\overline{\psi^-(t)}+2\mu g(t),\quad t\in L. \tag{4.8}$$

将(4.7)和(4.8)相加，便得 $\varphi(z)$ 所应满足的 Riemann 周期边值条件：

$$\varphi^+(t)=\frac{\kappa^-+1}{\kappa^++1}\varphi^-(t)+\frac{2\mu}{\kappa^++1}g(t). \tag{4.9}$$

对于 $\psi(z)$ 所应满足的条件，只要在(4.7)中取共轭值，并将(4.9)的结果代入：

$$\psi^+(t)-\psi^-(t)=-\left[\overline{\varphi^+(t)-\varphi^-(t)}+\bar{t}(\varphi'^+(t)-\varphi'^-(t))\right]. \tag{4.10}$$

现在再把 $\varphi(z),\psi(z)$ 改为周期函数 $\varphi_0(z),\psi_0(z)$ 如下. 这时(3.16)，(3.17)仍成立，但对于 S^+ 和 S^- 中的 z，公式中的 β 也有分别不同的值[见(3.18)]：

$$\beta^+=\frac{-\sigma+\mathrm{i}\tau}{1+\kappa^+},\quad \beta^-=\frac{-\sigma+\mathrm{i}\tau}{1+\kappa^-}, \tag{4.11}$$

这里 σ,τ 仍是 $z=\pm\infty\mathrm{i}$ 处的外应力. 由此易见

$$\beta^+(1+\kappa^+)=\beta^-(1+\kappa^-)=-\sigma+\mathrm{i}\tau. \tag{4.12}$$

现在代替(3.16)和(3.17)，在 S^+,S^- 中分别有

$$\varphi^+(z)=\varphi_0^+(z)+\beta^+z,\quad \varphi^-(z)=\varphi_0^-(z)+\beta^-z; \tag{4.13}$$

$$\psi^+(z)=\psi_0^+(z)-z\varphi'^+(z)+\kappa^+\overline{\beta^+}z,\quad \psi^-(z)=\psi_0^-(z)-z\varphi'^-(z)+\kappa^-\overline{\beta^-}z. \tag{4.14}$$

边界条件(4.9)现在成为对于 $\varphi_0(z)$ 的边界条件

$$\varphi_0^+(t)=\frac{\kappa^-+1}{\kappa^++1}\varphi_0^-(t)+\frac{2\mu}{\kappa^++1}g(t). \tag{4.15}$$

对于边界条件(4.10)，现在化为 $\psi_0(z)$ 的边界条件

$$\begin{aligned}\psi_0^+(t)-\psi_0^-(t)&=-[\overline{\varphi_0^+(t)}-\overline{\varphi_0^-(t)}]+(t-\bar{t})[\varphi_0'^+(t)-\varphi_0'^-(t)]+2(t-\bar{t})\mathrm{Re}(\beta^+-\beta^-)\\&=-[\overline{\varphi_0^+(t)}-\overline{\varphi_0^-(t)}]+(t-\bar{t})\left[\varphi_0'^+(t)-\varphi_0'^-(t)+\frac{(\kappa^+-\kappa^-)\sigma}{(1+\kappa^+)(1+\kappa^-)}\right].\end{aligned} \tag{4.16}$$

由(4.15)，立即可写出(不计刚性位移)

$$\varphi_0(z)=\begin{cases}\varphi_0^+(z)=\dfrac{\mu}{(\kappa^++1)a\pi\mathrm{i}}\displaystyle\int_{L_0}g(t)\cot\frac{t-z}{a}\,\mathrm{d}t, & z\in S^+;\\ \varphi_0^-(z)=\dfrac{\mu}{(\kappa^-+1)a\pi\mathrm{i}}\displaystyle\int_{L_0}g(t)\cot\frac{t-z}{a}\,\mathrm{d}t, & z\in S^-.\end{cases}\tag{4.17}$$

由此算出 $\varphi_0^+(t),\varphi_0^-(t)$ 代入(4.16)，便可求解 $\psi_0(z)$. 因此问题便最后解决.

如果 $g(t)$ 满足条件

$$\int_{L_0}g(t)\cot\frac{t-z}{a}\,\mathrm{d}t=0,\quad z\in S^-,\tag{4.18}$$

则 $\varphi_0^-(z)\equiv0$，而 $\varphi_0^+(z)$ 仍以(4.17)的前一式给出，且由(4.15)，

$$\varphi_0^+(t)=\frac{2\mu}{(\kappa^++1)}g(t).$$

以此代入(4.16)，并设 $\sigma=\tau=0$，则

$$\psi_0^+(t)-\psi_0^-(t)=\frac{2\mu}{\kappa^++1}[-\overline{g(t)}+(t-\bar t)g'(t)]=\frac{2\mu}{\kappa^++1}h_0(t),$$

结果与(4.3)相同——只是 κ 改为了 κ^+. 且因这时 $\beta^+=\beta^-=0$，所以(4.13)和(4.14)也与(3.16)和(3.17)当 $\beta=0$ 时相同，即 $\varphi(z),\psi(z)$ 也相同.

同理，如 $\sigma=\tau=0$，且对 $z\in S^+,g(t)$ 满足条件(4.18)，则也可得类似结果，但 κ 换作了 κ^-. 于是我们得到

定理 设弹性平面和垫圈有相同剪切模数 μ，而有不同的 κ，并设在 $z=\pm\infty\mathrm{i}$ 处外应力为0. 如孔边与垫圈间的相对位移 $g(t)$，对于 $z\in S^-$(或 S^+)，满足条件(4.18)，则在 S^+ 和 S^- 中应力状态的分布一如弹性平面和垫圈的 κ 均为 κ^+(或 κ^-)时的情况.

前段中的附注1和2这时也对.

例2 同例1，但 κ^+ 和 κ^- 不同. 现在 $\sigma=\tau=0$，又因 $g(t)=-\dfrac{\varepsilon t}{r}$，而

$$\int_{L_0}t\cot\frac{t-z}{a}\,\mathrm{d}t=0,\quad z\in S^-,$$

故由定理，立刻知道例1的结果仍旧是对的，只要把 κ 理解为垫圈的 κ^+：

$$\varphi(z)=\begin{cases}-\dfrac{2\mu\varepsilon z}{(\kappa^++1)r}, & z\in S_0^+;\\ 0, & z\in S^-,\end{cases}\quad \psi(z)=\begin{cases}\dfrac{4\mu\varepsilon r}{\kappa^++1}\left(\dfrac{1}{z}-\dfrac{1}{a}\cot\dfrac{z}{a}\right), & z\in S_0^+;\\ -\dfrac{4\mu\varepsilon r}{(\kappa^++1)a}\cot\dfrac{z}{a}, & z\in S^-.\end{cases}$$

§5 弹性半平面中的周期基本问题

1. 第一基本问题 设弹性体占有 z 的下半平面 S^-，在 x 轴上，记 $z=t$ (t 为实数). 设已知在 x 轴上的外应力分布

$$\sigma_y(t)=-P(t),\quad \tau_{xy}(t)=T(t),\tag{5.1}$$

它们分段地 $\in H$，且以 $a\pi$ 为周期. 这里把 σ_y 写成 $-P$，意指 P 是压力分布. 我们仍设应力、位移均以 $a\pi$ 为周期，且应力有界. 求整个弹性体应力(及位移)分布，称为第一基本问题. 我们要证明

定理1 在上述条件下，第一基本问题的解存在且唯一.

下面我们不仅证明这一定理，并指出求解的方法.

由(3.12)，现在的问题是求解周期 Riemann 边值问题

$$\Phi^{+}(t)-\Phi^{-}(t)=P(t)+\mathrm{i}T(t),\quad t\in L\ (x\text{ 轴}),\tag{5.2}$$

且要求 $\Phi(\pm\infty\mathrm{i})$ 有界. 因为现在指标 $k=0$，所以立刻知道

$$\Phi(z)=\frac{1}{2a\pi\mathrm{i}}\int_{L_0}[P(t)+\mathrm{i}T(t)]\cot\frac{t-z}{a}\,\mathrm{d}t+C,\tag{$*$}$$

其中 C 为某一常数，L_0 以后永远指线段 $-\frac{1}{2}a\pi\leqslant t\leqslant\frac{1}{2}a\pi$.

为要确定 C，只要利用位移的周期性假定. 为此，我们把($*$)式积分，如不计弹性体刚性平移，可得

$$\varphi(z)=-\frac{1}{2\pi\mathrm{i}}\int_{L_0}[P(t)+\mathrm{i}T(t)]\log\sin\frac{t-z}{a}\,\mathrm{d}t+Cz,$$

其中对数已任意取定一支.

注意，当 z 例如沿水平线增加 $a\pi$ 时，$Z=\frac{t-z}{a}$ 就减少 π，而 $\omega=\sin Z$ 就从 ω 变为 $-\omega$，且当 z 在下(上)半平面时，$\arg\omega$ 增加(减少)π (参看[10]，§ 3). 因此，当 $z\in S^{-}$ 时，

$$\begin{aligned}\varphi(z+a\pi)&=-\frac{1}{2\pi\mathrm{i}}\int_{L_0}[P(t)+\mathrm{i}T(t)]\left[\log\sin\frac{t-z}{a}+\mathrm{i}\,\pi\right]+C(z+a\pi)\\&=\varphi(z)-\frac{1}{2}\int_{L_0}(P+\mathrm{i}T)\mathrm{d}t+Ca\pi,\\\varphi(\bar{z}+a\pi)&=-\frac{1}{2\pi\mathrm{i}}\int_{L_0}[P(t)+\mathrm{i}T(t)]\left[\log\sin\frac{t-\bar{z}}{a}-\mathrm{i}\,\pi\right]+C(\bar{z}+a\pi)\\&=\varphi(\bar{z})+\frac{1}{2}\int_{L_0}(P+\mathrm{i}T)\mathrm{d}t+Ca\pi.\end{aligned}$$

因之，由(3.14)，为了位移的周期性，必须

$$\begin{aligned}&\kappa[\varphi(z+a\pi)-\varphi(z)]+[\varphi(\bar{z}+a\pi)-\varphi(\bar{z})]\\&\quad=-\frac{1}{2}(\kappa-1)\int_{L_0}(P+\mathrm{i}T)\mathrm{d}t-(\kappa+1)Ca\pi=0,\end{aligned}$$

从而

$$C=\frac{\kappa-1}{\kappa+1}\cdot\frac{1}{2a\pi}\int_{L_0}(P+\mathrm{i}T)\mathrm{d}t=\frac{1}{2}\frac{\kappa-1}{\kappa+1}(P^{*}+\mathrm{i}T^{*}),\tag{5.3}$$

其中已令

$$P^{*}=\frac{1}{a\pi}\int_{L_0}P(t)\mathrm{d}t,\quad T^{*}=\frac{1}{a\pi}\int_{L_0}T(t)\mathrm{d}t.\tag{5.4}$$

所以，把 C 的值代入($*$)式，便得问题的唯一解答：

$$\Phi(z)=\frac{1}{2a\pi\mathrm{i}}\int_{L_0}[P(t)+\mathrm{i}T(t)]\cot\frac{t-z}{a}\,\mathrm{d}t+\frac{1}{2}\frac{\kappa-1}{\kappa+1}(P^{*}+\mathrm{i}T^{*}).\tag{5.5}$$

定理证毕.

如果不计弹性体刚性旋转，则 $\Phi(z)$ 中的虚常数项还可略去，而有

$$\Phi(z)=\frac{1}{2a\pi\mathrm{i}}\int_{L_0}[P(t)+\mathrm{i}T(t)]\cot\frac{t-z}{a}\,\mathrm{d}t+\frac{1}{2}\frac{\kappa-1}{\kappa+1}P^{*}.\tag{5.5$'$}$$

P^*, T^* 有简单的力学意义，它们事实上分别是 $z=-\infty\mathrm{i}$ 处的外应力压力和剪切力. 这从(3.12)便能看出. 在(3.10)和(3.12)中令 $z\to-\infty\mathrm{i}$，注意到

$$\lim_{z\to-\infty\mathrm{i}}(z-\bar{z})\Phi'(z)=0, \tag{5.6}$$

立刻得知

$$\sigma_x(-\infty\mathrm{i})+\sigma_y(-\infty\mathrm{i})=4\mathrm{Re}\,\Phi(-\infty\mathrm{i})=-\frac{4}{\kappa+1}P^*,$$

$$\sigma_y(-\infty\mathrm{i})-\mathrm{i}\tau_{xy}(-\infty\mathrm{i})=-(P^*+\mathrm{i}T^*),$$

于是，

$$\sigma_y(-\infty\mathrm{i})=-P^*,\quad \tau_{xy}(-\infty\mathrm{i})=T^*, \tag{5.7}$$

$$\sigma_x(-\infty\mathrm{i})=-\frac{3-\kappa}{1+\kappa}P^*. \tag{5.8}$$

(5.7)式从力学上看是容易明白的：取出 S^- 中一个周期半带形，考虑到应力的周期性，左、右两边上应力彼此相消，立刻就得知上述结果.

注意到(3.7)，可知当 $P^*>0$ 时，由(5.8)知 $\sigma_x(-\infty\mathrm{i})<0$. 于是我们有下面的推论：

推论 *在半平面第一基本(周期)问题中，如在边界一个周期上正应力合力是压力：* $\int_{L_0}P(t)\mathrm{d}t=a\pi P^*>0$，*则在 $z=-\infty\mathrm{i}$ 处的 σ_x 也是压力，以(5.8)给出.*

附注 如果把水平方向位移周期性条件改为准周期的，即只要求 $u(z+a\pi)=u(z)+q$，其中 q 为一待定常数，而另外已经给出 $\sigma_x(-\infty\mathrm{i})$，则第一基本问题也有唯一解.

例 1 设在半平面边界上部分地受有周期均匀压力，求弹性平衡(图 6).

设 P 是一常数，而在边界的一个周期上，

$$P(t)=\begin{cases}P, & |t|\leqslant l;\\ 0, & l<|t|\leqslant\frac{1}{2}a\pi,\end{cases}\qquad T(t)=0.$$

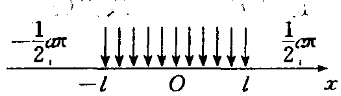

图6

记 γ_0 为线段 $-l\leqslant t\leqslant l$，且取自左至右为正向. 因这时 $P^*=\dfrac{2lP}{a\pi}$，故由(5.5)′，

$$\Phi(z)=\frac{P}{2a\pi\mathrm{i}}\int_{\gamma_0}\cot\frac{t-z}{a}\mathrm{d}t+\frac{\kappa-1}{\kappa+1}\frac{lP}{a\pi}. \tag{5.9}$$

代入(3.10)和(3.11)中，容易计算出

$$\begin{cases}\sigma_x+\sigma_y=\dfrac{2P}{\pi}\left[\arg\sin\dfrac{t-z}{a}\right]_{\gamma_0}-\dfrac{\kappa-1}{\kappa+1}\dfrac{4lP}{a\pi},\\[2ex] \sigma_y-\sigma_x+2\mathrm{i}\tau_{xy}=\dfrac{P(z-\bar{z})\sin\dfrac{2l}{a}}{a\pi\mathrm{i}\sin\dfrac{l+z}{a}\sin\dfrac{l-\bar{z}}{a}}-\dfrac{\kappa-1}{\kappa+1}\dfrac{4lP}{a\pi}.\end{cases} \tag{5.10}$$

此中 $\left[\arg\sin\dfrac{t-z}{a}\right]_{\gamma_0}$ 表示 $\sin\dfrac{t-z}{a}$ 的辐角当 t 沿着 γ_0 自 $-l$ 变到 $+l$ 时的增量.

如果在(5.9)中令 $l\to0$，$P\to+\infty$，但保持 $2lP=P_0$，这就变成在边界上有一组周期集中压力 P_0 时第一基本周期问题的解. 这时(5.9)成为

$$\Phi(z)=-\frac{P_0}{2a\pi\mathrm{i}}\cot\frac{z}{a}+\frac{\kappa-1}{\kappa+1}\frac{P_0}{2a\pi}, \tag{5.11}$$

而这时应力公式(5.10)就成为

$$\begin{cases}\sigma_x+\sigma_y=-\dfrac{2P_0}{a\pi}\operatorname{Im}\cot\dfrac{z}{a}+\dfrac{\kappa-1}{\kappa+1}\dfrac{2P_0}{a\pi},\\[2ex]\sigma_y-\sigma_x+2\mathrm{i}\tau_{xy}=\dfrac{P_0(\bar z-z)}{a^2\pi\mathrm{i}\sin^2\dfrac{z}{a}}-\dfrac{\kappa-1}{\kappa+1}\dfrac{2P_0}{a\pi}.\end{cases}\tag{5.12}$$

如果在(5.10)中令 $a\to+\infty$，便回到半平面上在一段区间$[-l,l]$上受有均匀压力 P 时的应力分布：

$$\begin{cases}\sigma_x+\sigma_y=\dfrac{2P}{\pi}[\arg(t-z)]_{\gamma_0},\\[2ex]\sigma_y-\sigma_x+2\mathrm{i}\tau_{xy}=\dfrac{2lP(z-\bar z)}{\pi\mathrm{i}(l^2-z^2)};\end{cases}\tag{5.10$'$}$$

这和[7]，§93a 中的公式一致.

同样，在(5.12)中令 $a\to+\infty$，便得到半平面上在原点受有集中压力 P_0 时的解：

$$\begin{cases}\sigma_x+\sigma_y=-\dfrac{2P_0}{\pi}\operatorname{Im}\dfrac{l}{z},\\[2ex]\sigma_y-\sigma_x+2\mathrm{i}\tau_{xy}=\dfrac{P_0(\bar z-z)}{\pi\mathrm{i}z^2}.\end{cases}\tag{5.12$'$}$$

例 2 同例 1，但在 γ_0 上受有均匀剪切力，即

$$P(t)=0,\quad T(t)=\begin{cases}T, & |t|\leqslant l,\\ 0, & l<|t|\leqslant\dfrac{1}{2}a\pi.\end{cases}$$

由于这时 $P^*=0$，故由(5.5)$'$，

$$\Phi(z)=\frac{T}{2a\pi}\int_{\gamma_0}\cot\frac{t-z}{a}\,\mathrm{d}t.\tag{5.13}$$

由此代入(3.10)和(3.12)，容易算出

$$\begin{cases}\sigma_x+\sigma_y=\dfrac{2T}{\pi}\ln\left|\dfrac{\sin\dfrac{l-z}{a}}{\sin\dfrac{l+z}{a}}\right|,\\[4ex]\sigma_y+\mathrm{i}\tau_{xy}=\dfrac{T\mathrm{i}}{\pi}\left[\arg\sin\dfrac{t-z}{a}\right]_{\gamma_0}+\dfrac{T(z-\bar z)\sin\dfrac{2l}{a}}{2a\pi\sin\dfrac{l+z}{a}\sin\dfrac{l-z}{a}}.\end{cases}\tag{5.14}$$

如在(5.13)，(5.14)中令 $l\to 0$，$T\to+\infty$，但保持 $2lT=T_0$ 是一常数，并利用与(5.10)类似的讨论，则得周期集中剪切力的解答：

$$\Phi(z)=-\frac{T_0}{2a\pi}\cot\frac{z}{a},\tag{5.15}$$

$$\begin{cases}\sigma_x+\sigma_y=-\dfrac{2T_0}{a\pi}\operatorname{Re}\cot\dfrac{z}{a},\\[2ex]\sigma_y+\mathrm{i}\tau_{xy}=\dfrac{T_0\mathrm{i}}{a\pi}\operatorname{Im}\cot\dfrac{z}{a}+\dfrac{T_0(\bar z-z)}{2a^2\pi\sin^2\dfrac{z}{a}}.\end{cases}\tag{5.16}$$

如再在(5.14)，(5.16)中令 $a\to+\infty$，可回到一段均匀剪切力和一个集中剪切力的公

式，我们便不再写出了.

2. 第二基本问题 设在弹性半平面 S^- 的边界 x 轴上，已知位移向量 $u^- + \mathrm{i}v^- = g(t)$，这里 $g(t)$ 以 $a\pi$ 为周期，连续(且允许含一任意常数项，相应于整个弹性体的刚性平移)，而 $g'(t)$ 分段地 $\in H$；又设在边界的一个周期上，外应力主矢量已知为 $X + \mathrm{i}Y$. 仍设应力、位移周期分布，且应力在 $-\infty\mathrm{i}$ 处有界. 求弹性体的平衡问题就是第二基本问题.

定理 2 在上述条件下，第二基本问题的解存在且唯一.

证 由(3.13)式，现在要求解周期 Riemann 边值问题：

$$\Phi^+(t) + \kappa\Phi^-(t) = 2\mu g'(t), \quad t \in L, \tag{5.17}$$

且要求 $\Phi(\pm\infty\mathrm{i})$ 有界. 这个问题的一般解是

$$\Phi(z) = \begin{cases} \Phi^+(z) = \dfrac{\mu}{a\pi\mathrm{i}}\displaystyle\int_{L_0} g'(t)\cot\frac{t-z}{a}\,\mathrm{d}t - \kappa C, & z \in S^+; \\ \varphi^-(z) = \dfrac{\mu}{\kappa a\pi\mathrm{i}}\displaystyle\int_{L_0} g'(t)\cot\frac{t-z}{a}\,\mathrm{d}t + C, & z \in S^-, \end{cases} \tag{5.18}$$

其中 C 为任意常数. 于是，如不计刚性平移，便有

$$\varphi(z) = \begin{cases} \varphi^+(z) = -\dfrac{\mu}{\pi\mathrm{i}}\displaystyle\int_{L_0} g'(t)\log\sin\frac{t-z}{a}\,\mathrm{d}t - \kappa Cz, & z \in S^+; \\ \varphi^-(z) = \dfrac{\mu}{\kappa\pi\mathrm{i}}\displaystyle\int_{L_0} g'(t)\log\sin\frac{t-z}{a}\,\mathrm{d}t + Cz, & z \in S^-, \end{cases} \tag{5.19}$$

其中对数可任意取定一支；由于

$$\int_{L_0} g'(t)\mathrm{d}t = [g(t)]_{L_0} = 0, \tag{*}$$

这根本不影响 $\varphi(z)$ 的值. 由于当 $z \in S^-$ 时，

$$\begin{aligned} \varphi(\bar{z}) &= -\frac{\mu}{\pi\mathrm{i}}\int_{L_0} g'(t)\log\sin\frac{t-\bar{z}}{a}\,\mathrm{d}t - \kappa C\bar{z} \\ &= -\frac{\mu}{\pi\mathrm{i}}\int_{L_0} g'(t)\overline{\log\sin\frac{t-z}{a}}\,\mathrm{d}t - \kappa C\bar{z}, \end{aligned}$$

代入(3.14)，得

$$2\mu(u + \mathrm{i}v) = \frac{2\mu}{\pi}\int_{L_0} g'(t)\arg\sin\frac{t-z}{a}\,\mathrm{d}t + \kappa C(z - \bar{z}) - (z - \bar{z})\overline{\Phi'(z)}.$$

由此可见，$u + \mathrm{i}v$ 不仅单值，且已以 $a\pi$ 为周期，因为当 z 改为 $z + a\pi$ 时，右边第一项根本不变，这还是由于(*)式之故.

为要决定 C，可考虑在 $z = -\infty\mathrm{i}$ 处的外应力情况. 根据平衡条件，应有

$$\sigma_y(-\infty\mathrm{i}) = \frac{Y}{a\pi}, \quad \tau_{xy}(-\infty\mathrm{i}) = \frac{X}{a\pi}. \tag{5.20}$$

又由(5.18)，易知

$$\Phi(+\infty\mathrm{i}) = -\kappa C, \quad \Phi(-\infty\mathrm{i}) = C;$$

此外，(5.6)式现在仍成立. 于是在(3.12)中，令 $z = -\infty\mathrm{i}$，注意到(5.20)，立得

$$\frac{Y - \mathrm{i}X}{a\pi} = (\kappa + 1)C,$$

亦即 $C = \dfrac{Y - \mathrm{i}X}{(\kappa + 1)a\pi}$. 于是最后得

$$\Phi(z)=\begin{cases}\dfrac{\mu}{a\pi\mathrm{i}}\displaystyle\int_{L_0}g'(t)\cot\frac{t-z}{a}\,\mathrm{d}t-\frac{\kappa(Y-\mathrm{i}X)}{(\kappa+1)a\pi}, & z\in S^+;\\[2ex] -\dfrac{\mu}{\kappa a\pi\mathrm{i}}\displaystyle\int_{L_0}g'(t)\cot\frac{t-z}{a}\,\mathrm{d}t+\frac{Y-\mathrm{i}X}{(\kappa+1)a\pi}, & z\in S^-.\end{cases}\tag{5.21}$$

定理已经得证.

由此还很容易得出下面两个推论：

推论 1　当且仅当 $Y=0$ 时，$\sigma_x(-\infty\mathrm{i})=0$.

推论 2　当且仅当 $X=0$ 时，u 有界；$Y=0$ 时，v 有界.

例 3　设半平面边界上有尖劈形周期位移，即在一个周期 L_0 上的位移已知为

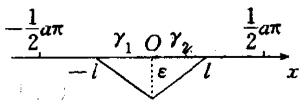

图7

$$g(t)=\begin{cases}\varepsilon(|t|/l-1)\mathrm{i}, & |t|\leqslant l;\\ 0, & l<|t|\leqslant\frac{1}{2}a\pi.\end{cases}$$

此外还设在 L_0 上的外应力主矢量为 0. 求弹性平衡(图 7).

这时 $C=0$. 故当 $z\in S^-$ 时，由(5.21)知(如果用 γ_1,γ_2 分别表示由 $-l$ 到 O 和 O 到 $+l$ 的二有向直线段)，

$$\Phi(z)=\frac{\mu\varepsilon}{\kappa la\pi}\left(\int_{\gamma_1}\cot\frac{t-z}{a}\,\mathrm{d}t-\int_{\gamma_2}\cot\frac{t-z}{a}\,\mathrm{d}t\right)$$

$$=\frac{\mu\varepsilon}{\kappa l\pi}\left\{\ln\left|\frac{\sin^2\frac{z}{a}}{\sin\frac{l+z}{a}\sin\frac{l-z}{a}}\right|+\mathrm{i}\left[\arg\sin\frac{t-z}{a}\right]_{\gamma_1}-\mathrm{i}\left[\arg\sin\frac{t-z}{a}\right]_{\gamma_2}\right\}.$$

由(5.21)也可知道，当 $z\in S^-$ 时，

$$\Phi(\bar{z})=\kappa\overline{\Phi(z)}\quad 或\quad \bar{\Phi}(z)=\kappa\Phi(z);$$

因此，最后代入(3.10)和(3.11)，得应力分布公式

$$\sigma_x+\sigma_y=4\mathrm{Re}\,\Phi(z)=\frac{4\mu\varepsilon}{\kappa l\pi}\ln\left|\frac{\sin^2\frac{z}{a}}{\sin\frac{l+z}{a}\sin\frac{l-z}{a}}\right|,$$

$$\sigma_y-\sigma_x+2\mathrm{i}\tau_{xy}=-2(\kappa+1)\Phi(z)-2(z-\bar{z})\Phi'(z)$$

$$=-\frac{2\mu\varepsilon(\kappa+1)}{\kappa l\pi}\left\{\ln\left|\frac{\sin^2\frac{z}{a}}{\sin\frac{l+z}{a}\sin\frac{l-z}{a}}\right|+\mathrm{i}\left[\arg\sin\frac{t-z}{a}\right]_{\gamma_1}-\mathrm{i}\left[\arg\sin\frac{t-z}{a}\right]_{\gamma_2}\right\}$$

$$-\frac{2\mu\varepsilon(z-\bar{z})}{\kappa l\pi a}\left|2\cot\frac{z}{a}+\frac{\sin\frac{2z}{a}}{\sin\frac{l+z}{a}\sin\frac{l-z}{a}}\right|.$$

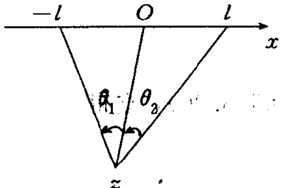

图8

如令 $a\to+\infty$，便得一个尖劈时的应力分布公式如下(其中 θ_1,θ_2 已取如图 8 中所示之角，且均取正锐角)：

$$\sigma_x+\sigma_y=\frac{4\mu\varepsilon}{\kappa l\pi}\ln\left|\frac{z^2}{l^2-z^2}\right|,$$

$$\sigma_y-\sigma_x+2\mathrm{i}\tau_{xy}$$

$$=-\frac{2\mu\varepsilon(\kappa+1)}{\kappa\pi l}\left\{\ln\left|\frac{z^2}{l^2-z^2}\right|+\mathrm{i}(\theta_2-\theta_1)\right\}-\frac{4\mu l\varepsilon(z-\bar{z})}{\kappa\pi z(l^2-z^2)}.$$

3. 基本混合问题　设在半平面边界的一个周期 L_0 上，其中一段 $\gamma_0(-l\leqslant t\leqslant l)$ 上已知位移

$$u^- + \mathrm{i}v^- = g(t),$$

仍设 $g(t)$ 连续，$g'(t)\in H$（且可差一常数项）。又设在 $L_0' = L_0-\gamma_0$ 上，外应力已知，例如不妨设为 0：

$$\sigma_y^-(t) = \tau_{xy}^-(t) = 0,\quad l<|t|\leqslant \frac{1}{2}a\pi.$$

此外，还已知 γ_0 上的外应力主矢量 $X+\mathrm{i}Y$. 以上诸条件都是以 $a\pi$ 为周期的. 仍假设应力、位移都是周期的，且应力在 $-\infty\mathrm{i}$ 处有界，求弹性体平衡. 这问题称为基本混合问题.

定理 3　*在上述条件下，基本混合问题的解存在唯一.*

一般地，在 $z=\pm l$ 附近，应力可能无界. 与前段所述理由相类似，现在要求解周期 Riemann 边值问题：

$$\Phi^+(t)+\kappa\Phi^-(t) = 2\mu g'(t),\quad t\in\gamma_0. \tag{5.22}$$

这与 §2,第 4 段中的特例相同，只要把那里的 K 改作 κ，并且这里要求的也是 h_0 类中的解.

由(2.14)式知，这问题的一般解是

$$\Phi(z) = \frac{\mu X(z)}{a\pi\mathrm{i}}\int_{\gamma_0}\frac{g'(t)}{X^+(t)}\cot\frac{t-z}{a}\mathrm{d}t + X(z)\left(C_0\tan\frac{z}{a}+C_1\right), \tag{5.23}$$

其中的典则函数

$$X(z) = \left(\tan\frac{z}{a}+\tan\frac{l}{a}\right)^{-\frac{1}{2}+\mathrm{i}\beta}\left(\tan\frac{z}{a}-\tan\frac{l}{a}\right)^{-\frac{1}{2}-\mathrm{i}\beta}, \tag{5.24}$$

这里

$$\beta = \frac{\ln\kappa}{2\pi}; \tag{5.25}$$

且 $X(z)$ 已取定一支，例如，使

$$\lim_{z\to\pm\frac{1}{2}a\pi}\tan\frac{z}{a}X(z) = 1, \tag{5.26}$$

这也容易证明就是这样的一支：令

$$L=\tan\frac{l}{a},\quad \zeta = \tan\frac{z}{a},$$

取 ζ 平面中把 $-L$ 到 $+L$ 沿实轴剖开后的如图 9 中所示的一支：

$$\arg(\zeta+L)=\theta_1,\quad \arg(\zeta-L)=\theta_2.$$

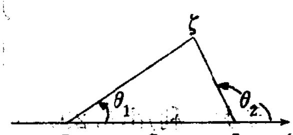

图9

不论 C_0, C_1 如何选取，因 $\Phi(z)$ 已以 $a\pi$ 为周期，故应力也已以 $a\pi$ 为周期无问题. 为要决定 C_0, C_1，只要利用位移周期性条件和在 $z=-\infty\mathrm{i}$ 处应力的平衡条件即可. 具体说明如下.

首先我们注意，

$$\begin{aligned}X(-\infty\mathrm{i}) &= \exp\left\{\left(-\frac{1}{2}+\mathrm{i}\beta\right)\log(-\mathrm{i}+L)-\left(\frac{1}{2}+\mathrm{i}\beta\right)\log(-\mathrm{i}-L)\right\}\\ &= \exp\left\{\left(\frac{1}{2}-\mathrm{i}\beta\right)\left[\ln\cos\frac{l}{a}+\mathrm{i}\left(\frac{\pi}{2}-\frac{l}{a}\right)\right]\right.\end{aligned}$$

$$+\left(\frac{1}{2}+\mathrm{i}\beta\right)\left[\ln\cos\frac{l}{a}+\mathrm{i}\left(\frac{\pi}{2}+\frac{l}{a}\right)\right]\Big\}$$
$$=\mathrm{i}\,\mathrm{e}^{-\frac{2\beta l}{a}}\cos\frac{l}{a},$$

同理，

$$X(+\infty\mathrm{i})=-\mathrm{i}\,\mathrm{e}^{-\frac{2\beta l}{a}}\cos\frac{l}{a}.$$

如果记 γ_0 对一个周期 L_0 的长度之比为 λ：$\lambda=\frac{2l}{a\pi}$，且令

$$A=\frac{2\beta l}{a}=\lambda\ln\sqrt{\kappa},\quad B=\beta\pi-\frac{2\beta l}{a}=(1-\lambda)\ln\sqrt{\kappa},\tag{5.27}$$

则有

$$X(+\infty\mathrm{i})=-\mathrm{i}\,\mathrm{e}^{A}\cos\frac{l}{a},\quad X(-\infty\mathrm{i})=\mathrm{i}\,\mathrm{e}^{-A}\cos\frac{l}{a}.\tag{5.28}$$

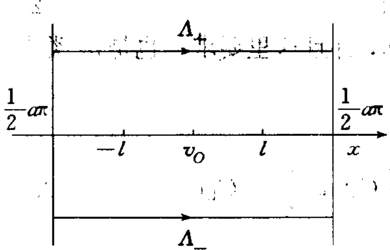

图10

如以 Λ_+，Λ_- 分别记上半平面和下半平面中的一水平周期线段(图 10)，则$\int_{\Lambda_\pm}X(z)\mathrm{d}z$ 显然和 $\Lambda_\pm$ 的高度无关. 因此，立刻可知

$$\begin{cases}\displaystyle\int_{\Lambda_+}X(z)\mathrm{d}z=a\pi X(+\infty\mathrm{i})=-a\pi\,\mathrm{i}\,\mathrm{e}^{A}\cos\frac{l}{a},\\[2ex] \displaystyle\int_{\Lambda_-}X(z)\mathrm{d}z=a\pi X(-\infty\mathrm{i})=a\pi\,\mathrm{i}\,\mathrm{e}^{-A}\cos\frac{l}{a}.\end{cases}\tag{5.29}$$

同理，

$$\begin{cases}\displaystyle\int_{\Lambda_+}X(z)\tan\frac{z}{a}\,\mathrm{d}z=a\pi\,\mathrm{e}^{A}\cos\frac{l}{a},\\[2ex] \displaystyle\int_{\Lambda_-}X(z)\tan\frac{z}{a}\,\mathrm{d}z=a\pi\,\mathrm{e}^{-A}\cos\frac{l}{a}.\end{cases}\tag{5.30}$$

又若 $t\in\gamma_0$，则还有

$$\begin{cases}\displaystyle\int_{\Lambda_+}X(z)\cot\frac{t-z}{a}\,\mathrm{d}z=a\pi\,\mathrm{e}^{A}\cos\frac{l}{a},\\[2ex] \displaystyle\int_{\Lambda_-}X(z)\cot\frac{t-z}{a}\,\mathrm{d}z=a\pi\,\mathrm{e}^{-A}\cos\frac{l}{a}.\end{cases}\tag{5.31}$$

于是，利用(5.29)～(5.31)，易见当 $z\in S^-$ 时，

$$\varphi(z+a\pi)-\varphi(z)=\int_{\Lambda_-}\Phi(z)\mathrm{d}z=a\pi\,\mathrm{e}^{-A}\cos\frac{l}{a}\left\{\frac{\mu}{a\pi\,\mathrm{i}}\int_{\gamma_0}\frac{g'(t)}{X^+(t)}\mathrm{d}t+C_0+\mathrm{i}C_1\right\},$$

$$\varphi(\bar{z}+a\pi)-\varphi(\bar{z})=\int_{\Lambda_+}\Phi(z)\mathrm{d}z=a\pi\,\mathrm{e}^{A}\cos\frac{l}{a}\left\{\frac{\mu}{a\pi\,\mathrm{i}}\int_{\gamma_0}\frac{g'(t)}{X^+(t)}\mathrm{d}t+C_0-\mathrm{i}C_1\right\}.$$

因而，由(3.14)，位移周期性条件为

$$\kappa\mathrm{e}^{-A}\left\{\frac{\mu}{a\pi\,\mathrm{i}}\int_{\gamma_0}\frac{g'(t)}{X^+(t)}\mathrm{d}t+C_0+\mathrm{i}C_1\right\}+\mathrm{e}^{A}\left\{\frac{\mu}{a\pi\,\mathrm{i}}\int_{\gamma_0}\frac{g'(t)}{X^+(t)}\mathrm{d}t+C_0-\mathrm{i}C_1\right\}=0,$$

或即(注意到 $\mathrm{e}^{B}=\sqrt{\kappa}\,\mathrm{e}^{-A}$)

$$C_0+\mathrm{i}C_1\tanh B=-\frac{\mu}{a\pi\,\mathrm{i}}\int_{\gamma_0}\frac{g'(t)}{X^+(t)}\mathrm{d}t.\tag{5.32}$$

再来看在 $z=-\infty\mathrm{i}$ 处应力平衡的条件：

$$\sigma_y(-\infty\mathrm{i})=\frac{Y}{a\pi},\quad \tau_{xy}(-\infty\mathrm{i})=\frac{X}{a\pi}.$$

现在

$$\Phi(+\infty\mathrm{i})=-\mathrm{i}\,\mathrm{e}^{A}\cos\frac{l}{a}\left\{\frac{\mu}{a\pi}\int_{\gamma_0}\frac{g'(t)}{X^+(t)}\mathrm{d}t+C_1+C_0\mathrm{i}\right\},$$

$$\Phi(-\infty\mathrm{i})=\mathrm{i}\,\mathrm{e}^{-A}\cos\frac{l}{a}\left\{-\frac{\mu}{a\pi}\int_{\gamma_0}\frac{g'(t)}{X^+(t)}\mathrm{d}t+C_1-C_0\mathrm{i}\right\},$$

又注意到(5.6)仍成立，故由(3.12)知，平衡条件成为

$$\frac{Y-\mathrm{i}X}{2a\pi}=\mathrm{i}\cos\frac{l}{a}\left\{\frac{\mu\sinh A}{a\pi}\int_{\gamma_0}\frac{g'(t)}{X^+(t)}\mathrm{d}t+C_1\cosh A+\mathrm{i}C_0\sinh A\right\},$$

或即

$$C_0\tanh A-\mathrm{i}C_1=\frac{-Y+\mathrm{i}X}{2a\pi\cos\frac{l}{a}\cosh A}-\frac{\mu\tanh A}{a\pi\mathrm{i}}\int_{\gamma_0}\frac{g'(t)}{X^+(t)}\mathrm{d}t. \tag{5.33}$$

注意到 A,B 均 >0，由(5.32)和(5.33)便可唯一确定 C_0,C_1. 因此问题的解存在且唯一.

例 4 水平直基底压头. 设弹性半平面上有一组周期为 $a\pi$ 的压头，每一个的基底为水平直线段，长 $2l$. 又假设在每一压头上作用有压力 P_0，而在压头以外的边界上外应力为 0. 求弹性体平衡.

这时，在 γ_0 上，已知 $g'(t)\equiv 0$；又 $X=0$，$Y=-P_0$. 现在(5.32)，(5.33)成为

$$C_0+\mathrm{i}C_1\tanh B=0,$$

$$C_0\tanh A-\mathrm{i}C_1=\frac{P_0}{2a\pi\cos\frac{l}{a}\cosh A}.$$

解出 C_0,C_1，注意到 $\cosh(A+B)=\cosh\beta\pi=\dfrac{\kappa+1}{2\sqrt{\kappa}}$，得知

$$C_0=\frac{\sqrt{\kappa}P_0\sinh B}{(\kappa+1)a\pi\cos\frac{l}{a}},\quad C_1=\frac{\mathrm{i}\sqrt{\kappa}P_0\cosh B}{(\kappa+1)a\pi\cos\frac{l}{a}}.$$

最后代入(5.23)式，得

$$\Phi(z)=\frac{\sqrt{\kappa}P_0X(z)}{(\kappa+1)a\pi\cos\frac{l}{a}}\left(\sinh B\tan\frac{z}{a}+\mathrm{i}\cosh B\right), \tag{5.34}$$

其中 $X(z)$ 以(5.24)给出.

我们来计算在压头正下方边界上的应力分布. 在 γ_0 上，$|t|\leqslant l$，显然当 z 自上半平面趋于 γ_0 上的 t 值时，$\theta_1=0$，$\theta_2=\pi$（图 9），于是

$$\begin{aligned}X^+(t)&=\exp\left\{\left(-\frac{1}{2}+\mathrm{i}\beta\right)\ln\left|\tan\frac{t}{a}+\tan\frac{l}{a}\right|-\left(\frac{1}{2}+\mathrm{i}\beta\right)\left[\ln\left|\tan\frac{t}{a}-\tan\frac{l}{a}\right|+\pi\mathrm{i}\right]\right\}\\&=-\frac{\mathrm{i}\sqrt{\kappa}}{\sqrt{\tan^2\frac{l}{a}-\tan^2\frac{t}{a}}}\exp\left\{\mathrm{i}\beta\ln\left|\frac{\sin\frac{l+t}{a}}{\sin\frac{l-t}{a}}\right|\right\}.\end{aligned} \tag{5.35}$$

代入(5.34)，得

$$\Phi^+(t)=\frac{\kappa P_0\exp\left\{\mathrm{i}\beta\ln\left|\dfrac{\sin\dfrac{l+t}{a}}{\sin\dfrac{l-t}{a}}\right|\right\}}{(\kappa+1)a\pi\sqrt{\sin\dfrac{l+t}{a}\sin\dfrac{l-t}{a}}}\left(\cosh B\cos\frac{t}{a}-\sinh B\sin\frac{t}{a}\right).$$

如用 $P(t)$, $T(t)$ 分别表示 γ_0 上的正压力和剪切力分布，则由(3.12)，

$$P(t)+\mathrm{i}T(t)=-\sigma_y(t)+\mathrm{i}\tau_{xy}(t)=\Phi^+(t)-\Phi^-(t)=\frac{\kappa+1}{\kappa}\Phi^+(t),$$

故最后得

$$P(t)+\mathrm{i}T(t)=\frac{P_0}{a\pi}\cdot\frac{\cosh B\cos\dfrac{t}{a}-\mathrm{i}\sinh B\sin\dfrac{t}{a}}{\sqrt{\sin\dfrac{l+t}{a}\sin\dfrac{l-t}{a}}}\left\{\cos\left[\beta\ln\left|\frac{\sin\dfrac{l+t}{a}}{\sin\dfrac{l-t}{a}}\right|\right]\right.$$

$$\left.+\mathrm{i}\sin\left[\beta\ln\left|\frac{\sin\dfrac{l+t}{a}}{\sin\dfrac{l-t}{a}}\right|\right]\right\}. \tag{5.36}$$

令 $a\to+\infty$ 时，便回到一个压头时的特例. 注意这时由(5.27)，$B\to\ln\sqrt{\kappa}$，故 $\cosh B\to\dfrac{\kappa+1}{2\sqrt{\kappa}}$, $\sinh B\to\dfrac{\kappa-1}{2\sqrt{\kappa}}$，因而

$$P(t)+\mathrm{i}T(t)=\frac{(\kappa+1)P_0}{2\pi\sqrt{\kappa}\sqrt{l^2-t^2}}\left\{\cos\left[\beta\ln\left|\frac{l+t}{l-t}\right|\right]+\mathrm{i}\sin\left[\beta\ln\left|\frac{l+t}{l-t}\right|\right]\right\}.$$

这与 B. M. Абрамов 的公式一致，见[7]，§ 114a①.

例 5　倾斜直基底压头. 该压头基底同前例，但与半平面边界作倾斜角 ε (图 11). 又设压头上外力主矢量为零：$X=Y=0$. 在压头以外仍设无外应力. 这时

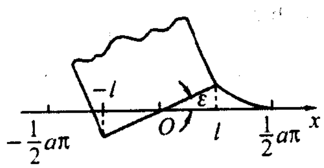

图 11

$$g(t)=\mathrm{i}\varepsilon t\ 或\ g'(t)=\mathrm{i}\varepsilon,\ t\in\gamma_0.$$

现在(5.32)，(5.33) 成为

$$C_0+\mathrm{i}C_1\tanh B=-\frac{\mu\varepsilon}{a\pi}\int_{\gamma_0}\frac{\mathrm{d}t}{X^+(t)},$$

$$C_0\tanh A-\mathrm{i}C_1=-\frac{\mu\varepsilon}{a\pi}\tanh A\int_{\gamma_0}\frac{\mathrm{d}t}{X^+(t)}.$$

显然，

$$C_1=0,\quad C_0=-\frac{\mu\varepsilon}{a\pi}\int_{\gamma_0}\frac{\mathrm{d}t}{X^+(t)}.$$

代入(5.23)，化简后得

$$\Phi(z)=\frac{\mu\varepsilon X(z)}{a\pi}\int_{\gamma_0}\left(\cot\frac{t-z}{a}-\tan\frac{z}{a}\right)\frac{\mathrm{d}t}{X^+(t)}. \tag{$*$}$$

① 该节公式(8) 原文本中右边分母上漏掉因子 2，中译本也未改正.

为了具体计算，令

$$I(z)=\int_{\gamma_0}\left(\cot\frac{t-z}{a}-\tan\frac{z}{a}\right)\frac{dt}{X^+(t)}. \tag{5.37}$$

作变换 $u=\tan\frac{t}{a}$，并令 $\zeta=\tan\frac{z}{a}$，$L=\tan\frac{l}{a}$，则

$$\frac{1}{a}\cos^2\frac{z}{a}\,I(z)=I_*(\zeta)=\int_{-L}^{L}\frac{du}{(1+u^2)(u-\zeta)X_*^+(u)},$$

其中

$$X_*(\zeta)=X(z)=(\zeta+L)^{-\frac{1}{2}+i\beta}(\zeta-L)^{-\frac{1}{2}-i\beta},$$

并且所取的支同前，即

$$\lim_{\zeta\to\infty}\zeta X_*(\zeta)=1.$$

为了计算 $I_*(\zeta)$，作辅助函数

$$\Omega(\zeta)=\frac{1}{2\pi i}\int_{\Lambda}\frac{dw}{(1+w^2)(w-\zeta)X_*(w)}=\frac{1}{2\pi i}\int_{\Lambda}\frac{f(w)}{w-\zeta}dw,$$

其中 Λ 为 w 平面中包围线段 $-L\leqslant u\leqslant L$ 的任一光滑闭路，但 ζ 在其外，并沿时针向积分. 让 Λ 无限地收缩于上述线段，并注意 $X_*^+(u)=-\kappa X_*^-(u)$，易见

$$\Omega(\zeta)=\frac{\kappa+1}{2\pi i}I_*(\zeta)\quad 或\quad I_*(\zeta)=\frac{2\pi i}{\kappa+1}\Omega(\zeta).$$

现在只要计算 $\Omega(\zeta)$. 因

$$f(\zeta)=\frac{1}{(1+\zeta^2)X_*(\zeta)},$$

注意到(5.28)，易见它在 $\zeta=i$ 处的主部为

$$\frac{1}{2i(\zeta-i)X_*(i)}=\frac{1}{2i(\zeta-i)X(+\infty i)}=\frac{e^{-A}}{2\cos\frac{l}{a}(\zeta-i)},$$

在 $\zeta=-i$ 处的主部为

$$-\frac{1}{2i(\zeta+i)X_*(-i)}=-\frac{1}{2i(\zeta+i)X(-\infty i)}=\frac{e^{A}}{2\cos\frac{l}{a}(\zeta+i)}.$$

于是由熟知的结果，知

$$\begin{aligned}\Omega(\zeta)&=\frac{1}{(1+\zeta^2)X_*(\zeta)}-\frac{e^{-A}}{2\cos\frac{l}{a}(\zeta-i)}-\frac{e^{A}}{2\cos\frac{l}{a}(\zeta+i)}\\&=\frac{\cos^2\frac{z}{a}}{X(z)}-\frac{\cos^2\frac{z}{a}}{\cos\frac{l}{a}}\left(\cosh A\tan\frac{z}{a}-i\sinh A\right).\end{aligned}$$

由它算出 $I_*(\zeta)$，再回到 $I(z)$，便得

$$I(z)=\frac{2a\pi i}{\kappa+1}\left\{\frac{1}{X(z)}-\sec\frac{l}{a}\left(\cosh A\tan\frac{z}{a}-i\sinh A\right)\right\}. \tag{5.38}$$

最后代入(*)式中，得

$$\Phi(z)=\frac{2\mu\varepsilon i}{\kappa+1}\left\{1-\frac{X(z)}{\cos\frac{l}{a}}\left(\cosh A\tan\frac{z}{a}-i\sinh A\right)\right\}, \tag{5.39}$$

因此问题已完全解决.

在压头正下方的边界上，仿前例易知，

$$P(t)+\mathrm{i}T(t)=\Phi^{+}(t)-\Phi^{-}(t)=\frac{\kappa+1}{\kappa}\Phi^{+}(t)-\frac{2\mu\varepsilon\mathrm{i}}{\kappa}$$
$$=-\frac{2\mu\varepsilon\mathrm{i}\,X^{+}(t)}{\kappa\cos\dfrac{l}{a}}\left(\cosh A\tan\frac{t}{a}-\mathrm{i}\sinh A\right).$$

再用(5.35)代入，得

$$P(t)+\mathrm{i}T(t)=-\frac{2\mu\varepsilon\left(\cosh A\sin\dfrac{t}{a}-\mathrm{i}\sinh A\cos\dfrac{t}{a}\right)}{\sqrt{\kappa}\sqrt{\sin\dfrac{l+t}{a}\sin\dfrac{l-t}{a}}}\left\{\cos\left[\beta\ln\left|\frac{\sin\dfrac{l+t}{a}}{\sin\dfrac{l-t}{a}}\right|\right]\right.$$
$$\left.+\mathrm{i}\sin\left[\beta\ln\left|\frac{\sin\dfrac{l+t}{a}}{\sin\dfrac{l-t}{a}}\right|\right]\right\}. \tag{5.40}$$

如在(5.39),(5.40)中令 $a\to+\infty$，并注意

$$\lim_{a\to+\infty}\frac{X(z)}{a}=X_0(z)=(z+l)^{-\frac{1}{2}+\mathrm{i}\beta}(z-l)^{-\frac{1}{2}-\mathrm{i}\beta},$$
$$\lim_{a\to+\infty}\cosh A=1,\quad \lim_{a\to+\infty}a\sinh A=2l\beta,$$

则得一个倾斜直基底压头的相应公式：

$$\Phi(z)=\frac{2\mu\varepsilon\mathrm{i}}{\kappa+1}\{1-(z-2l\beta\mathrm{i})X_0(z)\},$$

$$P(t)+\mathrm{i}T(t)=-\frac{2\mu\varepsilon\mathrm{i}}{\kappa}(t-2l\beta\mathrm{i})X_0^{+}(t)$$
$$=-\frac{2\mu\varepsilon}{\sqrt{\kappa}}\cdot\frac{t-2l\beta\mathrm{i}}{\sqrt{l^2-t^2}}\left\{\cos\left(\beta\ln\left|\frac{l+t}{l-t}\right|\right)+\mathrm{i}\sin\left(\beta\ln\left|\frac{l+t}{l-t}\right|\right)\right\};$$

这和[7], § 114 中的结果一致.

§ 6 周期压头在弹性半平面上的压力问题

1. 无摩擦存在时的情况 现设有一列周期压头(基底形状相同)压入弹性半平面中. 暂设不存在摩擦力①. 在压头之外，半平面边界上的条件仍与前节最后一段同，即 $\sigma_y=\tau_{xy}=0$；而在基底的正下方，只知道半平面边界上的纵向位移 $v(t)$，而不知道横向位移 $u(t)$，但由于无摩擦力，故仍有 $\tau_{xy}=0$，而 σ_y 则不知道. 此外，假设每一压头上的作用力是正压力 P_0，即 $Y=-P_0$，而 $X=0$. 求弹性平衡.

设 $y=f(x)$ 为压入弹性半平面的压头的基底方程，其中 $f(x)$ 以 $a\pi$ 为周期，且 $f'(x)\in H$. 在一个周期 L_0 上，受压区间仍设为 γ_0: $-l\leqslant t\leqslant l$. 于是，在 L_0 上，我们的边界条件是（参看[7], § 115)：

$$\tau_{xy}^{-}(t)=0,\quad t\in L_0;\quad \sigma_y^{-}(t)=0,\quad t\in L_0-\gamma_0;$$

① 前节末讨论的情况，当 $u^{-}(t)=0$ 即压头无水平滑动时，实际上就是有无穷大摩擦的情况.

$$v^-(t)=f(t),\quad t\in\gamma_0;$$

又在 γ_0 上的外应力主矢量为 $X+\mathrm{i}Y=-P_0\mathrm{i}$.

由这些边界条件，首先我们知道，在 $z=-\infty\mathrm{i}$ 处，根据平衡原理，应有

$$\sigma=\sigma_y(-\infty\mathrm{i})=-\frac{P_0}{a\pi},\quad \tau=\tau_{xy}(-\infty\mathrm{i})=0. \tag{6.1}$$

注意 $\Phi(z)$ 在沿 γ_0 及其周期对应线段剖开的平面上是全纯的. 由(3.12)，当 $t\in\gamma_0$ 时，因 $\tau_{xy}=0$，显然

$$\Phi^+(t)-\Phi^-(t)=\overline{\Phi}^-(t)-\overline{\Phi}^+(t),$$

于是

$$\Phi^+(t)+\overline{\Phi}^+(t)=\Phi^-(t)+\overline{\Phi}^-(t).$$

这就说明 $\Phi(z)+\overline{\Phi}(z)$ 在全平面上全纯. 又它在 $z=\pm\infty\mathrm{i}$ 处有界，且以 $a\pi$ 为周期，用一保角变换，立刻可以知道它恒等于一常数：

$$\Phi(z)+\overline{\Phi}(z)=2\beta_2. \tag{6.2}$$

由(3.11)，取 t 在 x 轴上时，因 $\tau_{xy}=0$，显然 β_2 是一实数.

由(3.13)，在 x 轴上，有

$$2\mu(u'+\mathrm{i}v')=\kappa\Phi^-(t)+\Phi^+(t),$$

取共轭值，并利用(6.2)，得

$$2\mu(u'-\mathrm{i}v')=-[\kappa\Phi^+(t)+\Phi^-(t)]+2(\kappa+1)\beta_2.$$

把这两式相减，并注意在 γ_0 上，$v'=f(t)$，于是得 Riemann 周期边值问题：

$$\Phi^+(t)+\Phi^-(t)=\frac{4\mu\mathrm{i}}{\kappa+1}f'(t)+2\beta_2,\quad t\in\gamma_0. \tag{6.3}$$

现在要求这一问题的一般周期解，在 $z=\pm\infty\mathrm{i}$ 有界者. 这时可利用 §2，第 4 段中(2.14)′ 式；但注意对于(6.3)中右端 $2\beta_2$ 这一项，显然有特解 β_2，因此得(6.3)的一般解为

$$\Phi(z)=\frac{2\mu}{(\kappa+1)\pi a\sqrt{R(z)}}\int_{\gamma_0}f'(t)\sqrt{R(t)}\cot\frac{t-z}{a}\,dt+\frac{\beta_0\tan\dfrac{z}{a}+\beta_1}{\sqrt{R(z)}}+\beta_2, \tag{6.4}$$

其中已把 $X(z)$ 改写如下：

$$X(z)=\frac{1}{\mathrm{i}\sqrt{R(z)}},\quad R(z)=\tan^2\frac{l}{a}-\tan^2\frac{z}{a}, \tag{6.5}$$

且当 z 由上半平面趋于 $t\in\gamma_0$ 时，$\sqrt{R(t)}$ 取正值. 从(6.4)还可立刻看出

$$\lim_{z\to\pm\infty\mathrm{i}}(z-\overline{z})\Phi'(z)=0 \tag{6.6}$$

仍成立.

现在来证明，在我们的条件下，β_0,β_1 必须是实数. 首先我们注意

$$\overline{X}(z)=X(z). \tag{6.7}$$

这可以这样看出：因为 $\overline{X}(z)=\overline{X(\overline{z})}$，所以它和 $X(z)$ 表示同一根式，因此只可能 $\overline{X}(z)=\pm X(z)$；但若比较(6.3)式和(5.21)式左边，知道这里的典则函数同那里的一样，只要令那里的 κ 改作1，也就是把(5.27)中的 A 改作0，于是现在我们有 $\overline{X}(-\infty\mathrm{i})=\overline{X(+\infty\mathrm{i})}=X(-\infty\mathrm{i})$，即上式中要取正号，亦即，(6.7)式成立. 再把(6.7)和(6.5)比较，便知道

$$\overline{\sqrt{R(\overline{z})}}=-\sqrt{R(z)}.$$

又因 $f'(t)$ 是实函数，故由(6.4)知，

$$\overline{\Phi}(z)=-\frac{2\mu}{(\kappa+1)\pi a\sqrt{R(z)}}\int_{\gamma_0}f'(t)\sqrt{R(t)}\cot\frac{t-z}{a}\,\mathrm{d}t+\frac{\overline{\beta}_0\tan\dfrac{z}{a}+\overline{\beta}_1}{\sqrt{R(z)}}+\beta_2.$$

再把它和(6.4)相加，根据(6.2)，便知道 β_0,β_1 都是实数.

剩下只要利用位移周期性条件和条件(6.1)来决定实常数 β_0,β_1,β_2.

先看位移周期性条件. 注意到

$$\sqrt{R(\pm\infty\mathrm{i})}=\pm\frac{1}{\cos\dfrac{l}{a}},\tag{6.8}$$

所以，$\displaystyle\int_{\Lambda_\pm}\frac{\mathrm{d}z}{\sqrt{R(z)}}=\pm a\pi\cos\frac{l}{a}$,

$$\int_{\Lambda_\pm}\frac{\tan\dfrac{z}{a}}{\sqrt{R(z)}}\mathrm{d}z=\int_{\Lambda_\pm}\frac{\cot\dfrac{t-z}{a}}{\sqrt{R(z)}}\mathrm{d}z=a\pi\mathrm{i}\cos\frac{l}{a},$$

其中 Λ_+ 和 Λ_- 见图 10. 于是，由(6.6)知，当 z 在下半平面时，

$$\begin{aligned}\varphi(z+a\pi)-\varphi(z)&=\int_{\Lambda_-}\Phi(z)\mathrm{d}z\\&=\frac{2\mu\mathrm{i}\cos\dfrac{l}{a}}{\kappa+1}\int_{\gamma_0}f'(t)\sqrt{R(t)}\mathrm{d}t+a\pi\cos\frac{l}{a}(\mathrm{i}\beta_0-\beta_1)+a\pi\beta_2,\\\varphi(\overline{z}+a\pi)-\varphi(\overline{z})&=\int_{\Lambda_+}\Phi(z)\mathrm{d}z\\&=\frac{2\mu\mathrm{i}\cos\dfrac{l}{a}}{\kappa+1}\int_{\gamma_0}f'(t)\sqrt{R(t)}\mathrm{d}t+a\pi\cos\frac{l}{a}(\mathrm{i}\beta_0+\beta_1)+a\pi\beta_2.\end{aligned}$$

故由(3.14)知，$u+\mathrm{i}v$ 的周期性条件为

$$\mathrm{i}(\kappa+1)\beta_0-(\kappa-1)\beta_1+\frac{\kappa+1}{\cos\dfrac{l}{a}}\beta_2=-\frac{2\mu\mathrm{i}}{a\pi}\int_{\gamma_0}f'(t)\sqrt{R(t)}\mathrm{d}t.$$

于是立刻知道

$$\beta_0=-\frac{2\mu}{(\kappa+1)a\pi}\int_{\gamma_0}f'(t)\sqrt{R(t)}\mathrm{d}t,\tag{6.9}$$

$$(\kappa-1)\beta_1-\frac{\kappa+1}{\cos\dfrac{l}{a}}\beta_2=0.\tag{6.10}$$

再看在 $z=-\infty\mathrm{i}$ 处应力平衡条件. 由(6.4)，

$$\Phi(-\infty\mathrm{i})=\frac{2\mu\mathrm{i}\cos\dfrac{l}{a}}{(\kappa+1)\pi a}\int_{\gamma_0}f'(t)\sqrt{R(t)}\mathrm{d}t-\beta_1\cos\frac{l}{a}+\beta_2+\mathrm{i}\beta_0\cos\frac{l}{a},$$

$$\Phi(+\infty\mathrm{i})=\frac{2\mu\mathrm{i}\cos\dfrac{l}{a}}{(\kappa+1)\pi a}\int_{\gamma_0}f'(t)\sqrt{R(t)}\mathrm{d}t+\beta_1\cos\frac{l}{a}+\beta_2+\mathrm{i}\beta_0\cos\frac{l}{a}.$$

所以由(3.12)，并注意(6.6)，知

$$\sigma_y(-\infty i) - i\tau_{xy}(-\infty i) = \Phi(-\infty i) - \Phi(+\infty i) = -2\beta_1 \cos\frac{l}{a}.$$

再由(6.1)，立刻知道

$$\beta_1 = \frac{P_0}{2a\pi\cos\frac{l}{a}}. \tag{6.11}$$

代入(6.10)，得

$$\beta_2 = \frac{\kappa-1}{\kappa+1}\cdot\frac{P_0}{2a\pi}. \tag{6.12}$$

把(6.9),(6.11),(6.12)代入(6.4)，最后得所求唯一解

$$\Phi(z) = \frac{2\mu}{(\kappa+1)a\pi\sqrt{R(z)}}\int_{\gamma_0} f'(t)\sqrt{R(t)}\left(\cot\frac{t-z}{a} - \tan\frac{z}{a}\right)dt$$
$$+ \frac{P_0}{2a\pi\cos\frac{l}{a}\sqrt{R(z)}} + \frac{\kappa-1}{\kappa+1}\frac{P_0}{2a\pi}. \tag{6.13}$$

注意，当压头基底 $f(t)$ 为偶函数(对称基底)时，上式中含 $\tan\frac{z}{a}$ 这一项可略去.

现在就容易计算在压头正下方边界上的压力分布 $P(t)$. 设 $t_0\in\gamma$，现在因 $\tau_{xy}^-(t_0)=0$，故

$$P(t_0) = -\sigma_y^-(t_0) = \Phi^+(t_0) - \Phi^-(t_0).$$

应用推广的 Plemelj 公式，不难得出

$$P(t_0) = \frac{4\mu}{(\kappa+1)a\pi\sqrt{R(t_0)}}\int_{-l}^{l} f'(t)\sqrt{R(t)}\left(\cot\frac{t-t_0}{a} - \tan\frac{t_0}{a}\right)dt$$
$$+ \frac{P_0}{a\pi\cos\frac{l}{a}\sqrt{R(t_0)}}. \tag{6.14}$$

我们来考虑当 P_0 甚小，不足使压头的角点接触弹性半平面的情况. 为简单起见，设压头基底是对称的，即 $f(t)$ 为偶函数，于是压头和弹性体接触的一周期线段由对称性可设为 γ_0：$-l\leqslant t\leqslant l$，但这时 l 为一未知正数$\left(l<\frac{1}{2}a\pi\right)$，决定它的条件是 $P(\pm l)=0$.

这时，(6.14) 成为

$$P(t_0) = \frac{4\mu}{(\kappa+1)a\pi\sqrt{R(t_0)}}\int_{-l}^{l} f'(t)\sqrt{R(t)}\cot\frac{t-t_0}{a}\,dt + \frac{P_0}{a\pi\cos\frac{l}{a}\sqrt{R(t_0)}}$$
$$= \frac{4\mu}{(\kappa+1)a\pi\sqrt{R(t_0)}}\int_{-l}^{l}\frac{f'(t)[R(t)-R(t_0)]}{\sqrt{R(t)}}\cot\frac{t-t_0}{a}\,dt$$
$$+ \frac{4\mu\sqrt{R(t_0)}}{(\kappa+1)a\pi}\int_{-l}^{l}\frac{f'(t)}{\sqrt{R(t)}}\cot\frac{t-t_0}{a}\,dt + \frac{P_0}{a\pi\cos\frac{l}{a}\sqrt{R(t_0)}}.$$

又因

$$[R(t)-R(t_0)]\cot\frac{t-t_0}{a} = -\tan\frac{t}{a}\sec^2\frac{t_0}{a} - \tan\frac{t_0}{a}\sec^2\frac{t}{a},$$

并注意 $f'(t)$ 是奇函数，于是有

$$P(t_0) = -\frac{4\mu\sec^2\frac{t_0}{a}}{(\kappa+1)a\pi\sqrt{R(t_0)}}\int_{-l}^{l}\frac{f'(t)\tan\frac{t}{a}}{\sqrt{R(t)}}dt + \frac{4\mu\sqrt{R(t_0)}}{(\kappa+1)a\pi}\int_{-l}^{l}\frac{f'(t)}{\sqrt{R(t)}}\cot\frac{t-t_0}{a}\,dt + \frac{P_0}{a\pi\cos\frac{l}{a}\sqrt{R(t_0)}}$$

$$= -\frac{4\mu\sec^2\frac{l}{a}}{(\kappa+1)a\pi\sqrt{R(t_0)}}\int_{-l}^{l}\frac{f'(t)\tan\frac{t}{a}}{\sqrt{R(t)}}dt + \frac{P_0}{a\pi\cos\frac{l}{a}\sqrt{R(t_0)}}$$

$$+ \frac{4\mu\sqrt{R(t_0)}}{(\kappa+1)a\pi}\int_{-l}^{l}\frac{f'(t)\tan\frac{t}{a}}{\sqrt{R(t)}}dt + \frac{4\mu\sqrt{R(t_0)}}{(\kappa+1)a\pi}\int_{-l}^{l}\frac{f'(t)}{\sqrt{R(t)}}\cot\frac{t-t_0}{a}\,dt.$$

当 $t_0\to\pm l$ 时，上式右边后两项皆 $\to 0$，其中第一项显然，而第二项中的积分(主值)含有对数奇异性，但 $\sqrt{R(t_0)}$ 为一阶无穷小.

所以，很明显，只要选取 l，使得下式成立：

$$P_0 = \frac{4\mu}{(\kappa+1)\cos\frac{l}{a}}\int_{-l}^{l}\frac{f'(t)\tan\frac{t}{a}}{\sqrt{R(t)}}dt. \tag{6.15}$$

这就是决定 l 的条件. 这方程也可这样解释：如果需要把周期压头加压使与弹性半平面接触一周期区间的长 $2l$，则所需的压力 P_0 由(6.15)给出. 这时压头下的压力公式是

$$P(t_0) = \frac{4\mu\sqrt{R(t_0)}}{(\kappa+1)a\pi}\int_{-l}^{l}\frac{f'(t)}{\sqrt{R(t)}}\left(\cot\frac{t-t_0}{a} + \tan\frac{t}{a}\right)dt. \tag{6.16}$$

如果在(6.13) ~ (6.16) 中令 $a\to+\infty$，便得一个压头时的相应公式：

$$\Phi(z) = \frac{2\mu}{(\kappa+1)\pi\sqrt{l^2-z^2}}\int_{-l}^{l}\frac{f'(t)\sqrt{l^2-t^2}}{t-z}dt + \frac{P_0}{2\pi\sqrt{l^2-z^2}}, \tag{6.13$'$}$$

$$P(t_0) = \frac{4\mu}{(\kappa+1)\pi\sqrt{l^2-t_0^2}}\int_{-l}^{l}\frac{f'(t)\sqrt{l^2-t^2}}{t-t_0}dt + \frac{P_0}{2\pi\sqrt{l^2-t_0^2}}, \quad -l<t_0<l, \tag{6.14$'$}$$

$$P_0 = \frac{4\mu}{\kappa+1}\int_{-l}^{l}\frac{tf'(t)}{\sqrt{l^2-t^2}}dt, \tag{6.15$'$}$$

$$P(t_0) = \frac{4\mu\sqrt{l^2-t_0^2}}{(\kappa+1)\pi}\int_{-l}^{l}\frac{f'(t)}{\sqrt{l^2-t^2}}\frac{dt}{t-t_0}; \tag{6.16$'$}$$

这些都与[7], § 115 的结果相同.

例 1 压头有直水平基底. 这时，$f'(t)=0$. 由(6.13),(6.14),

$$\Phi(z) = \frac{P_0}{2a\pi\cos\frac{l}{a}\sqrt{\tan^2\frac{l}{a}-\tan^2\frac{z}{a}}} + \frac{\kappa-1}{\kappa+1}\frac{P_0}{2a\pi}$$

$$= \frac{P_0 \cos \dfrac{z}{a}}{2a\pi \sqrt{\sin \dfrac{l+z}{a} \sin \dfrac{l-z}{a}}} + \frac{\kappa - 1}{\kappa + 1} \frac{P_0}{2a\pi},$$

$$P(t) = \frac{P_0}{a\pi \cos \dfrac{l}{a} \sqrt{\tan^2 \dfrac{l}{a} - \tan^2 \dfrac{t}{a}}} = \frac{P_0 \cos \dfrac{t}{a}}{a\pi \sqrt{\sin \dfrac{l+t}{a} \sin \dfrac{l-t}{a}}},$$

其中根号已取定这样一支：当 z 在上半平面趋于 γ_0 上的点 t 时，取正值.

例 2 压头有直倾斜基底. 仍设倾角为 ε(图 11). 这时 $f'(t) = \varepsilon$. 于是由(6.13)，

$$\begin{aligned}\Phi(z) = {} & \frac{2\mu\varepsilon}{(\kappa+1)\pi a \sqrt{R(z)}} \int_{-l}^{l} \left(\cot \frac{t-z}{a} - \tan \frac{z}{a} \right) \sqrt{R(t)} \mathrm{d}t \\ & + \frac{P_0}{2a\pi \cos \dfrac{l}{a} \sqrt{R(z)}} + \frac{\kappa-1}{\kappa+1} \frac{P_0}{2a\pi}.\end{aligned}$$

此式右边第一项实际上就是

$$\frac{2\mu\varepsilon X(z)}{(\kappa+1)\pi a} \int_{\gamma_0} \left(\cot \frac{t-z}{a} - \tan \frac{z}{a} \right) \frac{\mathrm{d}t}{X^+(t)},$$

其中积分就是(5.36)所表示的 $I(z)$，但需把那里的 κ 改作 1. 于是由(5.37)知（这时 $A = 0$），

$$\int_{\gamma_0} \left(\cot \frac{t-z}{a} - \tan \frac{z}{a} \right) \frac{\mathrm{d}t}{X^+(t)} = a\pi \mathrm{i} \left\{ \frac{1}{X(z)} - \sec \frac{l}{a} \tan \frac{z}{a} \right\}.$$

因而上式 $\Phi(z)$ 中的第一项成为

$$\frac{2\mu\varepsilon}{\kappa+1} \left\{ \mathrm{i} - \frac{\mathrm{i} \tan \dfrac{z}{a} X(z)}{\cos \dfrac{l}{a}} \right\} = \frac{2\mu\varepsilon}{\kappa+1} \left\{ \mathrm{i} - \frac{\tan \dfrac{z}{a}}{\cos \dfrac{l}{a} \sqrt{R(z)}} \right\}.$$

代入 $\Phi(z)$ 中，得

$$\Phi(z) = \frac{2\mu\varepsilon}{\kappa+1} \left[\mathrm{i} - \frac{\sin \dfrac{z}{a}}{\sqrt{\sin \dfrac{l+z}{a} \sin \dfrac{l-z}{a}}} \right] + \frac{P_0}{2a\pi} \left[\frac{\cos \dfrac{z}{a}}{\sqrt{\sin \dfrac{l+z}{a} \sin \dfrac{l-z}{a}}} + \frac{\kappa-1}{\kappa+1} \right].$$

为了计算在压头正下方边界上的压力分布，可应用(6.14)，但直接用上式更为方便. 事实上，设 $-l < t < l$，则有

$$\begin{aligned}P(t) &= \Phi^+(t) - \Phi^-(t) \\ &= -\frac{4\mu\varepsilon}{\kappa+1} \cdot \frac{\sin \dfrac{t}{a}}{\sqrt{\sin \dfrac{l+t}{a} \sin \dfrac{l-t}{a}}} + \frac{P_0}{a\pi} \frac{\cos \dfrac{t}{a}}{\sqrt{\sin \dfrac{l+t}{a} \sin \dfrac{l-t}{a}}}.\end{aligned}$$

如要求在物理上可能，即 $P(t) \geqslant 0$，只需 P_0 满足下一条件即可：

$$P_0 \geqslant \frac{4\mu a\pi\varepsilon}{\kappa+1} \tan \frac{l}{a}.$$

例 3 压头有圆基底. 设压头有半径为 r 的圆基底，且 r 甚大. 我们把基底曲线近似地

取作

$$f(t)=\frac{a^2}{2r}\tan^2\frac{t}{a},\quad -l\leqslant t\leqslant l$$

(曲线在 $t=0$ 处的曲率半径为 r)，于是

$$f'(t)=\frac{a}{r}\tan\frac{t}{a}\sec^2\frac{t}{a}.$$

代入(6.13)，并注意对称性，可知

$$\Phi(z)=\frac{2\mu}{(\kappa+1)\pi r\sqrt{R(z)}}\int_{-l}^{l}\tan\frac{t}{a}\sec^2\frac{t}{a}\sqrt{R(t)}\cot\frac{t-z}{a}\mathrm{d}t+\frac{P_0}{2a\pi}\left(\frac{1}{\cos\frac{l}{a}\sqrt{R(z)}}+\frac{\kappa-1}{\kappa+1}\right).$$

为了算出右端中的积分，记作 $J(z)$，作类似于前节例5中的代换，得知

$$J(z)=a\int_{-L}^{L}u\sqrt{L^2-u^2}\,\frac{1+\zeta u}{u-\zeta}\mathrm{d}u=a(1+\zeta^2)\int_{-L}^{L}\frac{u\sqrt{L^2-u^2}}{u-\zeta}\mathrm{d}u;$$

然后再利用那里所用的方法，最后可算出

$$J(z)=\frac{a\pi}{\cos^2\frac{z}{a}}\left\{\mathrm{i}\tan\frac{z}{a}\sqrt{R(z)}+\frac{1}{2}\tan^2\frac{l}{a}-\tan^2\frac{z}{a}\right\}.$$

代入前式，化简后得

$$\Phi(z)=\frac{\mu a}{(\kappa+1)r\cos^2\frac{z}{a}}\left\{2\mathrm{i}\tan\frac{z}{a}+\frac{\tan^2\frac{l}{a}-2\tan^2\frac{z}{a}}{\sqrt{\tan^2\frac{l}{a}-\tan^2\frac{z}{a}}}\right\}+\frac{P_0}{2a\pi}\left\{\frac{1}{\cos\frac{l}{a}\sqrt{\tan^2\frac{l}{a}-\tan^2\frac{z}{a}}}+\frac{\kappa-1}{\kappa+1}\right\}.$$

当 $-l<t<l$ 时，压头正下方压力分布为

$$P(t)=\Phi^+(t)-\Phi^-(t)=\frac{2\mu a\left(\tan^2\frac{l}{a}-2\tan^2\frac{t}{a}\right)}{(\kappa+1)r\cos^2\frac{t}{a}\sqrt{\tan^2\frac{l}{a}-\tan^2\frac{t}{a}}}+\frac{P_0}{a\pi\cos\frac{l}{a}\sqrt{\tan^2\frac{l}{a}-\tan^2\frac{t}{a}}}.$$

欲 $P(t)\geqslant 0$，只需

$$P_0\geqslant\frac{2\mu a^2\pi\sin^2\frac{l}{a}}{(\kappa+1)r\cos^3\frac{l}{a}}.$$

当 r 充分大时，这一定可以做到. 如给定 $P_0>0$，把上式改为等式，可以证明对于 l，在 $\left(0,\frac{1}{2}a\pi\right)$ 中有唯一解，这就是相应于压头和弹性半平面接触的一个周期线段的半长.

在以上三例中，令 $a\to+\infty$ 时，可得到[7]，§116a 中的已知结果.

2. 摩擦存在时的情况　现设周期压头与弹性半平面间摩擦系数 $k\neq 0$. 即设在压头正

下方，剪切力 $T(t)=\tau_{xy}(t)$ 和正压力 $P(t)=-\sigma_y(t)$ 间有关系式

$$T(t)=kP(t),\quad k\neq 0,\quad t\in\gamma_0. \tag{6.17}$$

此外，仍设已知$v^-(t)=f(t)$，$f'(t)\in H$，其中$t\in\gamma_0$；又在γ_0上，已知外压力合力为P_0，因而外应力主矢量 $X+\mathrm{i}Y=T_0-\mathrm{i}P_0=(k-\mathrm{i})P_0$. 在$L_0-\gamma_0$上，$T(t)=P(t)=0$.

由(3.12)和(3.13)容易看出（参见[7]，§117），当$t\in\gamma_0$时，

$$(1-\mathrm{i}k)\Phi^+(t)+(1+\mathrm{i}k)\overline{\Phi}^+(t)=(1-\mathrm{i}k)\Phi^-(t)+(1+\mathrm{i}k)\overline{\Phi}^-(t), \tag{6.18}$$

$$\kappa\Phi^-(t)+\Phi^+(t)-\kappa\overline{\Phi}^+(t)-\overline{\Phi}^-(t)=4\mathrm{i}\mu f'(t); \tag{6.19}$$

且当t不在γ_0及其周期性相应部分上时，$\Phi^+(t)=\Phi^-(t)$. 由(6.18)可知

$$(1-\mathrm{i}k)\Phi(z)+(1+\mathrm{i}k)\overline{\Phi}(z)=2\beta_2, \tag{6.20}$$

其中β_2是一常数. 如取$t\in L_0-\gamma_0$，则因$\Phi^+(t)=\Phi^-(t)=\Phi(t)$，故有

$$\overline{\Phi}(t)=\overline{\Phi(\bar{t})}=\overline{\Phi(t)},$$

所以立刻知道(6.20)中的β_2是实数.

用(6.20)化简(6.19)，消去$\overline{\Phi}(z)$，便得周期 Riemann 边值问题：

$$\frac{(\kappa+1)-\mathrm{i}k(\kappa-1)}{1+\mathrm{i}k}\Phi^+(t)+\frac{(\kappa+1)+\mathrm{i}k(\kappa-1)}{1+\mathrm{i}k}\Phi^-(t)$$
$$=4\mathrm{i}\mu f'(t)+\frac{2(\kappa+1)\beta_2}{1+\mathrm{i}k},$$

或

$$\Phi^+(t)=K\Phi^-(t)+f_0'(t)+\frac{2(\kappa+1)}{(\kappa+1)-\mathrm{i}k(\kappa-1)}\beta_2, \tag{6.21}$$

其中

$$K=-\frac{(\kappa+1)+\mathrm{i}k(\kappa-1)}{(\kappa+1)-\mathrm{i}k(\kappa-1)}, \tag{6.22}$$

$$f_0'(t)=\frac{4\mu\mathrm{i}(1+\mathrm{i}k)}{(\kappa+1)-\mathrm{i}k(\kappa-1)}f'(t). \tag{6.23}$$

如果令

$$\tan\pi\alpha=k\frac{\kappa-1}{\kappa+1}\quad(\text{且取 }|\alpha|\text{ 为最小正数}), \tag{6.24}$$

则由于

$$\kappa+1\pm\mathrm{i}k(\kappa-1)=\frac{(\kappa+1)\mathrm{e}^{\pm\pi\mathrm{i}\alpha}}{\cos\pi\alpha}, \tag{6.25}$$

(6.22)和(6.23)可改写为

$$K=-\mathrm{e}^{2\pi\mathrm{i}\alpha}, \tag{6.22$'$}$$

$$f_0'(t)=\frac{4\mu\mathrm{i}(1+\mathrm{i}k)\mathrm{e}^{\pi\mathrm{i}\alpha}\cos\pi\alpha}{\kappa+1}f'(t). \tag{6.23$'$}$$

而边值问题(6.21)可改写为

$$\Phi^+(t)=-\mathrm{e}^{2\pi\mathrm{i}\alpha}\Phi^-(t)+f_0'(t)+2\beta_2\mathrm{e}^{\pi\mathrm{i}\alpha}\cos\pi\alpha. \tag{6.21$'$}$$

注意，因$k>0$，$\kappa>1$，故

$$0<|\alpha|<\frac{1}{2}. \tag{6.26}$$

现在

$$\frac{\log K}{2\pi \mathrm{i}} = \frac{1}{2} + \alpha,$$

且我们要求(6.21)′在 h_0 类中的解，所以，类似于§2第4段中特例的讨论，易见问题的指标为1，且典则函数

$$X(z) = \left(\tan\frac{z}{a} + \tan\frac{l}{a}\right)^{-\frac{1}{2}-\alpha}\left(\tan\frac{z}{a} - \tan\frac{l}{a}\right)^{-\frac{1}{2}+\alpha}, \tag{6.27}$$

且不妨取这样的一支，使

$$\lim_{z\to\pm\frac{1}{2}a\pi} \tan\frac{z}{a} X(z) = 1. \tag{6.28}$$

和前一段相似，立刻可知问题(6.21)′的一般解为

$$\Phi(z) = \frac{2\mu(1+\mathrm{i}k)\mathrm{e}^{\pi\mathrm{i}\alpha}\cos\pi\alpha\, X(z)}{a\pi(\kappa+1)}\int_{\gamma_0}\frac{f'(t)}{X^+(t)}\cot\frac{t-z}{a}\,\mathrm{d}t$$
$$+ X(z)(1+\mathrm{i}k)\mathrm{i}\left(\beta_0\tan\frac{z}{a} + \beta_1\right) + \beta_2, \tag{6.29}$$

其中 β_0,β_1 为某二常数.

我们来证明，β_0,β_1 必须是实数方能适合条件(6.20). 首先我们注意到：因为 $X(z)$ 当 $z\in L_0-\gamma_0$ 时取实值，故 $X(\bar{z}) = \overline{X(z)}$，即 $\overline{X}(z) = X(z)$；又当 $t\in\gamma_0$ 时

$$\overline{X^+(t)} = \overline{X}^-(t) = X^-(t) = \frac{1}{K}X^+(t) = -\mathrm{e}^{-2\pi\mathrm{i}\alpha}X^+(t),$$

因此

$$\frac{\Phi(z)}{1+\mathrm{i}k} = \frac{2\mu\mathrm{e}^{\pi\mathrm{i}\alpha}\cos\pi\alpha\, X(z)}{a\pi(\kappa+1)}\int_{\gamma_0}\frac{f'(t)}{X^+(t)}\cot\frac{t-z}{a}\,\mathrm{d}t$$
$$+ X(z)\left(\beta_0\tan\frac{z}{a} + \beta_1\right)\mathrm{i} + \frac{\beta_2}{1+\mathrm{i}k},$$

而

$$\frac{\overline{\Phi}(z)}{1-\mathrm{i}k} = \overline{\left[\frac{\Phi(\bar{z})}{1+\mathrm{i}k}\right]} = -\frac{2\mu\mathrm{e}^{\pi\mathrm{i}\alpha}\cos\pi\alpha\, X(z)}{a\pi(\kappa+1)}\int_{\gamma_0}\frac{f'(t)}{X^+(t)}\cot\frac{t-z}{a}\,\mathrm{d}t$$
$$- X(z)\left(\overline{\beta}_0\tan\frac{z}{a} + \overline{\beta}_1\right)\mathrm{i} + \frac{\beta_2}{1-\mathrm{i}k}.$$

为要(6.20)成立，即

$$\frac{\Phi(z)}{1+\mathrm{i}k} + \frac{\overline{\Phi}(z)}{1-\mathrm{i}k} = \frac{2\beta_2}{1+k^2},$$

必须 β_0,β_1 为实数.

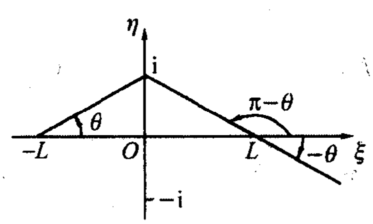

图 12

仍和上段一样，利用位移周期性和 $z=-\infty\mathrm{i}$ 处应力平衡条件可决定 β_0,β_1,β_2.

为此，注意图12（并和图9对照），其中已令

$$L = \tan\frac{l}{a},\quad \zeta = \tan\frac{z}{a},$$

$$\theta = \arctan\frac{1}{L} = \frac{\pi}{2} - \frac{l}{a},\quad 0<\theta<\frac{\pi}{2},$$

易见

$$X(+\infty \mathrm{i}) = X_*(\mathrm{i}) = \cos\frac{l}{a}\,\mathrm{e}^{(-\frac{1}{2}-\alpha)\theta\mathrm{i}}\mathrm{e}^{(-\frac{1}{2}+\alpha)(\pi-\theta)\mathrm{i}}$$

$$= -\mathrm{i}\cos\frac{l}{a}\,\mathrm{e}^{\alpha\pi\mathrm{i}}\mathrm{e}^{-2\alpha\theta\mathrm{i}} = -\mathrm{i}\cos\frac{l}{a}\,\mathrm{e}^{\frac{2l\alpha}{a}\mathrm{i}}, \tag{6.30}$$

同理，

$$X(-\infty \mathrm{i}) = \mathrm{i}\cos\frac{l}{a}\,\mathrm{e}^{-\frac{2l\alpha}{a}\mathrm{i}}. \tag{6.31}$$

于是，如令 $\Lambda_\pm$ 仍同前段意义，则有

$$\int_{\Lambda_\pm} X(z)\mathrm{d}z = a\pi X(\pm\infty\mathrm{i}) = \mp a\pi\mathrm{i}\cos\frac{l}{a}\,\mathrm{e}^{\pm\frac{2l\alpha}{a}\mathrm{i}},$$

$$\int_{\Lambda_\pm} X(z)\tan\frac{z}{a}\,\mathrm{d}z = \int_{\Lambda_\pm} X(z)\cot\frac{t-z}{a}\,\mathrm{d}z = \pm a\pi\mathrm{i}X(\pm\infty\mathrm{i}) = a\pi\cos\frac{l}{a}\,\mathrm{e}^{\pm\frac{2l\alpha}{a}\mathrm{i}}.$$

当 z 在下半平面时，由(6.29) 知，

$$\varphi(z+a\pi) - \varphi(z) = \int_{\Lambda_-}\Phi(z)\mathrm{d}z$$

$$= \frac{2\mu(1+\mathrm{i}k)\cos\pi\alpha\cos\frac{l}{a}\,\mathrm{e}^{(\pi-\frac{2l}{a})\alpha\mathrm{i}}}{\kappa+1}\int_{\gamma_0}\frac{f'(t)}{X^+(t)}\mathrm{d}t$$

$$+ a\pi(1+\mathrm{i}k)\cos\frac{l}{a}\,(\mathrm{i}\beta_0-\beta_1)\mathrm{e}^{-\frac{2l\alpha}{a}\mathrm{i}} + a\pi\beta_2,$$

$$\varphi(\bar{z}+a\pi) - \varphi(\bar{z}) = \int_{\Lambda_+}\Phi(z)\mathrm{d}z$$

$$= \frac{2\mu(1+\mathrm{i}k)\cos\pi\alpha\cos\frac{l}{a}\,\mathrm{e}^{(\pi+\frac{2l}{a})\alpha\mathrm{i}}}{\kappa+1}\int_{\gamma_0}\frac{f'(t)}{X^+(t)}\mathrm{d}t$$

$$+ a\pi(1+\mathrm{i}k)\cos\frac{l}{a}\,(\mathrm{i}\beta_0+\beta_1)\mathrm{e}^{\frac{2l\alpha}{a}\mathrm{i}} + a\pi\beta_2.$$

于是位移周期性条件就是

$$\mathrm{i}(\mathrm{e}^{\frac{2l\alpha}{a}\mathrm{i}} + \kappa\mathrm{e}^{-\frac{2l\alpha}{a}\mathrm{i}})\beta_0 + (\mathrm{e}^{\frac{2l\alpha}{a}\mathrm{i}} - \kappa\mathrm{e}^{-\frac{2l\alpha}{a}\mathrm{i}})\beta_1 + \frac{(\kappa+1)\beta_2}{(1+\mathrm{i}k)\cos\frac{l}{a}}$$

$$= -\frac{2\mu\cos\pi\alpha}{(\kappa+1)a\pi}[\mathrm{e}^{(\pi+\frac{2l}{a})\alpha\mathrm{i}} + \kappa\mathrm{e}^{(\pi-\frac{2l}{a})\alpha\mathrm{i}}]\int_{\gamma_0}\frac{f'(t)}{X^+(t)}\mathrm{d}t. \tag{6.32}$$

再看在 $z=-\infty\mathrm{i}$ 处应力平衡条件. 因为由(6.30),(6.31)，易见

$$\Phi(-\infty\mathrm{i}) = \frac{2\mu(1+\mathrm{i}k)\cos\pi\alpha\cos\frac{l}{a}\,\mathrm{e}^{(\pi-\frac{2l}{a})\alpha\mathrm{i}}}{a\pi(\kappa+1)}\int_{\gamma_0}\frac{f'(t)}{X^+(t)}\mathrm{d}t$$

$$+ (1+\mathrm{i}k)\cos\frac{l}{a}\,(\mathrm{i}\beta_0-\beta_1)\mathrm{e}^{-\frac{2l\alpha}{a}\mathrm{i}} + \beta_2,$$

$$\Phi(+\infty\mathrm{i}) = \frac{2\mu(1+\mathrm{i}k)\cos\pi\alpha\cos\frac{l}{a}\,\mathrm{e}^{(\pi+\frac{2l}{a})\alpha\mathrm{i}}}{a\pi(\kappa+1)}\int_{\gamma_0}\frac{f'(t)}{X^+(t)}\mathrm{d}t$$

$$+ (1+\mathrm{i}k)\cos\frac{l}{a}\,(\mathrm{i}\beta_0+\beta_1)\mathrm{e}^{\frac{2l\alpha}{a}\mathrm{i}} + \beta_2.$$

所以，注意到(6.6)仍成立，因而

$$\Phi(-\infty i)-\Phi(+\infty i)=\sigma_y(-\infty i)-i\tau_{xy}(-\infty i)=-\frac{(1+ik)P_0}{a\pi},$$

把上式代入，即得

$$-\frac{P_0}{a\pi}=\frac{2\mu\cos\pi\alpha\cos\frac{l}{a}}{a\pi(\kappa+1)}[e^{(\pi-\frac{2l}{a})\alpha i}-e^{(\pi+\frac{2l}{a})\alpha i}]\int_{\gamma_0}\frac{f'(t)}{X^+(t)}dt$$
$$+\cos\frac{l}{a}\left\{i\beta_0(e^{-\frac{2l\alpha}{a}i}-e^{\frac{2l\alpha}{a}i})-\beta_1(e^{\frac{2l\alpha}{a}i}+e^{-\frac{2l\alpha}{a}i})\right\},$$

亦即

$$\beta_0\sin\frac{2l\alpha}{a}-\beta_1\cos\frac{2l\alpha}{a}=-\frac{2\mu\cos\pi\alpha\sin\frac{2l\alpha}{a}e^{\pi\alpha i}}{a\pi i(\kappa+1)}\int_{\gamma_0}\frac{f'(t)}{X^+(t)}dt-\frac{P_0}{2a\pi\cos\frac{l}{a}}.\tag{6.33}$$

如果我们令

$$Q(z)=\left(\tan\frac{l}{a}+\tan\frac{z}{a}\right)^{\frac{1}{2}+\alpha}\left(\tan\frac{l}{a}-\tan\frac{z}{a}\right)^{\frac{1}{2}-\alpha},\tag{6.34}$$

并取定当 z 由上半平面趋于 $t\in\gamma_0$ 时，$Q(t)$ 取正值的那一支，则由于

$$\arg X^+(t)=\left(-\frac{1}{2}+\alpha\right)\pi,\quad t\in\gamma_0,$$

所以

$$X(z)=-\frac{i\,e^{\alpha\pi i}}{Q(z)},\quad X^+(t)=-\frac{i\,e^{\alpha\pi i}}{Q(t)},\ t\in\gamma_0,$$

且 $Q(t)$ 为正值. 因此，$\Phi(z)$ 的表达式(6.29)可改写为

$$\Phi(z)=\frac{2\mu(1+ik)e^{\pi i\alpha}\cos\pi\alpha}{a\pi(\kappa+1)Q(z)}\int_{\gamma_0}f'(t)Q(t)\cot\frac{t-z}{a}dt$$
$$+\frac{(1+ik)e^{\pi i\alpha}}{Q(z)}\left(\beta_0\tan\frac{z}{a}+\beta_1\right)+\beta_2,\tag{6.29$'$}$$

而 β_0,β_1,β_2 由(6.32),(6.33)或即下列两方程决定:

$$i(e^{\frac{2l\alpha}{a}i}+\kappa e^{-\frac{2l\alpha}{a}i})\beta_0+(e^{\frac{2l\alpha}{a}i}-\kappa e^{-\frac{2l\alpha}{a}i})\beta_1+\frac{\kappa+1}{(1+ik)\cos\frac{l}{a}}\beta_2$$
$$=\frac{2\mu\cos\pi\alpha}{(\kappa+1)a\pi i}(e^{\frac{2l\alpha}{a}i}+\kappa e^{\frac{2l\alpha}{a}i})\int_{\gamma_0}f'(t)Q(t)dt.\tag{6.32$'$}$$

$$\beta_0\sin\frac{2l\alpha}{a}-\beta_1\cos\frac{2l\alpha}{a}=-\frac{2\mu\cos\pi\alpha\sin\frac{2l\alpha}{a}}{(\kappa+1)a\pi}\int_{\gamma_0}f'(t)Q(t)dt-\frac{P_0}{2a\pi\cos\frac{l}{a}}.\tag{6.33$'$}$$

如果把(6.32)′的实部和虚部分开，得

$$\beta_0\cos\frac{2l\alpha}{a}+\beta_1\sin\frac{2l\alpha}{a}-\frac{k}{k^2+1}\frac{\beta_2}{\cos\frac{l}{a}}=-\frac{2\mu\cos\pi\alpha\cos\frac{2l\alpha}{a}}{(\kappa+1)a\pi}\int_{\gamma_0}f'(t)Q(t)dt,\tag{6.35}$$

$$\beta_0 \sin\frac{2l\alpha}{a} - \beta_1 \cos\frac{2l\alpha}{a} + \frac{\kappa+1}{\kappa-1}\frac{\beta_2}{(k^2+1)\cos\frac{l}{a}} = -\frac{2\mu\cos\pi\alpha\sin\frac{2l\alpha}{a}}{(\kappa+1)a\pi}\int_{\gamma_0} f'(t)Q(t)\mathrm{d}t. \tag{6.36}$$

比较(6.36)和(6.33)′，立得

$$\beta_2 = \frac{\kappa-1}{\kappa+1}\frac{(k^2+1)P_0}{2a\pi}. \tag{6.37}$$

把它代入(6.35)，再和(6.33)′联立，便可算出 β_0, β_1：

$$\beta_0 = -\frac{2\mu\cos\pi\alpha}{(\kappa+1)a\pi}\int_{\gamma_0} f'(t)Q(t)\mathrm{d}t + \frac{P_0\sin(1-\lambda)\pi\alpha}{2a\pi\cos\pi\alpha\cos\frac{l}{a}}, \tag{6.38}$$

$$\beta_1 = \frac{P_0\cos(1-\lambda)\pi\alpha}{2a\pi\cos\pi\alpha\cos\frac{l}{a}}, \tag{6.39}$$

这里已把 γ_0 对一个周期 L_0 的长度之比仍记为 λ：

$$\lambda = \frac{2l}{a\pi} \quad (0<\lambda<1). \tag{6.40}$$

把(6.37),(6.38),(6.39)代入(6.29)′，便得 $\Phi(z)$ 的最后表达式.

现在来计算压头正下方的压力分布. 因为现在

$$P(t_0) + \mathrm{i}T(t_0) = (1+\mathrm{i}k)P(t_0) = \Phi^+(t_0) - \Phi^-(t_0),$$

利用推广的 Plemelj 公式，并注意到

$$Q^-(t) = -Q(t)\mathrm{e}^{2\alpha\pi\mathrm{i}}, \tag{6.41}$$

可算出

$$P(t_0) = -\frac{2\mu\sin 2\pi\alpha}{\kappa+1}f'(t_0) + \frac{4\mu\cos^2\pi\alpha}{a\pi(\kappa+1)Q(t_0)}\int_{-l}^{l} f'(t)Q(t)\cot\frac{t-t_0}{a}\mathrm{d}t + \frac{2\cos\pi\alpha}{Q(t_0)}\left(\beta_0\tan\frac{t_0}{a} + \beta_1\right), \tag{6.42}$$

其中 β_0, β_1 仍由(6.38),(6.39)给出.

当 $k=0$(随之 $\alpha=0$)时，就回到上一段中曾讨论过的情况. 另一方面，如令 $a\to+\infty$，便可得[7]，§117 中的结果.

例 4 压头有直水平基底. 这时 $f'(t)=0$. 于是由(6.29)′和(6.37)～(6.39)，知

$$\Phi(z) = \frac{P_0(1+\mathrm{i}k)\mathrm{e}^{\pi\mathrm{i}\alpha}\cos\left[\frac{z}{a} - (1-\lambda)\pi\alpha\right]}{2a\pi\cos\pi\alpha\sin^{\frac{1}{2}+\alpha}\frac{l+z}{a}\sin^{\frac{1}{2}-\alpha}\frac{l-z}{a}} + \frac{\kappa-1}{\kappa+1}\frac{(k^2+1)P_0}{2a\pi}, \tag{6.43}$$

其中 $\sin^{\frac{1}{2}+\alpha}\frac{l+z}{a}\sin^{\frac{1}{2}-\alpha}\frac{l-z}{a}$ 已取定这样一支：当 z 在上半平面趋于 γ_0 上的点时，它取正值；而 λ 仍以(6.40)给出.

又由(6.42)知，在压头正下方的边界上

$$P(t) = \frac{P_0\cos\left[\frac{t}{a} - (1-\lambda)\pi\alpha\right]}{a\pi\sin^{\frac{1}{2}+\alpha}\frac{l+t}{a}\sin^{\frac{1}{2}-\alpha}\frac{l-t}{a}}, \quad t\in\gamma_0. \tag{6.44}$$

令 $k=0(\alpha=0)$ 时，仍回到例 1 的结果.

例 5 压头有直倾斜基底. 仍设倾角为 ε(图 11). 这时 $f'(t)=\varepsilon$. 代入(6.29)′ 并注意(6.37)～(6.39)，可得

$$\Phi(z)=\frac{2\mu\varepsilon(1+\mathrm{i}k)\mathrm{e}^{\pi\mathrm{i}\alpha}\cos\pi\alpha}{a\pi(\kappa+1)Q(z)}\int_{\gamma_0}Q(t)\left(\cot\frac{t-z}{a}-\tan\frac{z}{a}\right)\mathrm{d}t+\Phi_1(z),\qquad(6.45)$$

其中 $\Phi_1(z)$ 就是例 4 中的 $\Phi(z)$.

现在来计算包含在上式中的积分

$$I(z)=\int_{\gamma_0}Q(t)\left(\cot\frac{t-z}{a}-\tan\frac{z}{a}\right)\mathrm{d}t.$$

仍和以前一样，令 $u=\tan\dfrac{t}{a}$，$\zeta=\tan\dfrac{z}{a}$，$L=\tan\dfrac{l}{a}$，则易见

$$I(z)=a\sec^2\frac{z}{a}J(\zeta),$$

其中

$$J(\zeta)=\int_{-L}^{L}\frac{(L+u)^{\frac{1}{2}+\alpha}(L-u)^{\frac{1}{2}-\alpha}}{(1+u^2)(u-\zeta)}\mathrm{d}u.$$

再令

$$\Omega(\zeta)=\int_{\Lambda}\frac{(L+w)^{\frac{1}{2}+\alpha}(L-w)^{\frac{1}{2}-\alpha}}{(1+w^2)(w-\zeta)}\mathrm{d}w,$$

其中 Λ 仍为 w 平面上围住线段$[-L,L]$的闭路，但不围住 ζ 者，并沿时针向积分. 注意到(6.41)，易见

$$\Omega(\zeta)=(1+\mathrm{e}^{2\alpha\pi\mathrm{i}})J(\zeta),$$

故只要计算 $\Omega(\zeta)$.

注意在 $w=\infty$ 附近，$\Omega(\zeta)$ 右端被积式有重零点；而在 $w=\mathrm{i}$ 附近(参看图 12)，

$$\begin{aligned}(L+\mathrm{i})^{\frac{1}{2}+\alpha}(L-\mathrm{i})^{\frac{1}{2}-\alpha}&=|L+\mathrm{i}|^{\frac{1}{2}+\alpha}|L-\mathrm{i}|^{\frac{1}{2}-\alpha}\mathrm{e}^{(\frac{1}{2}+\alpha)\theta\mathrm{i}}\mathrm{e}^{-(\frac{1}{2}+\alpha)\theta\mathrm{i}}\\&=\sec\frac{l}{a}\,\mathrm{e}^{2\alpha\theta\mathrm{i}}=\sec\frac{l}{a}\,\mathrm{e}^{2\alpha(\frac{\pi}{2}-\frac{l}{a})\mathrm{i}}=\sec\frac{l}{a}\,\mathrm{e}^{(1-\lambda)\alpha\pi\mathrm{i}},\end{aligned}$$

在 $w=-\mathrm{i}$ 附近

$$\begin{aligned}(L-\mathrm{i})^{\frac{1}{2}+\alpha}(L+\mathrm{i})^{\frac{1}{2}-\alpha}&=\sec\frac{l}{a}\,\mathrm{e}^{-(\frac{1}{2}+\alpha)\theta\mathrm{i}}\mathrm{e}^{(\frac{1}{2}-\alpha)(-2\pi+\theta)\mathrm{i}}\\&=-\sec\frac{l}{a}\,\mathrm{e}^{2\pi\alpha\mathrm{i}}\mathrm{e}^{-2\alpha\theta\mathrm{i}}=-\sec\frac{l}{a}\,\mathrm{e}^{(1+\lambda)\alpha\pi\mathrm{i}}.\end{aligned}$$

于是，由熟知的公式，知

$$\frac{1}{2\pi\mathrm{i}}\Omega(\zeta)=\frac{(L+\zeta)^{\frac{1}{2}+\alpha}(L-\zeta)^{\frac{1}{2}-\alpha}}{1+\zeta^2}-\frac{\mathrm{e}^{(1-\lambda)\alpha\pi\mathrm{i}}}{2\mathrm{i}(\zeta-\mathrm{i})\cos\dfrac{l}{a}}-\frac{\mathrm{e}^{(1+\lambda)\alpha\pi\mathrm{i}}}{2\mathrm{i}(\zeta+\mathrm{i})\cos\dfrac{l}{a}},$$

故

$$\Omega(\zeta)=\frac{2\pi\mathrm{i}(L+\zeta)^{\frac{1}{2}+\alpha}(L-\zeta)^{\frac{1}{2}-\alpha}}{1+\zeta^2}-\frac{2\pi\,\mathrm{e}^{\alpha\pi\mathrm{i}}(\zeta\cos\lambda\alpha\pi+\sin\lambda\alpha\pi)}{(1+\zeta^2)\cos\dfrac{l}{a}}.$$

回到 $I(z)$，因

$$I(z)=\frac{a}{\cos^2\frac{z}{a}}\frac{1}{1+e^{2\alpha\pi i}}\Omega(\zeta)=\frac{ae^{-\alpha\pi i}}{2\cos^2\frac{z}{a}\cos\alpha\pi}\Omega(\zeta),$$

故最后得

$$I(z)=\frac{a\pi i\Omega(z)}{\cos\pi\alpha\, e^{\alpha\pi i}}-\frac{a\pi\sin\left(\frac{z}{a}+\lambda\alpha\pi\right)}{\cos\pi\alpha\cos\frac{l}{a}\cos\frac{z}{a}}.$$

把它代入(6.45)，得

$$\Phi(z)=\frac{2\mu\varepsilon i(1+ik)}{\kappa+1}-\frac{2\mu\varepsilon(1+ik)e^{\kappa i\alpha}\sin\left(\frac{z}{a}+\lambda\alpha\pi\right)}{(\kappa+1)\sin^{\frac{1}{2}+\alpha}\frac{l+z}{a}\sin^{\frac{1}{2}-\alpha}\frac{l-z}{a}}+\Phi_1(z), \tag{6.46}$$

其中 $\Phi_1(z)$ 要以(6.43)给出.

现在来计算压头正下方的压力分布. 我们不用(6.42)而更快地直接计算. 由于

$$P(t)=\frac{\Phi^+(t)-\Phi^-(t)}{1+ik},$$

并仍注意到(6.41)，立刻可算出

$$P(t)=-\frac{4\mu\varepsilon\cos\pi\alpha\sin\left(\frac{t}{a}+\lambda\alpha\pi\right)}{(\kappa+1)\sin^{\frac{1}{2}+\alpha}\frac{l+t}{a}\sin^{\frac{1}{2}-\alpha}\frac{l-t}{a}}+P_1(t),$$

其中 $P_1(t)$ 以(6.44)给出. 故最后可得

$$P(t)=\frac{P_0\cos\left[\frac{t}{a}-(1-\lambda)\pi\alpha\right]}{a\pi\sin^{\frac{1}{2}+\alpha}\frac{l+t}{a}\sin^{\frac{1}{2}-\alpha}\frac{l-t}{a}}-\frac{4\mu\varepsilon\cos\pi\alpha\sin\left(\frac{t}{a}+\lambda\pi\alpha\right)}{(\kappa+1)\sin^{\frac{1}{2}+\alpha}\frac{l+t}{a}\sin^{\frac{1}{2}-\alpha}\frac{l-t}{a}},\quad t\in\gamma_0. \tag{6.47}$$

显然，当 ε 充分小，例如当

$$\varepsilon\leqslant\frac{P_0(\kappa+1)\cos\left[(\alpha+\frac{1}{2})\lambda\pi-\pi\alpha\right]}{4\mu a\pi\cos\pi\alpha\sin(\alpha+\frac{1}{2})\lambda\pi}$$

时，可保证 $P(t)\geqslant 0$，即物理上可能.

如在(6.46),(6.47)中令 $k=0$，则回到例2中的结果；如令 $\alpha\to+\infty$，则得[7]，§117a中的结果.

附注1 运用[7]，§119以及本节中类似的方法，可解决周期弹性接触的平面问题，这里就不详谈了.

附注2 以上所论，都假定一个周期内只有一个压头，但显然都可推广到有几个压头的情形.

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开口弧段的双周期 Riemann 边值问题

一、问题、记号、引理

关于光滑封闭曲线的双周期 Riemann 边值问题已有充分的研究[1,2]. 这里讨论光滑开口弧段 L_0 上的类似问题，并将其应用于某些奇异积分方程. 我们限定 $L_0=\widehat{ab}$是一条光滑弧段，但所用的方法及结果不难推广到一般情形.

设双周期是 $2\omega_1, 2\omega_2$，其中 $\mathrm{Im}(\omega_2/\omega_1)>0$. 基本胞腔取成以$\pm\omega_1\pm\omega_2$ 为顶点的平行四边形，而设 L_0 位于这胞腔内，且不失一般性，设原点 $O\overline{\in} L_0$，L_0 的正向取作自 a 到 b 的方向. 基本胞腔中除去 L_0 后的区域记为 S_0. 把 L_0 作双周期平移后的曲线族的集合记为 L，全平面除去 L 后的区域记为 S.

我们的问题如下：在 S 中求一双周期的单值解析函数 $\Phi(z)$，使得

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t),\quad t\in L,\tag{1.1}$$

其中$\Phi^\pm(t)$ 表示从L 的正负侧趋于 L 上的点 t ($t\neq a, b$ 及其周期合同点，下同) 时的极限值，而 $G(t), g(t)$ 都是 L 上 $\in H$(Hölder 条件) 的已知双周期函数，且 $G(t)\neq 0$；在 L 的端点附近，要求

$$|\Phi(z)|\leqslant K|z-c|^{-\mu},\quad \mu<1, K>0\ (c=a \text{ 或 } b).\tag{1.2}$$

此外，我们还要求：或者 $\Phi(z)$ 在 S 中处处正则，记为问题(R_0)；或者允许 $\Phi(z)$ 在 $z=0$ 处可以有一阶极点，记为问题(R_1). 这两种情况在应用中特别重要，虽然也不难类似地考虑更一般的情况. 当然也可考虑在 a 或 b 附近有界的解，但方法也是类似的，这里一概从略.

我们将采用与 [3] 中类似的一些记号. 取定 $\log G(t)$ ($t\in L_0$) 的一支，使得

$$-\frac{1}{2\pi\mathrm{i}}\log G(a)=\alpha_a+\mathrm{i}\beta_a,\quad -1<\alpha_a\leqslant 0,\tag{1.3}$$

并设

$$\frac{1}{2\pi\mathrm{i}}\log G(b)=\alpha_b+\mathrm{i}\beta_b;\tag{1.4}$$

又取 κ 为一整数，使

$$-1<\alpha_b-\kappa\leqslant 0.\tag{1.5}$$

κ 就是问题(1.1) 的指标. 此外，我们记

$$G_*=\frac{1}{2\pi\mathrm{i}}\int_{L_0}\log G(t)\,\mathrm{d}t,\tag{1.6}$$

$$\gamma(z)=\frac{1}{2\pi\mathrm{i}}\int_{L_0}\log G(t)\cdot\zeta(t-z)\mathrm{d}t,\quad z\overline{\in} L.\tag{1.7}$$

这里 ζ 函数以及以下有关椭圆函数的记号都与[4] 中相同. 易证，在端点 c ($c=a$ 或 b) 附

近，有

$$e^{\gamma(z)} = (z-c)^{\alpha_c+i\beta_c}\Omega_c(z), \tag{1.8}$$

其中 $\Omega_c(z)$ 在 $z=c$ 除去 L_0 后的邻域内全纯，且 $\Omega_c(c)$ 有不等于零的定值；于是在 $z=c$ 附近

$$|e^{\gamma(z)}| \leqslant |z-c|^{\alpha_c} M \quad (M>0). \tag{1.9}$$

还应注意

$$e^{\gamma(z+2\omega_j)} = e^{-2\eta_j G_*}\cdot e^{\gamma(z)}, \quad j=1,2, \tag{1.10}$$

其中 $\eta_j=\zeta(\omega_j)$，所以 $e^{\gamma(z)}$ 一般为准周期的；当且仅当下式成立时它才是双周期的：

$$\eta_j G_* = k_j\pi i, \quad (j=1,2), \tag{1.11}$$

这里 k_j 是整数. 如再利用熟知的关系式

$$\eta_1\omega_2-\eta_2\omega_1=\frac{1}{2}\pi i,$$

则容易证明下一事实：

引理 1 当且仅当

$$\eta_1/\eta_2=k_1/k_2, \quad G_*=2k_1\omega_2-2k_2\omega_1$$

时(1.11)成立，亦即 $e^{\gamma(z)}$ 为双周期的；又若 $G_*=2l_1\omega_1+2l_2\omega_2$，而 $\eta_jG_*=k_j\pi i$，则必有

$$l_1=-k_2, \quad l_2=k_1.$$

以后我们要遇到如下的积分

$$\Psi(z)=\int_{L_0}(t-c)^r\varphi(t)\zeta(t-z)dt, \quad z\overline{\in} L, \tag{1.12}$$

其中 $\varphi(t)\in H$，$c=a$ 或 b，r 为一正整数. 我们有

引理 2 由(1.12)定义的 $\Psi(z)$，当且仅当

$$\int_{L_0}(t-c)^r\varphi(t)\zeta^{(s)}(t-c)dt=0, \quad s=0,1,\cdots,r-1 \tag{1.13}$$

成立时，$\Psi(z)$ 在 $z=c$ 处至少有几乎 r 阶的零点（即在 $z=c$ 附近，$\Psi(z)=(z-c)^{r-\varepsilon}\psi(z)$，其中 $\varepsilon>0$ 可任意小，$\psi(z)$ 有界）.

证 将 $\Psi(z)$ 改写为

$$\Psi(z)=(z-c)^r\int_{L_0}\varphi(t)\zeta(t-z)dt+\Psi_0(z), \tag{1.14}$$

其中

$$\Psi_0(z)=\sum_{k=0}^{r-1}C_k^r(z-c)^k\int_{L_0}(t-z)^{r-k}\varphi(t)\zeta(t-z)dt$$

已在 $z=c$ 处全纯. 我们可以直接验证

$$\Psi_0^{(s)}(c)=(-1)^s\int_{L_0}(t-c)^r\varphi(t)\zeta^{(s)}(t-c)dt. \tag{1.15}$$

这易于做到，因为

$$\Psi_0^{(s)}(c)=\sum_{k=0}^{s}C_k^r\left\{C_k^s k!\int_{L_0}\frac{\partial^{s-k}}{\partial z^{s-k}}[(t-z)^{r-k}\zeta(t-z)]_{z=c}\varphi(t)dt\right\}.$$

如果把上式中导数用Leibnitz公式展开，并交换求和次序，然后经化简就可得到(1.15)式.

注意(1.14)右端第一项中的积分在 $z=c$ 处有对数奇异性，因此这个项在 $z=c$ 处有几

乎 r 阶的零点，而当且仅当(1.13) 成立时，$\Psi_0(z)$ 在 $z=c$ 处有 r 阶零点，因此引理证毕.

由(1.15) 知，类似地还有

引理 3 函数

$$\Phi(z)=\int_{L_0}(t-c)^r\varphi(t)[\zeta(t-z)+\zeta(z)]\mathrm{d}t$$

在 $z=c$ 处有几乎 r 阶零点当且仅当

$$\int_{L_0}(t-c)^r\varphi(t)\zeta^{(s)}(t-c)\mathrm{d}t=(-1)^{s+1}\int_{L_0}(t-c)^r\varphi(t)\mathrm{d}t\cdot\zeta^{(s)}(c),$$

$$s=0,1,\cdots,r-1.$$

注意，如果 $c\in L_0$ 不是端点，则上二引理中的"几乎"二字均可去掉.

以后我们还要用到下面一引理：

引理 4 如果 $\Phi(z)$ 在 S 内双周期解析，在基本胞腔中只在 $z=0$ 处可能有单极点，且 $\Phi^{\pm}(t)\in H^*$，因而 $\Phi(z)$ 在 $z=a,b$ 附近可有不到一阶的奇异性(见[3])，则

$$\frac{1}{\pi\mathrm{i}}\int_{L_0}[\Phi^+(\tau)-\Phi^-(\tau)][\zeta(\tau-t)+\zeta(t)]\mathrm{d}\tau$$

$$=\Phi^+(t)+\Phi^-(t)-\frac{2\eta_1}{\pi\mathrm{i}}\int_{\gamma_1}\Phi(\tau)\mathrm{d}\tau+\frac{2\eta_2}{\pi\mathrm{i}}\int_{\gamma_2}\Phi(\tau)\mathrm{d}\tau,\quad t\in L,\tag{1.16}$$

其中 γ_1 为基本胞腔边界上 $\omega_1-\omega_2$ 到 $\omega_1+\omega_2$ 的线段，γ_2 为 $-\omega_1+\omega_2$ 到 $\omega_1+\omega_2$ 的线段.

证 设 $t\in L_0$. 记基本胞腔的边界为 Γ_0，并取定反时针方向为正向. 以下限定 $z\in S_0$. 令

$$\Psi(z)=\frac{1}{2\pi\mathrm{i}}\int_{L_0}[\Phi^+(\tau)-\Phi^-(\tau)][\zeta(\tau-z)+\zeta(z)]\mathrm{d}\tau,\quad z\overline{\in}L_0,$$

则(1.16) 左端就是 $\Psi^+(t)+\Psi^-(t)$. 另一方面，有

$$\frac{1}{2\pi\mathrm{i}}\Big(\int_{L_0^+}-\int_{L_0^-}+\int_{\Gamma_0^+}\Big)\Phi(\tau)[\zeta(\tau-z)+\zeta(z)]\mathrm{d}\tau=\Phi(z),\quad z\in S_0.$$

于是

$$\Phi(z)=\Psi(z)+\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0^+}\Phi(\tau)[\zeta(\tau-z)+\zeta(z)]\mathrm{d}\tau,\quad z\in S_0.$$

利用 $\Phi(\tau)$ 的双周期性，立刻可证明

$$\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0^+}\Phi(\tau)\mathrm{d}\tau=0,\quad \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0^+}\Phi(\tau)[\zeta(\tau-z)+\zeta(\tau)]\mathrm{d}t=0.$$

因此

$$\Phi(z)=\Psi(z)+\frac{1}{\pi\mathrm{i}}\int_{\Gamma_0^+}\Phi(\tau)\zeta(\tau)\mathrm{d}\tau,\quad z\in S_0.$$

于是

$$\Psi^+(t)+\Psi^-(t)=\Phi^+(t)+\Phi^-(t)-\frac{1}{\pi\mathrm{i}}\int_{\Gamma_0^+}\Phi(\tau)\zeta(\tau)\mathrm{d}\tau,$$

亦即

$$\frac{1}{\pi\mathrm{i}}\int_{L_0}[\Phi^+(\tau)-\Phi^-(\tau)][\zeta(\tau-t)+\zeta(t)]\mathrm{d}\tau$$

$$=\Phi^+(\tau)+\Phi^-(\tau)-\frac{1}{\pi\mathrm{i}}\int_{\Gamma_0^+}\Phi(\tau)\zeta(\tau)\mathrm{d}\tau.\tag{1.17}$$

再利用$\zeta(\tau)$的性质

$$\zeta(\tau+2\omega_j)=\zeta(\tau)+2\eta_j \quad (j=1,2),$$

立刻可得(1.16).

对于一般的$t\in L$，由于(1.16)式两端都是t的双周期函数，故它仍成立.

推论　设$\Phi(z)$同引理4，则

$$\Phi^+(t)+\Phi^-(t)=\frac{1}{\pi\mathrm{i}}\int_{L_0}[\Phi^+(\tau)-\Phi^-(\tau)][\zeta(\tau-t)+\zeta(t)]\mathrm{d}\tau$$

当且仅当

$$\int_{\Gamma_0^+}\Phi(\tau)\zeta(\tau)\mathrm{d}\tau=0 \tag{1.18}$$

或即

$$\eta_1\int_{\gamma_1}\Phi(\tau)\mathrm{d}\tau=\eta_2\int_{\gamma_2}\Phi(\tau)\mathrm{d}\tau \tag{1.19}$$

满足时成立.

注意，引理4及其推论中，如果$\Phi(z)$在基本胞腔中无极点，则(1.16)与(1.17)左端可改写为

$$\frac{1}{\pi\mathrm{i}}\int_{L_0}[\Phi^+(\tau)-\Phi^-(\tau)]\zeta(\tau-t)\mathrm{d}\tau.$$

二、齐次问题

先求解相应于(1.1)的齐次问题

$$\Phi^+(t)=G(t)\,\Phi^-(t). \tag{2.1}$$

(一) 设(1.11)成立，即$\mathrm{e}^{\gamma(z)}$已是双周期的. 由引理1，这时必然

$$G_*\equiv 0 \pmod{2\omega_1,2\omega_2}. \tag{2.2}$$

若$\Phi(z)$为(2.1)的解，则$\Phi(z)\mathrm{e}^{-\gamma(z)}$不再以$L$为跳跃曲线. 注意$\mathrm{e}^{\gamma(z)}$在$z=a$处有不到一阶的奇异性，而在$z=b$处有$(z-b)^{-\alpha_b}$的因子. 分下列情况讨论：

1°　设$\kappa=0$. 这时$\Phi(z)\mathrm{e}^{-\gamma(z)}$在$a,b$处都不会有一阶奇异性. 由于不存在非退化的一阶椭圆函数，所以(2.1)不论在(R_0)或(R_1)都有一般解

$$\Phi(z)=C\mathrm{e}^{\gamma(z)} \tag{2.3}$$

(C以及以后带附标的C都表示任意常数).

2°　设$\kappa=1$. 对于(R_0)，(2.3)仍是(2.1)的一般解；对于(R_1)，由于$0<\alpha_b\leqslant 1$，所以$\Phi(z)\mathrm{e}^{-\gamma(z)}$可以在$z=0$与$z=b$处有单极点，因此这时问题(2.1)的一般解为

$$\Phi(z)=\mathrm{e}^{\gamma(z)}[C_0+C_1\zeta(z)-C_1\zeta(z-b)]. \tag{2.4}$$

3°　设$\kappa\geqslant 2$. 这时$\Phi(z)\mathrm{e}^{-\gamma(z)}$在$z=b$处可以有$\kappa$阶奇异性，所以(2.1)在$(\mathrm{R}_0)$中有一般解

$$\Phi(z)=\mathrm{e}^{\gamma(z)}[C_0+C_1\zeta(z-b)+\cdots+C_{\kappa-1}\zeta^{(\kappa-1)}(z-b)]; \tag{2.5}$$

而在(R_1)中有一般解

$$\Phi(z)=\mathrm{e}^{\gamma(z)}[C_0-C_1\zeta(z)+C_1\zeta(z-b)+C_2\zeta'(z-b)+\cdots+C_\kappa\zeta^{(\kappa-1)}(z-b)]. \tag{2.6}$$

4° 设 $\kappa<0$. 这时 $\Phi(z)e^{-\gamma(z)}$ 至多只有一个单极点，且以 $z=b$ 为零点，故 (2.1) 在 (R_0) 或 (R_1) 中都只有零解.

（二）设 (1.11) 不成立. 由引理 1 这时 $G_*\neq 0$.

1° 设 $\kappa=0$. 我们令

$$h_*(z)=\sigma(z)/\sigma(z-G_*),\quad X_*(z)=h_*(z)e^{\gamma(z)},\tag{2.7}$$

后者就可起典则函数的作用，只是它在 $z=a,b$ 处都有不到一阶的奇异性，又分两种情况.

(i) 设 $G_*\equiv 0$（以 $2\omega_1,2\omega_2$ 为模的同余式，下同）. 这时 $h_*(z)$ 是一整函数，形为（除一非零常数因子外）

$$h_*(z)=\exp\{2(l_1\eta_1+l_2\eta_2)z\},\tag{2.8}$$

其中 l_1,l_2 由 $G_*=2l_1\omega_1+2l_2\omega_2$ 决定. 因此问题 (2.1) 不论在 (R_0) 或 (R_1) 中都有一般解

$$\Phi(z)=Ch_*(z)e^{\gamma(z)}.\tag{2.9}$$

(ii) 设 $G_*\not\equiv 0$. 先考虑 (R_1). 这时代替 (2.7) 要用

$$\tilde{h}_*(z)=\sigma(z+G_*)/\sigma(z),\quad \tilde{X}_*(z)=\tilde{h}_*(z)e^{\gamma(z)},\tag{2.7$'$}$$

于是 $\Phi(z)/\tilde{X}_*(z)$ 就是至多以 $-G_*$ 为一阶极点的椭圆函数，故只能是常数. 所以 (2.1) 在 (R_1) 中有一般解

$$\Phi(z)=Ce^{\gamma(z)}\sigma(z+G_*)/\sigma(z).\tag{2.10}$$

(2.1) 在 (R_0) 中的解必在 (R_1) 中，所以由 (2.10) 立刻知道在 (R_0) 中只有零解.

2° 设 $\kappa\neq 0$. 这时我们应该令

$$h_d(z)=\sigma(z-d)/\sigma(z-b),\quad d=b-G_*/\kappa,\tag{2.11}$$

以保证 $X_d(z)=h_d(z)^{\kappa}e^{\gamma(z)}$ 能起典则函数的作用.

(i) 设 $d\not\equiv b$ 即 $G_*/\kappa\not\equiv 0$. 又分几种情况：

1) 设 $\kappa=1$. 如果在 (R_1) 中求解，则当 $d\not\equiv 0$ 时，因 $\Phi(z)/X_d(z)$ 在 $z=0$ 与 d 处都可以有一阶极点，因此 (2.1) 的一般解为

$$\Phi(z)=e^{\gamma(z)}[C_0+C_1\zeta(z)-C_1\zeta(z-d)]\sigma(z-d)/\sigma(z-b),\tag{2.12}$$

而当 $d\equiv 0$ 时易见一般解为

$$\Phi(z)=e^{\gamma(z)}[C_0+C_1\zeta'(z)]\sigma(z-d)/\sigma(z-b).\tag{2.12$'$}$$

在 (R_0) 中，不论 $d\equiv 0$ 或否，显然 (2.1) 的一般解为

$$\Phi(z)=Ce^{\gamma(z)}\sigma(z-d)/\sigma(z-b).\tag{2.13}$$

2) 设 $\kappa\geqslant 2$. 易见，(2.1) 在 (R_1) 中的一般解，当 $d\not\equiv 0$ 时为

$$\begin{aligned}\Phi(z)=e^{\gamma(z)}[&C_0-C_1\zeta(z)+C_1\zeta(z-d)+C_2\zeta'(z-d)\\&+\cdots+C_\kappa\zeta^{(\kappa-1)}(z-d)]\sigma^{\kappa}(z-d)/\sigma^{\kappa}(z-b),\end{aligned}\tag{2.14}$$

而当 $d\equiv 0$ 时为

$$\Phi(z)=e^{\gamma(z)}[C_0+C_1\zeta'(z)+\cdots+C_\kappa\zeta^{(\kappa)}(z)]\sigma^{\kappa}(z-d)/\sigma^{\kappa}(z-b);\tag{2.14$'$}$$

在 (R_0) 中的一般解，不论 $d\equiv 0$ 或否，为

$$\Phi(z)=e^{\gamma(z)}[C_0+C_1\zeta'(z-d)+\cdots+C_{\kappa-1}\zeta^{(\kappa-1)}(z-d)]\sigma^{\kappa}(z-d)/\sigma^{\kappa}(z-b).\tag{2.15}$$

3) 设 $\kappa<0$. 这时 (2.1) 在 (R_0) 或 (R_1) 中都只有零解.

注意，(2.12)，(2.14)，(2.15) 诸式方括号中的 d 均可改为它在基本胞腔中的合同值 d_0.

(ii) 设 $d=b$ 即 $G_*/\kappa\equiv 0$. 此时必然 $d\neq a$，且由(2.11)定义的 $h_d(z)$ 又是 z 的指数函数，所以 $X_d(z)$ 在 $z=a,b$ 处的形状同 $e^{\gamma(z)}$. 因此(i)中所述结果现在完全成立，且 $l_0=b$.

这样，问题(2.1)已完全解决. 概括如下：

定理1　齐次问题(2.1)在(R_0)中当 $\kappa\geqslant 1$ 时恒有 κ 个线性无关的解，当 $\kappa<0$ 时只有零解，当 $\kappa=0$ 时如果 $G_*\not\equiv 0$ 则只有零解，而如 $G_*\equiv 0$ 则有一个非零解；在(R_1)中当 $\kappa\geqslant 0$ 时恒有 $\kappa+1$ 个解，当 $\kappa<0$ 时只有零解.

三、非齐次问题

现在讨论一般问题(1.1). 因为相应的齐次问题已经解决，所以只要弄清楚它的可解条件并求出一个特解即可. 仍分两种情况讨论.

(一) 设(1.11)满足. 如前所述，$e^{\gamma(z)}$ 已是双周期的.

1° 设 $\kappa=0$. 这时由于 $g(t)e^{-\gamma^+(t)}$ 在 $t=a,b$ 处有界，所以显然

$$\Psi(z)=\frac{e^{\gamma(z)}}{2\pi i}\int_{L_0}g(t)e^{-\gamma(t)}[\zeta(t-z)+\zeta(z)]dt \tag{3.1}$$

就是(1.1)在(R_1)中的一个特解. 如果要在(R_0)中求解，则当且仅当

$$\int_{L_0}g(t)e^{-\gamma^+(t)}dt=0 \tag{3.2}$$

成立时(1.1)才有解，且其一特解为

$$\Psi(z)=\frac{e^{\gamma(z)}}{2\pi i}\int_{L_0}g(t)e^{-\gamma^+(t)}\zeta(t-z)dt. \tag{3.3}$$

2° 设 $\kappa\geqslant 1$. 这时 $e^{\gamma(z)}/\sigma^\kappa(z-b)$ 在 $z=b$ 处有不到一阶的奇异性. 为了构作一个特解，例如，可以考虑以 $t\in L$ 为参数的 z 的椭圆函数

$$\frac{\sigma^\kappa(z)\sigma(t-z+\kappa b)}{\sigma(t-z)\sigma^\kappa(z-b)}\cdot\frac{\sigma^\kappa(t-b)}{\sigma^\kappa(t)\sigma(\kappa b)}$$

(不失一般性，我们已假定 $\kappa b\neq 0$)，它在 $z=t$ 处有留数 -1，故仍保有Cauchy核的性质. 可以直接验证

$$\Psi(z)=\frac{\sigma^\kappa(z)e^{\gamma(z)}}{\sigma(\kappa b)\sigma^\kappa(z-b)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{\sigma^\kappa(t-b)}{\sigma^\kappa(t)e^{\gamma^+(t)}}\cdot\frac{\sigma(t-z+\kappa b)}{\sigma(t-z)}g(t)dt \tag{3.4}$$

就是问题(1.1)在(R_0)或(R_1)中的一个特解.

3° 设 $\kappa<0$. 由于 $\alpha_b\leqslant\kappa<-1$，在($R_1$)中，$\Phi(z)e^{-\gamma(z)}$ 至多只在 $z=0$ 处有一阶极点，故若(1.1)有解，必定唯一，且只能是(3.1)之形. 为要它在 $z=b$ 处有不到一阶奇异性，就应要求(3.1)中的积分在 $z=b$ 处至少有几乎 $-\kappa$ 阶零点，但因 $e^{-\gamma^+(t)}$ 在 $t=b$ 处已知有 $-\kappa$ 阶零点，故由引理3，必须且只需

$$\int_{L_0}g(t)e^{-\gamma^+(t)}\zeta^{(s)}(t-b)dt=(-1)^{s+1}\int_{L_0}g(t)e^{-\gamma^+(t)}dt\cdot\zeta^{(s)}(b),$$

$$s=0,1,\cdots,-\kappa-1. \tag{3.5}$$

此即(1.1)在(R_1)中的可解条件，当它满足时，有唯一解(3.1).

如果在(R_0)中求解，则可解条件就是(3.2)以及

$$\int_{L_0} g(t)e^{-\gamma^+(t)}\zeta^{(s)}(t-b)dt=0,\quad s=0,1,\cdots,-\kappa-1. \tag{3.5'}$$

当它们满足时，(1.1) 有唯一解(3.3).

(二) 设(1.11) 不满足. 记住这时 $G_* \neq 0$.

1° 设 $\kappa=0$.

(i) 设 $G_* \equiv 0$. 这时(1.1) 在(R_1) 中有特解

$$\Psi(z)=\frac{h_*(z)e^{\gamma(z)}}{2\pi i}\int_{L_0}\frac{g(t)}{h_*(t)e^{\gamma^+(t)}}[\zeta(t-z)+\zeta(z)]dt, \tag{3.6}$$

其中 $h_*(z)$ 由(2.8) 给出. 如果在(R_0) 中求解，则

$$\int_{L_0}\frac{g(t)dt}{h_*(t)e^{\gamma^+(t)}}=0 \tag{3.7}$$

为可解条件，当它满足时，(1.1) 有唯一解

$$\Phi(z)=\frac{h_*(z)e^{\gamma(z)}}{2\pi i}\int_{L_0}\frac{g(t)}{h_*(t)e^{\gamma^+(t)}}\zeta(t-z)dt. \tag{3.8}$$

(ii) 设 $G_* \not\equiv 0$. 这时(1.1) 在(R_0) 中无条件可解，且显然有唯一解

$$\Phi(z)=\frac{h_*(z)e^{\gamma(z)}}{2\pi i}\int_{L_0}\frac{g(t)}{h_*(t)e^{\gamma^+(t)}}[\zeta(t-z)+\zeta(z)-\zeta(t-G_*)-\zeta(G_*)]dt, \tag{3.9}$$

其中 $h_*(z)$ 由(2.7) 给出. 也容易把它改写为

$$\Phi(z)=\frac{e^{\gamma(z)}}{\sigma(G_*)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{g(t)}{e^{\gamma^+(t)}}\cdot\frac{\sigma(z-t+G_*)}{\sigma(t-z)}dt. \tag{3.9'}$$

(3.9) 或 (3.9′) 也是(1.1) 在(R_1) 中的一个特解.

2° 设 $\kappa \geqslant 1$. 由于这时 $e^{\gamma(z)}\sigma^\kappa(z-d)/\sigma^\kappa(z-b)$ 已是双周期的，故有

(i) 如 $d \overline{\in} L$，则显然

$$\Psi(z)=\frac{\sigma^\kappa(z-d)e^{\gamma(z)}}{\sigma^\kappa(z-b)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{\sigma^\kappa(t-b)g(t)}{\sigma^\kappa(t-d)e^{\gamma^+(t)}}[\zeta(t-z)+\zeta(z-d)]dt \tag{3.10}$$

已是(1.1) 在(R_0) 或(R_1) 中的特解.

(ii) 如 $d \in L$，则(3.10) 中的积分当 $\kappa=1$ (且 $d \neq a$) 时在主值意义下仍可用；但当 $\kappa \geqslant 2$ 时它已发散.

为了避免这一困难，可改用下列作法. 引进在 $z=t$ 处留数为 -1 的一个 $\kappa+1$ 阶椭圆函数. 例如，当 $\kappa d \not\equiv 0$ 时，可用

$$\frac{\sigma^\kappa(z)\sigma(z-t-\kappa d)}{\sigma(t-z)\sigma^\kappa(z-d)}\cdot\frac{\sigma^\kappa(t-d)}{\sigma^\kappa(t)\sigma(-\kappa d)}, \tag{3.11}$$

立刻就可知道，(1.1) 在(R_0) 或(R_1) 中有一特解

$$\Psi(z)=\frac{\sigma^\kappa(z)e^{\gamma(z)}}{\sigma(\kappa d)\sigma^\kappa(z-b)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{\sigma^\kappa(t-b)g(t)}{\sigma^\kappa(t)e^{\gamma^+(t)}}\cdot\frac{\sigma(t-z+\kappa d)}{\sigma(t-z)}dt. \tag{3.12}$$

而如 $\kappa d \equiv 0$，则不用(3.11) 而任取一点 $c \overline{\in} L$，且 $\kappa c \not\equiv 0$，并改用

$$\frac{\sigma^\kappa(z-c)\sigma(z-t+\kappa(c-d))}{\sigma(t-z)\sigma^\kappa(z-d)}\cdot\frac{\sigma^\kappa(t-d)}{\sigma^\kappa(t-c)\sigma(\kappa(c-d))}$$

代替(3.12)，有特解

$$\Psi(z)=\frac{\sigma^{\kappa}(z-c)\mathrm{e}^{\gamma(z)}}{\sigma(\kappa(c-d))\sigma^{\kappa}(z-b)}\frac{1}{2\pi\mathrm{i}}\int_{L_0}\frac{\sigma^{\kappa}(t-b)g(t)}{\sigma^{\kappa}(t-c)\mathrm{e}^{\gamma^+(t)}}\cdot\frac{\sigma(z-t+\kappa(c-d))}{\sigma(t-z)}\mathrm{d}t. \quad (3.13)$$

此外，我们注意，如果 $g(t)$ 有属于 H 的 $\kappa-1$ 阶导数，则采用[5,6]中关于积分的推广，(3.10) 式仍可用作(1.1) 在(R_0) 或(R_1) 中的特解.

3° 设 $\kappa<0$. 先假定 $d\not\equiv b$ 即 $G_*/\kappa\not\equiv 0$. 这时显然如果(1.1) 在(R_1) 中有解，则必唯一，且必为形如

$$\Phi(z)=\frac{\sigma^{-\kappa}(z-b)\mathrm{e}^{\gamma(z)}}{\sigma^{-\kappa}(z-d)}\cdot\frac{1}{2\pi\mathrm{i}}\int_{L_0}\frac{\sigma^{-\kappa}(t-b)g(t)}{\sigma^{-\kappa}(t-d)\mathrm{e}^{\gamma^+(t)}}[\zeta(t-z)+\zeta(z)]\mathrm{d}t. \quad (3.14)$$

但它一般并不在(R_1) 中，因为它在 $z=d$ 处可能有 $-\kappa$ 阶奇异性(当 $d\not\equiv a$ 时) 或者不到 $-\kappa+1$ 阶的奇异性(当 $d\equiv a$ 时). 为要使它在 $z=d$ 处有界(当 $d\not\equiv a$ 时) 或有不到一阶的奇异性(当 $d\equiv a$ 时)，须且必须(3.14) 右端的积分在 $z=d$ 处直到 $-\kappa-1$ 阶的导数为零. 由引理 3，这就要求

$$\int_{L_0}\frac{\sigma^{-\kappa}(t-d)g(t)}{\sigma^{-\kappa}(t-b)\mathrm{e}^{\gamma^+(t)}}\zeta^{(s)}(t-d)\mathrm{d}t=(-1)^{s+1}\int_{L_0}\frac{\sigma^{-\kappa}(t-d)g(t)}{\sigma^{-\kappa}(t-b)\mathrm{e}^{\gamma^+(t)}}\mathrm{d}t\cdot\zeta^{(s)}(d),$$
$$s=0,1,\cdots,-\kappa-1. \quad (3.15)$$

当这些条件满足时，(3.14) 就是(1.1) 在(R_1) 中的唯一解. 如果在(R_0) 中求解，可解条件应是

$$\int_{L_0}\frac{\sigma^{-\kappa}(t-d)g(t)}{\sigma^{-\kappa}(t-b)\mathrm{e}^{\gamma^+(t)}}\mathrm{d}t=0, \quad (3.16)$$

$$\int_{L_0}\frac{\sigma^{-\kappa}(t-d)g(t)}{\sigma^{-\kappa}(t-b)\mathrm{e}^{\gamma^+(t)}}\zeta^{(s)}(t-d)\mathrm{d}t=0,\quad s=0,1,\cdots,-\kappa-1. \quad (3.17)$$

当它们满足时，(1.1) 在(R_0) 中有唯一解

$$\Phi(z)=\frac{\sigma^{-\kappa}(z-b)\mathrm{e}^{\gamma(z)}}{\sigma^{-\kappa}(z-d)}\cdot\frac{1}{2\pi\mathrm{i}}\int_{L_0}\frac{\sigma^{-\kappa}(t-d)g(t)}{\sigma^{-\kappa}(t-b)\mathrm{e}^{\gamma^+(t)}}\zeta(t-z)\mathrm{d}t. \quad (3.18)$$

再设 $d\equiv b$ 即 $G_*/\kappa\equiv 0$. 这时 $\sigma^{-\kappa}(z-b)/\sigma^{-\kappa}(z-d)$ 是 z 的指数函数，而 $\mathrm{e}^{-\gamma^+(t)}$ 在 $t=b$ 处有 $-\kappa$ 阶零点，因此以上的论述和结果仍成立.

综合以上结果，我们得到：

定理 2 对于非齐次问题(1.1)，如果在(R_0) 中求解，则当 $\kappa\geqslant 1$ 时无条件可解；当 $\kappa\leqslant 0$ 时有 $-\kappa+1$ 个可解条件；当 $\kappa=0$ 时，如 $G_*\not\equiv 0$ 则无条件可解，如 $G_*\equiv 0$ 则有一个可解条件. 如果在(R_1) 中求解，则当 $\kappa\geqslant 0$ 时无条件可解，当 $\kappa<0$ 时有 $-\kappa$ 个可解条件.

前面设 L_0 全在基本胞腔内. 今讨论 L_0 的两端在基本胞腔边界上的二合同点处而其余部分在其内. 不失一般性，可以认为 $b=a+2\omega_1$. 为简明起见，设 L_0 在 a 与 b 处有平行的切线. 当 L_0 以周期 $2\omega_1$ 延拓时，成一无限延伸的光滑曲线 L_0^*，所以 L 现在是由无限条互相平行的曲线 L_n^* $(n=0,\pm 1,\pm 2,\cdots)$ 构成，这里 L_n^* 是 L_0^* 经平移 $2n\omega_2$ 后形成的. 求解(1.1) 时，由于 a 点在 L_0^* 上与别的点处于同等地位，我们要求 $\Phi(z)$ 在 $z=a$ 处也有界. 在 L_0 上任取 $\log G(t)$ 的一连续支(例如不妨认定 $0\leqslant\alpha<1$)，则有

$$\frac{1}{2\pi\mathrm{i}}\log G(a)=\alpha+\mathrm{i}\beta,\quad \frac{1}{2\pi\mathrm{i}}\log G(a+2\omega_1)=\kappa+\alpha+\mathrm{i}\beta.$$

仍定义 $G_*,\gamma(z)$ 如(1.6),(1.7).

于是以上所讨论的一切方法与结果也都成立. 还可注意，现在 $d = a - G_*/\kappa + 2\omega_1$，而且所有结果中的 b 与 d 一律可分别改为 a 与 $a - G_*/\kappa$.

四、应用于求解奇异积分方程

1° 我们应用上述结果来求解奇异积分方程

$$A(t)\varphi(t) + \frac{B(t)}{\pi \mathrm{i}}\int_{L_0}\varphi(\tau)[\zeta(\tau - t) + \zeta(t)]\mathrm{d}\tau = f(t), \quad t \in L_0, \tag{4.1}$$

其中 $A,B,f \in H$，且 $A^2 - B^2 \neq 0$，并允许 $\varphi(t)$ 在 a,b 处可有不到一阶的奇异性.

像通常那样，令

$$\Phi(z) = \frac{1}{2\pi \mathrm{i}}\int_{L_0}\varphi(\tau)[\zeta(\tau - z) + \zeta(z)]\mathrm{d}\tau, \quad z \in L, \tag{4.2}$$

则(4.1) 成为在(R_1) 中求(1.1) 的解，其中

$$G = (A - B)/(A + B), \quad g = f/(A + B), \tag{4.3}$$

且已作了双周期延拓.

如果(4.1) 有解，则由(4.2) 定义的 $\Phi(z)$ 必为问题(1.1) 在(R_1) 中的解；反之，如 $\Phi(z)$ 是这样的解，则由下式求出的

$$\varphi(t) = \Phi^+(t) - \Phi^-(t) \tag{4.4}$$

要想是(4.1) 的解，需且必须

$$A(t)[\Phi^+(t)-\Phi^-(t)]+\frac{B(t)}{\pi \mathrm{i}}\int_{L_0}[\Phi^+(\tau)-\Phi^-(\tau)][\zeta(\tau-t)+\zeta(t)]\mathrm{d}\tau=f(t), \quad t\in L_0,$$

但这等价于

$$\frac{1}{\pi \mathrm{i}}\int_{L_0}[\Phi^+(\tau)-\Phi^-(\tau)][\zeta(\tau-t)+\zeta(t)]\mathrm{d}\tau=\Phi^+(t)+\Phi^-(t), \quad t\in L_0.$$

而由引理 4 的推论，这又等价于(1.18) 或(1.19) 成立. 因此，我们得到

定理 3 *方程(4.1) 等价于在(R_1) 中求解(1.1) 以及附加条件(1.18) 或(1.19)；当附加条件满足时，解由(4.4) 给出.*

同样地可考虑 L_0 是(三) 中末段时的情况来求解方程(4.1). 容易验证，现在引理 4 及其推论仍成立，故其求解问题与上面全同，即定理 3 也成立(但要求 $\varphi(t)$ 在 $t = a$ 处从而 $\Phi(z)$ 在 $z = a$ 处有界). 但应注意，如果 L_0 是基本胞腔从 $-\omega_1 - \omega_2$ 到 $\omega_1 - \omega_2$ 的边，则条件(1.16) 与(1.19) 里 $\int_{\gamma_1}\Phi(t)\mathrm{d}t$ 中的 $\Phi(t)$ 要理解为 $\Phi^-(t)$. 而要用(1.18) 时，$\int_{\Gamma_0^+}\Phi(t)\zeta(t)\mathrm{d}t$ 中的 $\Phi(t)$ 不论 t 在 L_0 或 γ_1 上，都应理解为 $\Phi^-(t)$.

2° 也可考虑另一类奇异积分方程

$$A(t)\varphi(t) + \frac{B(t)}{\pi \mathrm{i}}\int_{L_0}\varphi(\tau)\zeta(\tau - t)\mathrm{d}\tau = f(t), \quad t \in L_0. \tag{4.5}$$

它可改写为

$$A(t)\varphi(t)+\frac{B(t)}{\pi \mathrm{i}}\int_{L_0}\varphi(\tau)[\zeta(\tau-t)+\zeta(t)]\mathrm{d}\tau=f(t)+\lambda B(t)\zeta(t), \quad t\in L_0, \tag{4.6}$$

其中

$$\lambda = \frac{1}{\pi i}\int_{L_0} \varphi(\tau) d\tau. \tag{4.7}$$

显然，如果$\varphi(t)$是(4.5)的解，则由(4.7)求出λ后，$\varphi(t)$必为(4.6)的解. 反之，如果(4.6)可解，其中λ待定（如果有解的话，解中当然含有λ），而如又能选取λ使满足(4.7)，便可求得(4.5)的解. 于是我们有

定理 4 *方程(4.5)等价于方程(4.6)以及附加条件(4.7).*

当L_0是(三)中末段的情况时，关于方程(4.5)的定理4仍成立.

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ON DOUBLY PERIODIC RIEMANN BOUNDARY VALUE PROBLEMS ALONG OPEN ARCS

Abstract

In this paper, we consider the doubly periodic Riemann boundary value problem (1.1) along a set L of smooth open arcs, any two of which are congruent to each other with respect to the two periods. In (1.1), $G(t)\neq 0$ and $g(t)$ are given functions on L, continuous in Hölder sense and doubly periodic, and $\Phi^{\pm}(t)$ are the boundary values of the unknown doubly periodic analytic function $\Phi(z)$ along the different sides of L. Such problems are solved effectively so that both the solutions and the conditions of solvability are obtained in explicit form. The results are then applied to solving certain classes of singular integral equations like (4.1) and (4.5) with kernels involving Weierstrass ζ functions. The case in which the two ends of each arc are congruent is also considered and similarly solved.

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双准周期 Riemann 边值问题

对于双周期的 Riemann 边值问题

$$\Phi^+(t) = G(t)\,\Phi^-(t) + g(t), \quad t \in L, \tag{0.1}$$

其中 L 是基本胞腔 S_0 中某一光滑曲线 L_0 及其所有周期合同曲线的并，不论 L_0 是封闭或开口的情况，都已有较完善的研究[1~3]. 本文将讨论双准周期的类似问题. 这种问题在实际应用中也是很有用的，因为在许多周期现象中，某些量是准周期的；例如周期应力的平面弹性问题，其位移一般就是准周期的[4,5]. 在讨论这个问题之前，我们对双准周期解析函数的一些性质作些说明.

§1 双准周期解析函数

取定基本周期为 $2\omega_1, 2\omega_2$，其中 $\mathrm{Im}(\omega_2/\omega_1) > 0$（有关椭圆函数论中的记号，我们采用[6]）. 下面给出一些名称. 复平面中单值解析函数（可以有孤立奇点）$f(z)$，如满足

$$f(z + 2\omega_j) = f(z) + \alpha_j, \quad j = 1,2, \tag{1.1}$$

则称它是加法双准周期的(解析）函数，而 α_1, α_2 称为它的加数；如果此外奇点只可能是极点，则称它是加法准椭圆函数. 同样，如果把(1.1) 改为

$$f(z + 2\omega_j) = \beta_j f(z), \quad \beta_j \neq 0, \quad j = 1,2, \tag{1.2}$$

则根据情况，称 $f(z)$ 为乘法双准周期的或乘法准椭圆的，而 β_1, β_2 称为其乘数. 对于准椭圆函数(加法的或乘法的)，其阶数就是它在基本胞腔 S_0(取成以 $\pm\omega_1, \pm\omega_2$ 为顶点的平行四边形) 中各极点阶数的和. W. T. Koiter[5] 曾讨论平面中挖掉一些周期合同孔后所成域中加法双准周期解析函数的某些性质. 我们将讨论全平面的双准周期解析函数的一般性质.

对于加法双准周期函数，首先我们有

引理 1 *设 $f(z)$ 是以 α_1, α_2 为加数的双准周期解析函数，以 $z = z_0$ 为其一奇点，则它必有如下形式：*

$$f(z) = \lambda z + \mu \zeta(z - z_0) + E(z), \tag{1.3}$$

其中 $E(z)$ 为一双周期解析函数，这里

$$\lambda = \frac{1}{\pi \mathrm{i}}(\alpha_2 \eta_1 - \alpha_1 \eta_2), \qquad \mu = \frac{1}{\pi \mathrm{i}}(\omega_2 \alpha_1 - \omega_1 \alpha_2), \tag{1.4}$$

而 $\eta_j = \zeta(\omega_j)$ $(j = 1,2)$；又若 $f(z)$ 为 k 阶的加法准椭圆函数，则 $E(z)$ 是 k 阶或 $k-1$ 阶椭圆函数，且后一情况只可能出现在 z_0 为单极点时. 表示式(1.3) 唯一.

此由等式

$$\eta_1 \omega_2 - \eta_2 \omega_1 = \frac{1}{2}\pi \mathrm{i} \tag{1.5}$$

容易直接验证. 下面的推论也极明显：

推论 1　如果上述 $f(z)$ 是一阶的准椭圆函数，且在 S_0 中以 z_0 为单极点，则

$$f(z) = C + \lambda z + \mu\zeta(z - z_0) \quad (C\text{ 为常数}). \tag{1.6}$$

推论 2　如上述 $f(z)$ 没有奇点，则必有 $\alpha_2/\alpha_1 = \omega_2/\omega_1$，且

$$f(z) = C + \lambda z \quad (C\text{ 为常数}), \tag{1.7}$$

其中 $\lambda = \dfrac{\alpha_1}{2\omega_1} = \dfrac{\alpha_2}{2\omega_2}$；因此，如果 $\alpha_2/\alpha_1 \neq \omega_2/\omega_1$，则 $f(z)$ 一定有奇点.

对于乘法双准周期函数，我们有

引理 2　如果 $f(z)$ 为 n 阶乘法准椭圆函数，以 β_1,β_2 为乘数，在 S_0 中以 $z_1,\cdots,z_n$（各个 z_k 可以其重数重复出现）为极点，则必有

$$f(z) = \left[C_0 + \sum_{k=1}^{n-1} C_k\zeta^{(k)}(z - z_0)\right]\frac{\sigma^n(z - z_0)}{\prod\limits_{k=1}^{n}\sigma(z - z_k)}\mathrm{e}^{\lambda z}, \tag{1.8}$$

其中 $C_0,\cdots,C_k$ 为常数，而

$$\lambda = \frac{1}{\pi\,\mathrm{i}}(\gamma_2\eta_1 - \gamma_1\eta_2), \tag{1.9}$$

$$z_0 = \frac{1}{n\pi\,\mathrm{i}}(\gamma_2\omega_1 - \eta_1\omega_2) + \frac{1}{n}\sum_{k=1}^{n} z_k, \tag{1.10}$$

这里已记 $\gamma_j = \log\beta_j (j = 1,2)$，且对数可以各自任意取定一值.

通过直接验证，知

$$g(z) = \frac{\sigma^n(z - z_0)}{\prod\limits_{k=1}^{n}\sigma(z - z_k)}\mathrm{e}^{\lambda z}$$

为 n 阶准椭圆函数，以 β_1,β_2 为乘数，以 $z_1,\cdots,z_n$ 为极点，以 z_0 为 n 阶零点，便得此引理. 注意，总可以适当选取 γ_1,γ_2，使 $nz_0 \in S_0$，从而也有 $z_0 \in S_0$.

引理 3　设 $f(z) \not\equiv 0$ 为乘法准椭圆函数，以 β_1,β_2 为乘数，则当且仅当可以适当选择 $\gamma_j = \log\beta_j$ 能使 $\gamma_2/\gamma_1 = \omega_2/\omega_1$ 时 $f(z)$ 才没有奇点，且这时

$$f(z) = C\mathrm{e}^{\lambda z} \quad (C\text{ 为常数}),$$

其中 $\lambda = \dfrac{\gamma_1}{2\omega_1} = \dfrac{\gamma_2}{2\omega_2}$.

证　条件的充分性显然. 现证必要性.

暂时命 $\gamma_1 = \log\beta_1$ 任意取定一值，并令

$$\tau(z) = \mathrm{e}^{\gamma_1 z/2\omega_1}.$$

因此 $F(z) = f(z)/\tau(z)$ 以 $2\omega_1$ 为周期且无奇点，易见

$$F(z + 2\omega_2) = \beta_2\,\mathrm{e}^{-\gamma_1\omega_2/\omega_1}F(z) \equiv \alpha F(z).$$

如果 $\alpha = 1$，则 $F(z)$ 已是双周期的，又无奇点，故必为常数，且这时 $\beta_2 = \mathrm{e}^{\gamma_1\omega_2/\omega_1}$，故可令 $\gamma_2 = \log\beta_2 = \gamma_1\omega_2/\omega_1$，必要性已证.

现设 $\alpha \neq 1$. 写出 $F(z)$ 的 Fourier 展式：

$$F(z) = \sum_{m=-\infty}^{+\infty} A_m \exp\frac{m\pi\,\mathrm{i}}{\omega_1}z,$$

代入上式并比较系数，可知

$$(\alpha - e^{2m\pi i\omega_2/\omega_1})A_m = 0, \quad m = 0, \pm 1, \pm 2, \cdots$$

因 $\alpha \neq 1$，显然 $A_0 = 0$. 又因 $f(z) \not\equiv 0$，所以至少有某个 $A_k \neq 0$；从而必然只有这个 $A_k \neq 0$，因为 ω_2/ω_1 不可能是整实数. 于是 $F(z) = A_k e^{k\pi iz/\omega_1}$，而

$$\alpha = \beta_2 e^{-\gamma_1\omega_2/\omega_1} = e^{2k\pi i\omega_2/\omega_1},$$

故任取 $\gamma_2 = \log \beta_2$ 的一个值时，恒有

$$\gamma_2\omega_1 - \gamma_1\omega_2 \equiv 0 \quad (\mathrm{mod}\ 2\pi i\omega_1, 2\pi i\omega_2).$$

因此总可改变 γ_1, γ_2 的值使 $\gamma_2/\gamma_1 = \omega_2/\omega_1$.

推论 1 $f(z)$ 同引理(但不要求 $\not\equiv 0$). 如果 $f(z)$ 有零点，则必 $f(z) \equiv 0$.

推论 2 $f(z)$ 同引理. 如果 $f(z)$ 以 $2\omega_1$ 为周期，则必 $f(z) \equiv$ 常数.

这两推论都很明显.

如果允许双准周期解析函数可以有本性奇点，我们也可求出其一般表示式；由于本文不用，故从略，只指出下一明显的

引理4 如果 $f(z)$ 是双准周期解析的，以 β_1, β_2 为乘数，在 S_0 中只以 $z=0$ 为奇点，则必有

$$f(z) = E(z)e^{\lambda z + \mu\zeta(z)}, \tag{1.11}$$

其中 $E(z)$ 为双周期的，在 S_0 中至多以 $z=0$ 为奇点，且

$$\lambda = \frac{1}{\pi i}(\gamma_2\eta_1 - \gamma_1\eta_2), \quad \mu = \frac{1}{\pi i}(\omega_2\gamma_1 - \omega_1\gamma_2). \tag{1.12}$$

这里 $\gamma_j = \log \beta_j$ 均仍可任意取定.

注意表示式(1.11)不是唯一的，因为 γ_j 可取不同的对数值，从而 μ 就要换成周期合同的别的值. 因此，如果限定 $\mu \in S_0$，则(1.11)就是唯一的.

注 类似地可定义分区全纯(或解析)双准周期函数，以 L 为跳跃曲线. 这在下面经常要用到.

§2 加法双准周期 Riemann 边值问题

对于加法问题(0.1)，G 只可能是一常数：

$$\Phi^+(t) = G\Phi^-(t) + g(t), \quad t \in L, \tag{2.1}$$

其中已设 $G \neq 0$，$g(t) \in H$ 且为双准周期的，并以 g_1, g_2 为加数，并设要求 $\Phi(z)$ 也是加法双准周期的，并分区全纯①.

Ⅰ. 先讨论 L_0 为 S_0 中一条光滑封闭曲线的情况，并设 L_0 已取定反时针向为正向. 不失一般性，可认为 $G=1$. 如果用记号 $[z]_0$ 表示关于模 $2\omega_1, 2\omega_2$ 在 S_0 内与 z 同余的点，并记 $g([t]_0) = g_0(t)$，则 $g_0(t)$ 是双周期的. 又若令

$$\Phi_0(z) = \begin{cases} \Phi(z) - mg_1 - ng_2, & \text{当 } z = [z]_0 + 2m\omega_1 + 2n\omega_2 \in S^+ \text{ 时}, \\ \Phi(z), & \text{当 } z \in S^- \text{ 时}, \end{cases}$$

这里 S^- 是 L 所围的外域，S^+ 是 $S^- + L$ 的补，则

$$\Phi_0^+(t) = \Phi_0^-(t) + g_0(t),$$

① 当然也可推广到 $\Phi(z)$ 有奇点的情况，下面方法完全适用.

且 $\Phi_0(z)$ 仍为分区全纯的加法双准周期函数，于是

$$\Phi_0(z)-\frac{1}{2\pi\mathrm{i}}\int_{L_0}g_0(t)\zeta(t-z)\mathrm{d}t=\Phi_0(z)-\frac{1}{2\pi\mathrm{i}}\int_{L_0}g(t)\zeta(t-z)\mathrm{d}t$$

就不再有跳跃也无奇点. 故由引理 1 推论 2,

$$\Phi_0(z)=C_0+C_1z+\frac{1}{2\pi\mathrm{i}}\int_{L_0}g(t)\zeta(t-z)\mathrm{d}t,$$

其中 C_0,C_1 为任意常数. 最后便得

$$\Phi(z)=\begin{cases}C_0+mg_1+ng_2+C_1z+\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_{L_0}g(t)\zeta(t-z)\mathrm{d}t,\\ \qquad\qquad 当\ z=[z_0]+2m\omega_1+2n\omega_2\in S^+\ 时,\\ C_0+C_1z+\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_{L_0}g(t)\zeta(t-z)\mathrm{d}t,\quad 当\ z\in S^-\ 时.\end{cases}\tag{2.2}$$

Ⅱ. L_0 为 S_0 中一开口光滑弧段 $\widehat{ab}$（取定 a 到 b 的方向为正向）；不失一般性，假定 $O\in L_0$. 要求问题(2.1)在 h_0 类中求解，即允许 $\Phi(z)$ 在 $z=a,b$ 处可以有不到一阶的奇异性. 设 $\Phi(z)$ 的加数为 Φ_1,Φ_2，于是

$$\Phi_j=G\Phi_j+g_j\quad(j=1,2).\tag{2.3}$$

如果 $G=1$，则必 $g_j=0$，这时 $g(t)$ 已是双周期的，立刻可知

$$\Phi(z)=C_0+C_1z+\frac{1}{2\pi\mathrm{i}}\int_{L_0}g(t)\zeta(t-z)\mathrm{d}t.\tag{2.4}$$

如果 $G\neq1$，问题并不如此简单. 这时，由(2.3)知，Φ_j 已一意确定:

$$\Phi_j=g_j/(1-G)\quad(j=1,2).\tag{2.5}$$

令 $\Psi(z)=\lambda z+\mu\zeta(z)$，其中

$$\lambda=\frac{1}{\pi\mathrm{i}}(\eta_1\Phi_2-\eta_2\Phi_1),\quad\mu=\frac{1}{\pi\mathrm{i}}(\omega_2\Phi_1-\omega_1\Phi_2),\tag{2.6}$$

则 $\Psi(z)$ 也以 Φ_1,Φ_2 为加数，从而 $\Phi_0(z)=\Phi(z)-\Psi(z)$ 已是双周期的. 又令

$$g_0(t)=g(t)-(1-G)\Psi(t),\tag{2.7}$$

则 $g_0(t)$ 也是双周期的，于是(2.1)化为了双周期的 Riemann 边值问题:

$$\Phi_0^+(t)=G\Phi_0^-(t)+g_0(t),\quad t\in L;\tag{2.8}$$

但要注意，$\Phi_0(z)$ 在 $z=0$ 处一般有一阶极点. 按照[3]中的记号，应在(R_1)中求解. 如果取定($\ln|G|$ 为实值)

$$\log G=\ln|G|+\mathrm{i}\theta,\quad 0\leqslant\theta<2\pi,$$

并记 $G_0=\frac{1}{2\pi}\mathrm{ldg}G$，则由[3]知，

$$\begin{aligned}&\alpha_a+\mathrm{i}\beta_a=-G_0=-\frac{\theta}{2\pi}-\frac{1}{2\pi\mathrm{i}}\ln|G|,\quad -1<\alpha_a\leqslant0,\\ &\alpha_b+\mathrm{i}\beta_b=G_0=\frac{\theta}{2\pi}+\frac{1}{2\pi\mathrm{i}}\ln|G|,\quad 0\leqslant\alpha_b<1,\\ &G_*=\frac{1}{2\pi\mathrm{i}}\int_{L_0}\log G\,\mathrm{d}t=G_0(b-a),\\ &\gamma(z)=\frac{1}{2\pi\mathrm{i}}\int_{L_0}\log G\cdot\zeta(t-z)\mathrm{d}t=G_0\frac{\sigma(z-b)}{\sigma(z-a)},\end{aligned}\tag{2.9}$$

因此

$$e^{\gamma(z)} = \left[\frac{\sigma(z-b)}{\sigma(z-a)}\right]^{G_0}; \tag{2.10}$$

这里最后这函数应理解为：以 L 为剖线将平面割开后任意取定一支．下面根据[3]中结果进行讨论

1° 如果

$$\eta_j G_* = k_j \pi \mathrm{i} \quad (j=1,2), \tag{2.11}$$

亦即 $\eta_2/\eta_1 = k_2/k_1$，且 $G_* = 2k_1\omega_2 - 2k_2\omega_1$（$k_1,k_2$ 为整实数），则：

(a) 如果 $G(\neq 1)$ 是正实数，则(2.8)的指标 $\kappa=0$，从而在(R_1)中无条件可解，且其一般解为

$$\Phi_0(z) = e^{\gamma(z)}\left\{C + \frac{1}{2\pi \mathrm{i}}\int_{L_0} g_0(t)e^{-\gamma^+(t)}[\zeta(t-z)+\zeta(z)]dt\right\},$$

其中 C 为任意常数；回到 $\Phi(z)$，则有

$$\Phi(z)=e^{\gamma(z)}\left\{C+\frac{1}{2\pi \mathrm{i}}\int_{L_0} g_0(t)e^{-\gamma^+(t)}[\zeta(t-z)+\zeta(z)]dt\right\}+\lambda z+\mu\zeta(z), \tag{2.12}$$

其中 $e^{\gamma(z)}, g_0(t)$ 以及 λ,μ 分别由(2.10)，(2.7)和(2.6)给出．但实际上 $\Phi(z)$ 不能在 $z=0$ 处有极点，故需

$$\mu + \frac{e^{\gamma(0)}}{2\pi \mathrm{i}}\int_{L_0} g_0(t)e^{-\gamma^+(t)}dt = 0, \tag{2.13}$$

或即

$$\frac{1-G}{2\pi \mathrm{i}}\int_{L_0}[\lambda t+\mu\zeta(t)]e^{-\gamma^+(t)}dt - e^{-\gamma(0)}\mu = \frac{1}{2\pi \mathrm{i}}\int_{L_0} g(t)e^{-\gamma^+(t)}dt. \tag{2.13$'$}$$

这是 λ,μ 从而也是 g_1,g_2 的线性方程；这就是说，$g(t)$ 的加数要受一线性约束（当 g_1,g_2 的系数不同时为零时）或者 $g(t)$ 在 L_0 上的值要受一约束（当(2.12)$'$ 左端恒为零时），这时一般解为(2.12)．

(b) 如果 G 不是正实数，则(2.8)的指标 $\kappa=1$．这时它在(R_1)中的一般解为

$$\begin{aligned}\Phi_0(z) &= e^{\gamma(z)}[C_0 + C_1\zeta(z) - C_1\zeta(z-b)] \\ &\quad + \frac{\sigma(z)e^{\gamma(z)}}{\sigma(b)\sigma(z-b)}\cdot\frac{1}{2\pi \mathrm{i}}\int_{L_0}\frac{\sigma(t-b)}{e^{\gamma^+(t)}\sigma(t)}g_0(t)\frac{\sigma(t-z+b)}{\sigma(t-z)}dt,\end{aligned}$$

而 $\Phi(z)=\Phi_0(z)+\lambda z+\mu\zeta(z)$．为要它在 $z=0$ 处无极点，应取 $C_1 = -\mu e^{-\gamma(0)}$．所以这时(2.1)无条件可解，且一般解为

$$\begin{aligned}\Phi(z)&=\lambda(z)+\mu\zeta(z)+e^{\gamma(z)}\{C_0-\mu e^{-\gamma(0)}[\zeta(z)-\zeta(z-b)] \\ &\quad +\frac{\sigma(z)}{\sigma(b)\sigma(z-b)}\cdot\frac{1}{2\pi \mathrm{i}}\int_{L_0}\frac{\sigma(t-b)}{e^{\gamma^+(t)}\sigma(t)}g_0(t)\frac{\sigma(t-z+b)}{\sigma(t-z)}dt\right\}.\end{aligned} \tag{2.14}$$

2° 如果(2.11)不满足，也分两种情况讨论：

(a) 如果 G 是正实数，则(2.8)的 $\kappa=0$.

(i) 如果 $G_* \equiv 0 \pmod{2\omega_1, 2\omega_2}$，则在($R_1$)中可求出 $\Phi_0(z)$，从而求出

$$\Phi(z)=\lambda z+\mu\zeta(z)+h_*(z)e^{\gamma(z)}\left\{C+\frac{1}{2\pi \mathrm{i}}\int_{L_0}\frac{g_0(t)}{e^{\gamma^+(t)}h_*(t)}[\zeta(t-z)+\zeta(z)]dt\right\}, \tag{2.15}$$

其中 $h_*(z)=\exp\{2(l_1\eta_1+l_2\eta_2)z\}$，这里已设 $G_* = 2l_1\omega_1+2l_2\omega_2$．为要它在 $z=0$ 处无极

点，应要求

$$\mu=-\frac{e^{\gamma(0)}}{2\pi i}\int_{L_0}\frac{g_0(t)}{e^{\gamma^+(t)}h_*(t)}dt, \tag{2.16}$$

这又是 g_1,g_2 应满足的线性关系或 $g_0(t)$ 要满足的一积分条件. 当它满足时，(2.1) 的一般解为(2.15).

(ii) 如果 $G_*\not\equiv 0$，则可得

$$\Phi(z)=\lambda z+\mu\zeta(z)+C\frac{\sigma(z+G_*)}{\sigma(z)}e^{\gamma(z)}+\frac{e^{\gamma(z)}}{\sigma(G_*)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{g(t)}{e^{\gamma^+(t)}}\cdot\frac{\sigma(z-t+G_*)}{\sigma(t-z)}dt. \tag{2.17}$$

为要使它在 $z=0$ 处无极点，必须

$$C=-\mu/\sigma(G_*)e^{\gamma(0)}.$$

以此代入(2.17)，便得(2.1) 的唯一解.

(b) 如果 G 不是正实数,则 $\kappa=1$.

(i) 如果 $G_*\equiv 0$，则[3] 中的 $d_0=b$，故

$$\begin{aligned}\Phi(z)=&\lambda z+\mu\zeta(z)+e^{\gamma(z)}[C_0+C_1\zeta(z)-C_1\zeta(z-b)]\\&+\frac{h_*(z)e^{\gamma(z)}}{2\pi i}\int_{L_0}\frac{g_0(t)}{h_*(t)e^{\gamma^+(t)}}[\zeta(t-z)+\zeta(z-b+G_*)]dt,\end{aligned} \tag{2.18}$$

其中 $h_*(z)$ 同前. 为使它在 $z=0$ 处正则，应取

$$C_1=-\mu/e^{\gamma(0)}.$$

故问题(2.1)无条件可解，且一般解中含一个任意常数.

(ii) 如果 $G_*\not\equiv 0$，则令 $d=b-G_*$. 由[3]:

1) 若 $b\not\equiv G_*$，则

$$\begin{aligned}\Phi(z)=&\lambda z+\mu\zeta(z)+\frac{\sigma(z-b+G_*)e^{\gamma(z)}}{\sigma(z-b)}\Big\{C_0+C_1\zeta(z)-C_1\zeta(z-b+G_*)\\&+\frac{1}{2\pi i}\int_{L_0}\frac{\sigma(t-b)g_0(t)}{\sigma(t-b+G_*)e^{\gamma^+(t)}}[\zeta(t-z)+\zeta(z-b+G_*)]dt\Big\},\end{aligned} \tag{2.19}$$

其中应取

$$C_1=-\mu\sigma(b)/\sigma(b-G_*)e^{\gamma(0)}.$$

2) 若 $b\equiv G_*$，则

$$\begin{aligned}\Phi(z)=&\lambda z+\mu\zeta(z)+\frac{\sigma(z)e^{\gamma(z)}}{\sigma(z-b)}[C_0+C_1\zeta'(z)]\\&+\frac{\sigma(z-b+G_*)e^{\gamma(z)}}{\sigma(z-b)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{\sigma(t-b)g_0(t)}{\sigma(t-b+G_*)e^{\gamma^+(t)}}[\zeta(t-z)+\zeta(z-b+G_*)]dt,\end{aligned} \tag{2.20}$$

其中应取

$$C_1=-\mu\sigma(b)/e^{\gamma(0)}.$$

综合以上情况，我们得到

定理 1 对于开口弧段($L_0=\widehat{ab}\in S_0$) 的加法双准周期问题(2.1)，当 $G=1$ 时，无条件可解，且一般解中含两个任意常数. 当 $G\neq 1$ 时，如果它是正实数，一般说来问题无条件可解，且有唯一解，例外情况是 $G_*\equiv 0$，这时 $g(t)$ 的加数 g_1,g_2 要受一线性约束或者 $g(t)$

在 L_0 上的值要受一积分约束，当它满足时，解中含一个任意常数；如果 G 不是正实数，则问题无条件可解，解中也含一常数.

以上讨论的方法完全适用于 L 是具结点的逐段光滑曲线的情况，也可讨论在某些端点处要求解有界的情况，这些在原则上都没有困难.

Ⅲ. 现设 $L_0=\widehat{ab}\in\overline{S}_0$ 且 $b=a+2\omega_1$ 的情况，并设 L_0 在 a,b 处有同向的切线. 记 L_0 经 $2\omega_1$ 作周期延拓的光滑曲线为 $\hat{L}_0$，把 $\hat{L}_0$ 经 $2k\omega_2$ 平移后的曲线为 $\hat{L}_k(k=\pm 1,\pm 2,\cdots)$，故 $L=\sum\limits_{-\infty}^{+\infty}\hat{L}_k$. 又记 $\hat{L}_k$ 与 $\hat{L}_{k+1}$ 间的带形域为 $\hat{S}_k$，并设 $O\in\hat{S}_0$.

在以上情况下我们来求解(2.1)，仍假定 $g(t)\in H$ 于整个 L 上；但由于 a 点在 $\hat{L}_0$ 上已无特殊地位，因此我们将要求 $\Phi(a)$ 有界. 分两种情况.

如果 $G=1$，则仍有 $g_j=0$，(2.4) 仍为一般解(容易验证 $\Phi(a)$ 确实有界).

如果 $G\neq 1$，则问题仍化为双周期问题(2.8). 现在 $G_*=2\omega_1 G_0$. 在(2.10) 中如限定 $z\in\hat{S}_0$，并取定一支，则不计一非零常数因子，有

$$e^{\gamma(z)}=e^{-2G_0\eta_1 z},\quad z\in\hat{S}_0;$$

从而一般应有

$$e^{\gamma(z)}=G^n e^{-2G_0\eta_1 z},\quad z\in\hat{S}_n;\tag{2.21}$$

由此易见

$$e^{\gamma(z+2\omega_j)}=e^{-2\eta_j G_*}e^{\gamma(z)},\quad z\overline{\in}L.\tag{2.22}$$

1° 如果(2.11) 成立，这时 $e^{\gamma(z)}$ 已是双周期的，由此易得(2.12)，可解条件为(2.13) 或(2.13)′，不过现在 $e^{\gamma(0)}=1$；此外，易见 $\Phi(z)$ 在 L_0 上 $t=a$ 处确实有界.

2° 如果(2.11) 不成立，则又分两种情况.

(i) 设 $G_*\equiv 0$. 这时仍可得 $\Phi(z)$ 以 (2.15) 给出，可解条件仍为(2.16). 不过这时可以求出

$$h_*(z)e^{\gamma(z)}=\exp\left(-\frac{l_2\pi i}{\omega_1}z\right),\quad z\in\hat{S}_0,$$

从而

$$h_*(z)e^{\gamma(z)}=G^n\exp\left(-\frac{l_2\pi i}{\omega_1}z\right),\quad z\in\hat{S}_n.$$

因此所求一般解为

$$\begin{aligned}\Phi(z)=&\lambda z+\mu\zeta(z)\\&+G^*\exp\left(-\frac{l_2\pi i}{\omega_1}z\right)\left\{C+\frac{1}{2\pi i}\int_{L_0}g_0(t)\exp\frac{l_2\pi it}{\omega_1}[\zeta(t-z)+\zeta(z)]dt\right\},\\&\qquad z\in\hat{S}_n,\end{aligned}\tag{2.15′}$$

而可解条件则为

$$\mu=-\frac{1}{\pi i}\int_{L_0}g_0(t)\exp\frac{l_2\pi it}{\omega_1}dt.\tag{2.16′}$$

(ii) 设 $G_*\neq 0$. 这时问题有唯一解(2.17)，其中

$$C=-\mu/\sigma(G_*).$$

所以在这种情况下结论是:

定理 1′ 对于 $L_0=\widehat{ab}\in\overline{S_0}$，$b=a+2\omega_1$ 的情况，当 $G=1$ 时，(2.1) 无条件可解，一般解中含两个任意常数. 当 $G\neq 1$ 时，如果 $G_*\not\equiv 0$，则问题有唯一解；如果 $G_*\equiv 0$，则或者 $g(t)$ 的加数要受一线性约束，或者 $g(t)$ 在 L_0 上要受一积分约束，且这时一般解中含有一个任意常数.

§3 乘法双准周期 Riemann 边值问题

现在考虑乘法双准周期问题(0.1). 我们将认为 $G(t),g(t)\in H$，$G(t)\neq 0$，它们都是乘法双准周期的，分别以 G_j,g_j 为乘数 ($g_j,G_j\neq 0$，$j=1,2$). 要求 $\Phi^{\pm}(z)$ 也是乘法双准周期的.

I. 设 L_0 是 S_0 中一条光滑封闭曲线. 仍设 $O\in S_0^+$. 并记

$$P=\frac{1}{\pi\,\mathrm{i}}(\eta_2\log G_1-\eta_1\log G_2),\quad Q=\frac{1}{\pi\,\mathrm{i}}(\omega_1\log G_2-\omega_2\log G_1),\tag{3.1}$$

其中 $\log G_j$ 已各取定一值. 于是

$$G_0(t)=G(t)\mathrm{e}^{Pt+Q\zeta(t)},\tag{3.2}$$

就是双周期的. 如再令

$$\Psi^+(z)=\Phi^+(z),\quad \Psi^-(z)=\mathrm{e}^{-Pz-Q\zeta(z)}\,\Phi^-(z).\tag{3.3}$$

则(0.1) 成为

$$\Psi^+(t)=G_0(t)\,\Psi^-(t)+g(t),\quad t\in L.\tag{3.4}$$

其中$\Psi^{\pm}(z)$ 仍是乘法双准周期的分区全纯函数.

先考虑齐次问题：$g(t)=0$. 这时(3.4) 成为

$$\Psi^+(t)=G_0(t)\,\Psi^-(t),\quad t\in L,\tag{3.4$'$}$$

其指标为

$$\kappa=\frac{1}{2\pi\,\mathrm{i}}[\log G_0(t)]_{L_0}=\frac{1}{2\pi\,\mathrm{i}}[\log G(t)]_{L_0}.$$

用[2] 中结果，令 $G(t)=G_0(t)\mu^{-\kappa}(t)$，其中

$$\mu(z)=\frac{\sigma(z)\sigma(z-\omega_1-\omega_2)}{\sigma(z-\omega_1)\sigma(z-\omega_2)},$$

又令

$$\Psi_*^+(z)=\Psi^+(z)\mu^{-\kappa}(z),\quad \Psi_*^-(z)=\Psi^-(z),\tag{3.5}$$

则(3.4)′ 成为

$$\Psi_*^+(t)=G_*(t)\,\Psi_*^-(t),\quad t\in L,\tag{3.4$''$}$$

其中$\Psi^{\pm}(z)$ 仍是乘法双准周期的，但$\Psi_*^+(z)$ 在 $z=0$ 处的阶数为 κ. 再令

$$\Gamma(z)=\frac{1}{2\pi\,\mathrm{i}}\int_{L_0}\log G(t)\cdot\zeta(t-z)\mathrm{d}t,\quad X_*(z)=\exp(\Gamma(z)).$$

则 $X_*(z)$ 是乘法双准周期的典则函数，$\Psi_*(z)/X_*(z)$ 已无跳跃曲线，但在 $z=0$ 处有 κ 阶.

如果 $\kappa\geqslant 0$，则由引理 2 知.

$$\Psi_*(z) = \exp(\lambda z + \Gamma(z)) \frac{\sigma^\kappa(z - z_0)}{\sigma^\kappa(z)} \Big[C_0 + \sum_{k=1}^{\kappa-1} C_k \zeta^{(k)}(z - z_0)\Big], \tag{3.6}$$

其中 $C_0, \cdots, C_{\kappa-1}, \lambda, z_0$ 均为任意常数. 将(3.5), (3.3) 代入(3.6), 就得出齐次问题的一般解.

如果 $\kappa < 0$, 则因 $\Psi_*(z)$ 在 $z = 0$ 处有 $-\kappa$ 阶零点, 则由引理 3 推论 1, 必 $\Psi_* \equiv 0$. 所以, 这时齐次问题只有零解.

现在考虑非齐次问题(0.1), 它已化为(3.4). 设 $\Psi^\pm(z)$ 的乘数分别为 $\Psi_j^\pm$, 由 (3.4) 立刻知道

$$\Psi_j^+ \Psi^+(t) = \Psi_j^- G_0(t) \Psi^-(t) + g_j g(t), \quad j = 1, 2.$$

与(3.4) 比较, 因 $g(t)$ 不恒为零, 故必

$$\begin{pmatrix} 1 & 1 & 1 \\ \Psi_1^+ & \Psi_1^- & g_1 \\ \Psi_2^+ & \Psi_2^- & g_2 \end{pmatrix}$$

降秩. 如果其秩为 2, 则必

$$\Psi^+(t) = \alpha g(t), \quad G_0(t) \Psi^-(t) = (1 - \alpha) g(t),$$

其中 α 为某常数. 这是平凡情况, 不必讨论. 因此, 如果不考虑平凡情况, 则上面矩阵的秩为 1, 于是 $\Psi_j^\pm = g_j$, 即 $\Psi(z)$ 与 $g(t)$ 有相同的乘数.

定义 $\Psi_*(z), X_*(z)$ 如前, 则(3.4) 成为

$$\frac{\Psi_*^+(t)}{X_*^+(t)} = \frac{\Psi_*^-(t)}{X_*^-(t)} + \frac{g_*(t)}{X_*^+(t)}, \quad t \in L, \tag{3.7}$$

其中已令

$$g_*(t) = g(t) \mu^{-\kappa}(t). \tag{3.8}$$

令

$$\hat{G}_* = \frac{1}{2\pi i} \int_{L_0} \log G_*(t)\, dt, \tag{3.9}$$

则有

$$X_*(z + 2\omega_j) = e^{-2\eta_j \hat{G}_*} X_*(z), \tag{3.10}$$

因此 $\Psi_*(z)/X_*(z)$ 的乘数为 $g_j e^{2\eta_j \hat{G}_*}$, 且在 $z = 0$ 处的阶数为 κ. 引进二常数 α, β, 使

$$2\alpha\omega_j - 2\beta\eta_j = \log g_j, \quad j = 1, 2, \tag{3.11}$$

亦即

$$\alpha = \frac{1}{\pi i}(\eta_1 \log g_2 - \eta_2 \log g_1), \quad \beta = \frac{1}{\pi i}(\omega_1 \log g_2 - \omega_2 \log g_1). \tag{3.11$'$}$$

其中 $\log g_j$ 已任意取定; 不妨设这样取定, 使 $\beta \in S_0$(或在 S_0 的确定两邻边上), 这样 α, β 都一意确定. 分两种情况讨论.

1° 设 $\hat{G}_* \neq \beta$. 我们令

$$\chi^*(z) = \frac{e^{\alpha' z}}{\sigma(\beta')} \frac{1}{2\pi i} \int_{L_0} \frac{g_*(t)}{e^{\alpha' t} X_*^+(t)} \cdot \frac{\sigma(t - z + \beta')}{\sigma(t - z)} dt, \tag{3.12}$$

其中 α', β' 为待定常数, 要使 $\chi_*(z)$ 也有乘数 $g_j e^{2\eta_j \hat{G}_*}$. 为此, 由(3.11), 不妨取

$$\alpha' = \alpha, \quad \beta' = \beta - \hat{G}_*.$$

这样，$\Omega_*(z)=\Psi_*(z)/X_*(z)-\chi_*(z)$ 已无跳跃，在 $z=0$ 处有 κ 阶，乘数为 $g_j e^{2\eta_j\hat{G}_*}$.

(a) 设 $\kappa>0$. 由引理 2，

$$\Omega_*(z)=e^{\lambda z}\frac{\sigma^\kappa(z-z_0)}{\sigma^\kappa(z)}\left[C_0+\sum_{k=1}^{\kappa-1}C_k\zeta^{(k)}(z-z_0)\right].$$

为要使它有上述乘数，应取 z_0 与 λ 使满足

$$2\lambda\omega_j-2\eta_j(\kappa g_0+\hat{G}_*)=\log g_j,\quad j=1,2;$$

与(3.11)比较，可知应取

$$\lambda=\alpha,\quad z_0=(\beta-\hat{G}_*)/\kappa.$$

这样，(3.4) 无条件可解，且解为

$$\Psi_*(z)=e^{\alpha z+\Gamma(z)}\left\{\frac{\sigma^\kappa\left(z-\dfrac{\beta-\hat{G}_*}{\kappa}\right)}{\sigma^\kappa(z)}\left[C_0+\sum_{k=1}^{\kappa-1}C_k\zeta^{(k)}\left(z-\frac{\beta-\hat{G}_*}{\kappa}\right)\right]\right.$$
$$\left.+\frac{1}{\sigma(\beta-\hat{G}_*)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{g_*(t)}{e^{\alpha+\Gamma^+(t)}}\frac{\sigma(t-z+\beta-\hat{G}_*)}{\sigma(t-z)}dt\right\}.\tag{3.13}$$

(b) 设 $\kappa=0$. 现在$\Omega_*(z)$ 在 $z=0$ 处无极点，故由引理 3，必然$\Omega_*(z)=Ce^{\lambda z}$. 如果 $C\neq 0$，则它是乘法双准周期的；由于其乘数应为 $g_j e^{2\eta_j\hat{G}_*}$，故必可选定 λ，使

$$2\lambda\omega_j-2\eta_j\hat{G}_*=\log g_j,\quad j=1,2.$$

但与(3.11) 比较，可见$\hat{G}_*\equiv\beta$. 此与假设相反，因此$\Omega_*(z)=0$. 所以这时(3.4) 有唯一解

$$\Psi_*(z)=\frac{e^{\alpha z+\Gamma(z)}}{\sigma(\beta-G_*)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{g_*(t)}{e^{\alpha+\Gamma^+(t)}}\frac{\sigma(t-z+\beta-\hat{G}_*)}{\sigma(t-z)}dt.\tag{3.14}$$

(c) 设 $\kappa<0$. 这时$\Psi_*(z)/X_*(z)$在 $z=0$ 处有$-\kappa$ 阶零点. 但 $\chi_*(0)$有限，因此$\Omega_*(0)$也有限，所以同上理，仍应有$\Omega_*(z)=0$. 因此，如问题有解，必定唯一，且为(3.14)之形. 但由于$\Psi_*(z)$应在 $z=0$ 处有$-\kappa$ 阶零点，故有下列可解条件：

$$\int_{L_0}\frac{g_*(t)}{e^{\alpha+\Gamma^+(t)}}\left[\frac{\partial^s}{\partial z^s}\frac{\sigma(t-z+\beta-G^*)}{\sigma(t-z)}\right]_{z=0}dt=0,\quad s=0,1,\cdots,-\kappa-1.\tag{3.15}$$

2° 设 $\hat{G}_*\equiv\beta$.① 这时可选取 $\log g_j$ 使(3.11) 中的 $\beta=\hat{G}_*$. 因此

$$2\alpha\omega_j=\log g_j+2\eta_j\hat{G}_*.\tag{3.16}$$

易见 $e^{\alpha z}$ 的乘数就是 $g_j e^{2\eta_j\hat{G}_*}$. 这样，不用(3.12) 而可改用

$$\chi_*(z)=\frac{e^{\alpha z}}{2\pi i}\int_{L_0}\frac{g_*(t)}{e^{\alpha t}X_*^+(t)}[\zeta(t-z)+\zeta(z)]dt.\tag{3.17}$$

(a) 设 $\kappa>0$. 问题(3.4) 无条件可解，且解为

$$\Psi_*(z)=e^{\alpha z+\Gamma(z)}\left\{C_0+\sum_{k=0}^{\kappa-1}C_k\zeta^{(k)}(z)+\frac{1}{2\pi i}\int_{L_0}\frac{g_*(t)}{e^{\alpha+\Gamma^+(t)}}[\zeta(t-z)+\zeta(z)]dt\right\}.\tag{3.18}$$

(b) 设 $\kappa=0$. 这时$\Omega_*(z)=\Psi_*(z)/X_*(z)-\chi_*(z)$ 必为$Ce^{\alpha z}$之形，故若(3.4) 有解，必为$X_*(z)[Ce^{\alpha z}+\chi_*(z)]$之形，其中$\chi_*(z)$ 由(3.17) 给出. 但$\Psi_*(z)$ 不能有奇点，故有可解条件

$$\int_{L_0}\frac{g_*(t)}{e^{\alpha+\Gamma^+(t)}}dt=0;\tag{3.19}$$

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① 这里包括了$\Psi_*(z)/X_*(z)$ 已是双周期即 $g_j=e^{-2\eta_j\hat{G}_*}$ 的情况.

当它满足时，(3.4) 的唯一解为

$$\Psi_*(z)=\mathrm{e}^{\alpha z+\Gamma(z)}\left\{C+\frac{1}{2\pi\,\mathrm{i}}\int_{L_0}\frac{g_*(t)}{\mathrm{e}^{\alpha t+\Gamma^+(t)}}\zeta(t-z)\mathrm{d}t\right\}. \tag{3.20}$$

(c) 设 $\kappa<0$. 可解条件(3.19)仍属必要，且解必为(3.20)之形. 但因$\Psi_*(z)$ 在 $z=0$ 处应有 $-\kappa$ 阶零点，故除(3.19)外，可解条件还应有

$$\int_{L_0}\frac{g_*(t)}{\mathrm{e}^{\alpha t+\Gamma^+(t)}}\zeta^{(k)}(t)\mathrm{d}t=0,\quad k=1,\cdots,\kappa-1; \tag{3.21}$$

且(3.20) 中的常数 C 应取成

$$C=-\frac{1}{2\pi\,\mathrm{i}}\int_{L_0}\frac{g_*(t)}{\mathrm{e}^{\alpha t+\Gamma^+(t)}}\zeta(t)\mathrm{d}t.$$

所以问题的可解条件为(3.19) 与(3.21)，当它们满足时，问题有唯一解

$$\Psi_*(z)=\frac{\mathrm{e}^{\alpha z+\Gamma(z)}}{2\pi\,\mathrm{i}}\int_{L_0}\frac{g_*(t)}{\mathrm{e}^{\alpha t+\Gamma^+(t)}}[\zeta(t-z)-\zeta(t)]\mathrm{d}t. \tag{3.22}$$

通过(3.3),(3.5),(3.8) 就可回到最终所要的结果.

综上所述，我们得到

定理 2 对于光滑封闭曲线，非齐次乘法双准周期问题(0.1) 当 $\kappa>0$ 时无条件可解，且一般解中含有 κ 个任意常数；当 $\kappa<0$ 时有 $-\kappa$ 个可解条件，且只能有唯一解；当 $\kappa=0$ 时一般说来无条件可解且有唯一解，但有一例外情况：$\hat{G}_*\equiv\beta$，这时有一个可解条件，且解中含一个任意常数.

Ⅱ. 设 $L_0=\widehat{ab}$ 是 S_0 中一光滑开口弧段，并设 $O\,\overline{\in}\,L_0$. 先考虑齐次问题

$$\Phi^+(t)=G(t)\,\Phi^-(t),\quad t\in L. \tag{3.23}$$

设 $\Phi(z)$ 的乘数为 $\Phi_j(\neq 0)$，易见这时 $G(t)$ 是双周期的. 取定 $\log G(t)$ 沿 L_0 的一支，用[3]中记号，

$$-\frac{\log G(a)}{2\pi\,\mathrm{i}}=\alpha_a+\mathrm{i}\beta_a,\quad -1<\alpha_a\leqslant 0,$$

$$\frac{\log G(b)}{2\pi\,\mathrm{i}}=\alpha_b+\mathrm{i}\beta_b,$$

问题的指标为 κ，$-1<\alpha_b-\kappa\leqslant 0$；并记

$$G_*=\frac{1}{2\pi\,\mathrm{i}}\int_{L_0}\log G(t)\,\mathrm{d}t, \tag{3.24}$$

$$\gamma(z)=\frac{1}{2\pi\,\mathrm{i}}\int_{L_0}\log G(t)\cdot\zeta(t-z)\mathrm{d}t,\quad z\,\overline{\in}\,L. \tag{3.25}$$

于是典则函数为 $X(z)=\mathrm{e}^{\gamma(z)}$. 它在 $z=b$ 处有 $(z-b)^{\alpha_b}$ 的因子，因而 $\Phi(z)c^{-\gamma(z)}$ 在 $z=b$ 处至少也有这一因子，从而至少有 $(z-b)^{-\kappa}$ 的因子，且无别的奇点.

如果 $\kappa>0$，由引理 2 知，

$$\Phi(z)=\mathrm{e}^{\lambda z+\gamma(z)}\frac{\sigma^\kappa(z-z_0)}{\sigma^\kappa(z-b)}\left[C_0+\sum_{k=1}^{\kappa-1}C_k\zeta^{(k)}(z-z_0)\right], \tag{3.26}$$

其中 $C_0,\cdots,C_{\kappa-1},\lambda,z_0$ 都是任意常数.

如果 $\kappa=0$，则由引理 3，

$$\Phi(z)=C\mathrm{e}^{\lambda z+\gamma(z)}. \tag{3.27}$$

其中C,λ都是任意常数.

如果$\kappa<0$，则由引理3推论1，必$\Phi(z)=0$.

因此我们有

定理3　开口光滑弧段的乘法双准周期齐次问题(3.23)，当$\kappa\geqslant 0$时一般解中含有$\kappa+2$个任意常数，当$\kappa<0$时只有零解.

现在考虑非齐次问题(0.1)，并设$\Phi(z),G(t),g(t)$都是乘法双准周期的，乘数分别为$\Phi_j,G_j,g_j(j=1,2)$. 由于

$$\Phi_j\Phi^+(t)=\Phi_jG_jG(t)\Phi^-(t)+g_jg(t),\quad j=1,2,\tag{*}$$

故矩阵

$$\begin{pmatrix}1 & 1 & 1\\ \Phi_1 & \Phi_1G_1 & g_1\\ \Phi_2 & \Phi_2G_2 & g_2\end{pmatrix}$$

为降秩的. 其秩不可能是2. 因若如此，将有

$$\Phi^+(t)=CG(t)\Phi^-(t)\quad(C\neq 0),$$

由此立即可得$G_j=1$；但由(*)式，知必$\Phi_j=g_j$，矛盾.

所以上面矩阵的秩必为1，因此$G(t)$是双周期的，且$\Phi(z)$与$g(t)$必有相同的乘数g_j. 这时(0.1)可化为

$$\frac{\Phi^+(t)}{e^{\gamma^+(t)}}=\frac{\Phi^-(t)}{e^{\gamma^-(t)}}+\frac{g(t)}{e^{\gamma^+(t)}},\tag{3.28}$$

其中$\Phi(z)/e^{\gamma(z)}$是乘法双准周期的，与$g(t)/e^{\gamma^\pm(t)}$有相同的乘数$g_je^{2\eta_jG_*}$，仍以(3.11)′定义α,β，且$\beta\in S_0$.

1°　设$\kappa=0$. 分两种情况：

(a)　$G_*\not\equiv\beta$. 与(3.12)类似，定义

$$\chi(z)=\frac{e^{\alpha z}}{\sigma(\beta-G_*)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{g(t)}{e^{\alpha t+\gamma^+(t)}}\frac{\sigma(t-z+\beta-G_*)}{\sigma(t-z)}dt,$$

则$\Omega(z)=\Phi(z)/e^{\gamma(z)}-\chi(z)$已无跳跃,除在$a,b$处有不到一阶的奇异性外无别的奇点，以$g_je^{2\eta_jG_*}$为乘数，故必$\Omega(z)=Ce^{\lambda z}$；和前面一样理由，由于$G_*\not\equiv\beta$，必定$c=0$. 因此问题(0.1)有唯一解

$$\Phi(z)=\frac{e^{\alpha z+\gamma(z)}}{\sigma(\beta-G_*)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{g(t)}{e^{\alpha t+\gamma^+(t)}}\frac{\sigma(t-z+\beta-G_*)}{\sigma(t-z)}dt.\tag{3.29}$$

(b)　$G_*\equiv\beta$. 可选取$\log g_j$使$G_*=\beta$，因此，

$$2\alpha\omega_j=\log g_j+2\eta_jG_*,\quad j=1,2.$$

于是$e^{\alpha z}$也以$g_je^{2\eta_jG_*}$为乘数，且在a,b处有不到一阶的奇异性，因此$\Psi(z)=\Phi(z)/e^{\alpha z+\gamma(z)}$就是双周期的. 这时问题已完全化为双周期的，可按[3]中方法讨论，从略.

2°　设$\kappa>0$. 引进

$$\Psi_0(z)=\frac{\sigma^\kappa(z-b)}{\sigma^\kappa(z)}\frac{\Phi(z)}{e^{\gamma(z)}},\quad g_0(t)=\frac{\sigma^\kappa(t-b)}{\sigma^\kappa(t)}\frac{g(t)}{e^{\gamma^+(t)}}.$$

在$z=b$处，$\Psi_0(z)$的阶数在-1与$+1$之间，而$g_0(b)$是有界的，它们的乘数都是$ge^{2\eta_j(G_*-\kappa b)}$，这时问题(0.1)化为了

$$\Psi^+(t) = \Psi_0^-(t) + g_0(t). \tag{0.1}$$

(a) 设$G_* \not\equiv \beta + \kappa b$. 令

$$\chi_0(z) = \frac{e^{\alpha z}}{\sigma(\beta + \kappa b - G_*)} \cdot \frac{1}{2\pi i}\int_{L_0} \frac{\sigma^\kappa(t-b)g(t)}{e^{\alpha+\gamma^+(t)}\sigma^\kappa(t)} \frac{\sigma(t-z+\beta-\kappa b-G_*)}{\sigma(t-z)}dt,$$

则它除在a,b处有对数奇异性外处处有界，所以$\Omega_0(z) = \Psi_0(z) - \chi_0(z)$无跳跃，乘数同前，在$a,b$处至多有不到一阶奇异性从而没有奇异性，在$z=0$处有$\kappa$阶极点. 故由引理 2,

$$\Omega_0(z) = e^{\lambda z}\frac{\sigma^\kappa(z-z_0)}{\sigma^\kappa(z)}\left[C_0 + \sum_{k=1}^{\kappa-1}C_k\zeta^{(k)}(z-z_0)\right],$$

但它的乘数$2\omega_j\lambda - 2\eta_j\kappa z_0$应该和$\log g_j + 2\eta_j(G_* - \kappa b)$相差$2\pi i$的整数倍，故不妨取

$$\lambda = \alpha, \quad z_0 = b + \frac{\beta - G_*}{\kappa},$$

因此(0.1)的一般解为

$$\Phi(z) = e^{\alpha z+\gamma(z)}\left\{C_0 + \sum_{k=1}^{\kappa-1}C_k\zeta^{(k)}(z-z_0) + \frac{\sigma(z)}{\sigma^\kappa(\beta+\kappa b-G_*)\sigma^\kappa(z-b)}\right.$$
$$\left.\cdot\frac{1}{2\pi i}\int_{L_0}\frac{\sigma^\kappa(t-b)g(t)}{e^{\alpha+\gamma^+(t)}\sigma^\kappa(t)}\frac{\sigma(t-z+\beta+\kappa b-G_*)}{\sigma(t-z)}dt\right\}. \tag{3.30}$$

(b) 设$G_* \equiv \beta + \kappa b$. 这时可选定$\log g_j$使

$$\alpha = \frac{\log g_j + 2\eta_j(G_* - \kappa b)}{2\omega_j}, \quad j = 1,2, \tag{3.31}$$

从而$e^{\alpha z}$以$g_je^{2\eta_j(G_*-\kappa b)}$为乘数，因此在(0.1)′中遍乘$e^{-\alpha z}$后问题又化为了双周期的，且指标$\kappa$不变. 再就不难按[3]中分析，从略.

3° 设$\kappa < 0$. 也分两种情况.

(a) 设$G_* \not\equiv \beta + \kappa b$. 令$\Psi_0(z), g_0(t), \chi_0(z), \Omega_0(z)$仍同前，则$\Psi_0(z)$在$z=0$处有$-\kappa$阶零点，而$\chi_0(0)$有界，因此$\Omega_0(0)$也有界，故必$\Omega_0(z) = Ce^{\lambda z}$. 若$C\neq 0$，则其乘数也应是$g_je^{2\eta_j(G_*-\kappa b)}$，这就是说可取定$\log g_j$使$\lambda$等于(3.31)中的$\alpha$，但这只有$\beta + \kappa b - G_* \equiv 0$才可以，这已排除在外. 故$C=0$. 所以，如果(0.1)′有解，则必$\Psi_0(z) = \chi_0(z)$，但因$\Psi_0(z)$以$z=0$为$-\kappa$阶零点，故还必须

$$\int_{L_0}\frac{\sigma^\kappa(t-b)g(t)}{e^{\alpha+\gamma^+(t)}\sigma^\kappa(t)}\frac{\partial^s}{\partial z^s}\left[\frac{\sigma(t-z+\beta+\kappa b-G_*)}{\sigma(t-z)}\right]_{z=0}dt = 0, \quad s=0,1,\cdots,-\kappa-1, \tag{3.32}$$

当这些条件满足时，问题(0.1)有唯一解

$$\Phi(z) = \frac{e^{\alpha z+\gamma(z)}\sigma^\kappa(z)}{\sigma(\beta+\kappa b-G_*)\sigma^\kappa(z-b)}\cdot\frac{1}{2\pi i}\int_{L_0}\frac{\sigma^\kappa(t-b)g(t)}{e^{\alpha+\gamma^+(t)}\sigma^\kappa(t)}\frac{\sigma(t-z+\beta+\kappa b-G_*)}{\sigma(t-z)}dt.$$

(b) 设$G_* \equiv \beta + \kappa b$. 如前，又可化为双周期问题进行讨论.

总之，我们有

定理 4 对于开口光滑弧段的非齐次乘法双准周期问题(0.1)，设其指标为κ. 如果$G_* \equiv \beta + \kappa b \pmod{2\omega_j}$，则问题可化为$\Phi(z)/e^{\alpha z+\gamma(z)}$的双周期类似问题；如果$G_* \not\equiv \beta + \kappa b$. 则当$\kappa \geq 0$时问题无条件可解，且一般解中含有$\kappa$个任意常数，当$\kappa < 0$时要满足$-\kappa$个可解条件才有唯一解.

Ⅲ. 设$L_0 = \widehat{ab} \in \overline{S}_0$, $b = a + 2\omega_1$. 其余关于L的假设及记号均同§2中Ⅰ段. 这

时除平凡情况外，$\Phi^{\pm}(z)$ 的乘数必相同，$G(t)$ 是双周期的. 现应设

$$\frac{\log G(a)}{2\pi\mathrm{i}}=A+\mathrm{i}B,\ \frac{\log G(a+2\omega_1)}{2\pi\mathrm{i}}=\kappa+A+\mathrm{i}B,\quad 0\leqslant A<1,$$

而 G_*，$\gamma(z)$ 仍分别如(3.24)，(3.25)，在 $z=a$ 处，$\mathrm{e}^{\gamma(z)}$ 仍有 $(z-a)^{\kappa}$ 的因子.

对于齐次问题($g=0$)，Ⅱ段中开始所论包括定理3完全成立，且(3.26)中的 $\sigma^{\kappa}(z-b)$ 可改为 $\sigma^{\kappa}(z-a)$.

对于非齐次问题，仍可化为(3.28)，并仍以(3.11)′定义 α,β. 以下讨论均与前段同，且定理4这时也成立，均从略.

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DOUBLY QUASI-PERIODIC RIEMANN BOUNDARY VALUE PROBLEMS

Abstract

By additive and multiplicative doubly quasi-periodic analytic functions we mean functions satisfying (1.1) and (1.2) respectively. In case they have only poles as singularities, we call them as quasi-elliptic functions. Similarly we may define sectionally holomorphic and doubly quasi-periodic functions. In this paper, we solve the corresponding doubly quasi-periodic Riemann boundary value problems by reducing them to doubly periodic ones which were considered and solved in [2] and [3] both for closed contours and open arcs.

[原载于数学物理学报，1981，1(1)：13～30.]

关于双准周期解析函数的 DIRICHLET 问题*

§1 引 言

在[1]中，我们讨论了双周期解析函数的 Dirichlet 问题，将其化为 Fredholm 积分方程而给出了其求解方法，并得出了可解条件. 我们也曾提到了双准周期解析函数的类似问题以及有关结果，但未予证明. 本文将给出这些结果的详细论证.

我们将沿用[1]中记号. 设周期为 $2\omega_1, 2\omega_2$，且 $\mathrm{Im}(\omega_2/\omega_1)>0$. L_0 为基本胞腔(以 $\pm\omega_1 \pm\omega_2$ 为顶点) S_0 中的一条 Liapunov 封闭曲线，围成一内域 S_0^+. 设 $O\in S_0^+$. 记 $S_0^-=S_0-\overline{S}_0^+$，$S^-$ 为由 L_0 及其所有周期合同曲线之并所围的外域.

关于双周期解析函数的 Dirichlet 问题有下述结果 (参看[1]，§3). 设 $\nu_j(t)$ $(j=1,2,3)$ 为 Fredholm 积分方程 $\boldsymbol{K}'\nu=0$ (算子 $\boldsymbol{K}$ 与 $\boldsymbol{K}'$ 的定义均见[1]) 的规则化解，即满足条件 $\nu_1^*=\nu_2^*=0$，$\nu_3^*=1$ 者，其中

$$\nu_j^*=\int_{L_0}\nu_j(t)\mathrm{d}s,\quad j=1,2,3.$$

在 L 上已给一双周期实函数 $f(t)\in H$ (Hölder 条件)，求在 S^- 中的一个双周期解析函数 $\Phi^-(z)$ 满足边值条件

$$\mathrm{Re}\,\Phi^-(t)=f(t),\quad t\in L \tag{1.1}$$

者，此问题有唯一解(可差一个任意复常数项)当且仅当下列两个实的可解条件成立：

$$\int_{L_0}f(t)\nu_j(t)\mathrm{d}s=0,\quad j=1,2. \tag{1.2}$$

双准周期解析函数的 Dirichlet 问题可表述如下. 在 L_0 上已给一实函数 $f(t)\in H$，求一个在 S^- 中解析的函数 $\Phi^-(z)$，具有双准周期性

$$\Phi^-(z+2\omega_j)=\Phi^-(z)+a_j,\quad j=1,2 \tag{1.3}$$

(a_1,a_2 为二复常数)，满足边值条件

$$\mathrm{Re}\,\Phi^-(t)=f(t),\quad t\in L_0. \tag{1.4}$$

加数 a_1,a_2 可以事先指定或否，但必须预先明确说明.

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§2 一个重要引理

我们首先建立一个有关双周期解析函数的引理，它在后面的讨论中起着重要作用.

引理1 设 $C\neq0$ 是一复常数. 则 S^- 中双周期解析函数 $\Phi^-(z)$ 满足边值条件

$$\operatorname{Re}\Phi^-(t)=\operatorname{Re}\{Ct\},\quad t\in L_0,\tag{2.1}$$

的 Dirichlet 问题无解.

注意(2.1)实际上是说：在(1.1)中，对于 $t\in L_0$，有 $f(t)=\operatorname{Re}\{Ct\}$；而对于 $t\in L$，$f(t)$ 等于其周期延拓而不再是 $\operatorname{Re}\{Ct\}$.

证 设若在 S^- 中存在这样的函数 $\Phi^-(z)$. 记 $\Phi^-(z)=u(z)+\mathrm{i}v(z)$. 则

$$\Phi^-(t)=\alpha x+\beta y+\mathrm{i}v(t),\quad t=x+\mathrm{i}y\in L_0,\tag{2.2}$$

其中 $C=\alpha-\mathrm{i}\beta$，因此 α,β 不同时为零. 由于 L_0 是一 Liapunov 曲线从而 $u'(t)$ 连续，故由解析函数边界性质的一些定理（参看[3]，第九章，§1，定理2与第十章，§1，定理6)，可以证明，在 L_0 外面靠近它的平准线 L_ϵ（即 L_ϵ 是圆周 $|w|=1-\epsilon$ 在 F 下的逆像，这里 F 是把 L_0 所围的外域保形映照到 $|w|<1$ 上的映射，使 $F(\infty)=0$ 者）上 $\frac{\partial u}{\partial s}$ 可连续延拓到 L_0 上的 $\frac{\partial u}{\partial s}$. 于是，由 Cauchy-Riemann 方程与 Green 公式，我们有

$$\left(\int_\Gamma-\int_{L_\epsilon}\right)v\frac{\partial u}{\partial s}\mathrm{d}s=-\iint_{S_\epsilon^-}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y,$$

其中 Γ 是 S_0 的边界，S_ϵ^- 是 L_ϵ 与 Γ 所围的区域. 注意 u,v 是双周期的，令 $\epsilon\to0$ 求极限，便得

$$\int_{L_0}v\mathrm{d}u=\iint_{S_0}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y\geqslant0.\tag{2.3}$$

另一方面，因为

$$\int_{L_0}\Phi^-(t)\mathrm{d}t=\int_\Gamma\Phi^-(\tau)\mathrm{d}\tau=0,$$

易证

$$\int_{L_0}v(t)\mathrm{d}t=-\alpha\int_{L_0}x\mathrm{d}y+\beta\mathrm{i}\int_{L_0}y\mathrm{d}x=-(\alpha+\beta\mathrm{i})|S_0^+|,$$

其中 $|S_0^+|$ 是 S_0^+ 的面积. 所以

$$\int_{L_0}v(t)\mathrm{d}x=-\alpha|S_0^+|,\quad\int_{L_0}v(t)\mathrm{d}y=-\beta|S_0^+|.$$

因此，

$$\int_{L_0}v\,\mathrm{d}u=\alpha\int_{L_0}v\,\mathrm{d}x+\beta\int_{L_0}v\,\mathrm{d}y=-(\alpha^2+\beta^2)|S_0^+|<0,$$

与(2.3)矛盾. 引理得证.

§3 双准周期问题(A)

在本节中考虑 S^- 中双准周期解析函数 $\Phi^-(z)$ 的 Dirichlet 问题(1.4)，要求(1.3)中的

加数 a_1,a_2 都不事先指定. 把此问题称作(A).

在[1]中曾指出，问题(A)可转换为 S^- 中的双周期解析函数 $\Psi^-(z)$ 的 Dirichlet 问题，满足边值条件

$$\operatorname{Re}\Psi^-(t)=f(t)-\operatorname{Re}\{A\zeta(t)+Bt\},\quad t\in L_0, \tag{3.1}$$

其中 $\zeta(t)$ 为 Weierstrass 的 ζ 函数，而 A,B 为二待定复常数，和 a_1,a_2 以下列关系式相关联[2]：

$$A=\frac{1}{\pi\,\mathrm{i}}(\omega_2a_1-\omega_1a_2),\quad B=\frac{1}{\pi\,\mathrm{i}}(a_2\eta_1-a_1\eta_2), \tag{3.2}$$

或者，完全一样，

$$a_j=2(\eta_jA+\omega_jB),\quad j=1,2, \tag{3.2$'$}$$

因为熟知 $\eta_1\omega_2-\eta_2\omega_1=\pi\,\mathrm{i}/2$ (参看[4])，这里 $\eta_j=\zeta(\omega_j)$. 因此，在问题(A)中，常数 A,B 也未事先指定. 如果问题(3.1)解出，则问题(A)的解由下式给出：

$$\Phi^-(z)=\Psi^-(z)+A\zeta(z)+Bz.$$

如前所述，(3.1)的可解条件为

$$\operatorname{Re}\int_{L_0}\{A\zeta(t)+Bt\}\nu_j(t)\mathrm{d}s=\int_{L_0}f(t)\nu_j(t)\mathrm{d}s,\quad j=1,2. \tag{3.3}$$

记

$$\int_{L_0}\zeta(t)\nu_j(t)\mathrm{d}s=c_{j1},\ \int_{L_0}t\nu_j(t)\mathrm{d}s=c_{j2},\ \int_{L_0}f(t)\nu_j(t)\mathrm{d}s=\gamma_j,\quad j=1,2. \tag{3.4}$$

于是，$c_{jk}(j,k=1,2)$ 是与 $f(t)$ 无关的复常数，而 γ_j 是由 $f(t)$ 一意确定的实常数. 为要确定 A,B，就要求解下列线方程组：

$$\begin{cases}c_{11}A+c_{12}B=\gamma_1+\mathrm{i}\delta_1,\\ c_{21}A+c_{22}B=\gamma_2+\mathrm{i}\delta_2,\end{cases} \tag{3.5}$$

其中 δ_1,δ_2 是两个待定实常数. 我们应把它们适当选取使(3.5)可以对 A,B 求解. 一旦求得 A,B，按[1]中结果，(3.1)的唯一解将由下式给出：

$$\Psi^-(z)=\frac{1}{\pi\,\mathrm{i}}\int_{L_0}\mu(t)[\zeta(t-z)+\zeta(z)]\mathrm{d}t+\alpha+\mathrm{i}\beta,$$

其中 $\mu(t)$ 是方程

$$\boldsymbol{K}\mu=-f(t)+A\zeta(t)+Bt+\alpha$$

的任一解（且不论取哪个解，前式右端积分是同一函数），且

$$\alpha=\int_{L_0}[f(t)-A\zeta(t)-Bt]\nu_3(t)\mathrm{d}s,$$

而 β 为任意实常数. 这里 $\boldsymbol{K}$ 为 Fredholm 算子，其表达式见[1]中(3.1)式.

在求解(3.5)之前，我们证明另一引理.

引理 2 c_{12},c_{22} 不同时为零.

证 设若 $c_{12}=c_{22}=0$. 则当 $f(t)\equiv 0$ 从而 $\gamma_1=\gamma_2=0$ 时，若取 $\delta_1=\delta_2=0$，则方程组(3.5)将有一组解 $A=0$，$B=1$. 与(3.1)相比较，这就表明，S^- 中双周期解析函数 $\Psi^-(z)$ 的 Dirichlet 问题满足边值条件

$$\operatorname{Re}\Psi^-(t)=\operatorname{Re}\{-t\},\quad t\in L_0,$$

者可解，与引理 1 矛盾．引理证毕．

不失一般性，可设 $c_{22}\neq 0$．令

$$\Delta=c_{11}c_{22}-c_{12}c_{21}.$$

如 $\Delta\neq 0$，则任取 δ_1,δ_2，(3.5)对 A,B 恒一意可解，于是问题(A)恒可解，且一般解中含有两个任意实常数．

如 $\Delta=0$，则必存在一复常数 $k=\kappa_1+\mathrm{i}\kappa_2$，使得

$$c_{11}A+c_{12}B=k(c_{21}A+c_{22}B)$$

对任何 A,B 均成立．这样，方程组(3.5)可改写为

$$\begin{cases}k(c_{21}A+c_{22}B)=\gamma_1+\mathrm{i}\delta_1,\\ c_{21}A+c_{22}B=\gamma_2+\mathrm{i}\delta_2.\end{cases}\tag{3.5$'$}$$

我们来证明 $\kappa_2\neq 0$．设若不是这样，则 $k=\kappa_1$ 是一实常数．那么，对于 $f(t)\equiv 0$（从而 $\gamma_1=\gamma_2=0$）以及任取的 $\delta_2\neq 0$，我们得到 $A=0$，$B=\mathrm{i}\delta_2/c_{22}$是(3.5)$'$中第二个方程的一组解．于是如再取 $\delta_1=k\delta_2$，则它们也是第一个方程的一组解．因此 S^- 中的双周期解析函数$\Psi^-(z)$的 Dirichlet 问题具边值条件

$$\operatorname{Re}\Psi^-(t)=\operatorname{Re}\{\mathrm{i}\delta_2 t/c_{22}\},\quad t\in L_0,\tag{3.6}$$

者可解，又与引理 1 矛盾．这样得知 $\kappa_2\neq 0$．

为要在这种情况下求解(3.5)$'$，必须取 δ_1,δ_2 使得

$$k(\gamma_2+\mathrm{i}\delta_2)=\gamma_1+\mathrm{i}\delta_1,$$

亦即

$$\begin{cases}\kappa_1\gamma_2-\kappa_2\delta_2=\gamma_1,\\ \kappa_1\delta_2+\kappa_2\gamma_2=\delta_1.\end{cases}\tag{3.7}$$

当 $f(t)$从而 γ_1,γ_2 已给，因 $\kappa_2\neq 0$，故 δ_1,δ_2 可由(3.7)一意确定．于是(3.5)$'$的解为

$$B=\frac{1}{c_{22}}(\gamma_2+\mathrm{i}\delta_2-c_{21}A),\tag{3.8}$$

这里 $A=\alpha_1+\mathrm{i}\alpha_2$ 可以任意．因此又得到：问题(A)的一般解中含有两个实任意常数 α_1,α_2．

这样，我们便证得

定理 1 S^- 中的问题(A)恒可解，且其一般解中含有两个实任意常数．

§4 双准周期问题(B)与(C)

现再考虑问题(1.4)，但设

$$\operatorname{Re}a_j=\varepsilon_j,\quad j=1,2,\tag{4.1}$$

已事先指定．换句话说，我们已设 $\operatorname{Re}\Phi^-(t)=f(t)$已双准周期地给出在 L 上：

$$f(t+2\omega_j)=f(t)+\varepsilon_j,\quad j=1,2,\ t\in L.$$

将此问题记为(B)．

当 $\Delta\neq 0$ 时，由§3 中的讨论知，A,B 是 $\gamma_1,\gamma_2,\delta_1,\delta_2$ 的线性齐次函数，所以令 $A=\alpha_1+\mathrm{i}\alpha_2$，$B=\beta_1+\mathrm{i}\beta_2$ 时，$\alpha_1,\alpha_2,\beta_1,\beta_2$ 将是 $\gamma_1,\gamma_2,\delta_1,\delta_2$ 的实线性组合．这些组合的系数是与 $f(t)$无关的常数．另一方面，在(3.2)$'$两端取实部，便得 $\varepsilon_1,\varepsilon_2$ 是 $\alpha_1,\alpha_2,\beta_1,\beta_2$ 的实线性组合．因此，我们可以写

$$P\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}=\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}+Q\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}, \tag{4.2}$$

其中$\boldsymbol{P},\boldsymbol{Q}$是$2\times2$实常数矩阵，其各个元与$f(t)$无关．今证$\det \boldsymbol{P}\neq0$．当$f(t)\equiv0$从而$\gamma_1=\gamma_2=0$并给定$\varepsilon_1=\varepsilon_2=0$时，问题(B)成为$S^-$中的双准周期解析函数$\Phi^-(z)$的Dirichlet问题，其边值条件为

$$\operatorname{Re}\Phi^-(t)=0,\quad t\in L_0,$$

且已知$\operatorname{Re}\Phi^-(z+2\omega_j)=\operatorname{Re}\Phi^-(z)$．所以$\operatorname{Re}\Phi^-(z)$是$S^-$中一个在$L$上具有零边值的双周期调和函数．由最大模原理，可知在S^-中$\operatorname{Re}\Phi^-(z)\equiv0$．因此$\Phi^-(z)\equiv\mathrm{i}\lambda$在$S^-$中等于一虚常数．这样，$\Phi^-(z)$本身已是双周期的．因此一定有$a_1=a_2=0$．于是由(3.2)，$A=B=0$．再由(3.5)便知$\delta_1=\delta_2=0$．这样，相应于(4.2)的齐次方程只有平凡解．这就证明了我们的论断．因而(4.2)对任意的γ_1,γ_2以及给定的$\varepsilon_1,\varepsilon_2$一意可解．

如$\Delta=0$，则由(3.7)，δ_1,δ_2是γ_1,γ_2的实线性组合．再由(3.8)，便知β_1,β_2是$\alpha_1,\alpha_2,\gamma_1,\gamma_2$的实线性组合．于是可以写

$$R\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}+S\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}, \tag{4.3}$$

其中$\boldsymbol{R},\boldsymbol{S}$也是$2\times2$矩阵，与上面所讲的为同种类型．如前同样推理，可证$\det\boldsymbol{R}\neq0$．所以(4.3)对于任意的$\gamma_1,\gamma_2$以及$\varepsilon_1,\varepsilon_2$也一意可解．

于是我们有

定理2 问题(B)恒唯一可解．

如果a_1,a_2都事先指定，则相应问题记为(C)．因为如前所述，A,B可一意地由γ_1,γ_2和$\operatorname{Re}a_1$，$\operatorname{Re}a_2$确定，故由(3.2)知，问题(C)(唯一)可解当且仅当下列二实的条件满足时：

$$\operatorname{Im}a_j=2\operatorname{Im}(\eta_jA+\omega_jB),\quad j=1,2. \tag{4.4}$$

(4.4)实际上间接地是加于$f(t)$上的两个实的条件．

这样，我们有

定理3 问题(C)(唯一)可解当且仅当$f(t)$满足两个实的可解条件．

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THE HILBERT BOUNDARY PROBLEM OF DOUBLY PERIODIC ANALYTIC FUNCTIONS*

Abstract

The doubly periodic Hilbert boundary value problem is discussed in this paper. First, certain kind of integral representations of doubly quasi-periodic analytic functions in multiplication is established so that the Dirichlet problem of such functions is solved. Then, by the method of regularization, the Hilbert boundary value problem is transferred to such a problem, and it is reduced at length to some Fredholm integral equation. The number of solutions and conditions of solvability as well as the form of the general solution are obtained.

We studied the Riemann boundary value problem of doubly periodic analytic functions in [1,2] and that of doubly quasi-periodic analytic functions in [3]. The corresponding Hilbert problem has not been investigated yet. The special case for the Dirichlet problem of analytic functions, doubly periodic or doubly quasi-periodic in addition, was discussed in [4,5].

In this paper, the Hilbert boundary value problem of doubly periodic analytic functions (DH problem) is discussed. We first give some integral representation for doubly quasi-periodic analytic functions in multiplication (MQ-function) so as to solve the Dirichlet problem for such functions. Then, by using the method of regularization, the DH problem is solved by reducing it to this solved problem.

§ 1 Definitions and Notations

We shall recall some notations used in [2]. Let the periods be $2\omega_1, 2\omega_2$ with $\mathrm{Im}(\omega_2/\omega_1)>0$ and S_0 be the fundamental cell (the parallelogram with vertices $\pm\omega_1\pm\omega_2$). L_0 is a Liapunov contour in S_0 with usual positive sense, the interior region bounded by which is denoted by S_0^+. We always assume $O\in S_0^+$. Denote $S_0^-=S_0-S_0^+$. The union of all the contours congruent to $L_0(\mathrm{mod}\ 2\omega_j)$ is denoted by L, the exterior region bounded by which is denoted by S^-.

Let $w=\omega(z)$ be the function conformally mapping S_0^+ to $|w|<1$ with $\omega(0)=0$,

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$\omega'(0)>0$.

We shall use the Weierstrass ζ-function $\zeta(z)$ and σ-function $\sigma(z)$ (cf. [6]). It is well known that $\zeta(z)$ has the single simple pole $z=0$ in S_0 with $1/z$ as its principal part and the property

$$\zeta(z+2\omega_j)=\zeta(z)+2\eta_j, \quad j=1,2, \tag{1.1}$$

where $\eta_j=\zeta(\omega_j)$, satisfying

$$2\eta_1\omega_2-2\eta_2\omega_1=\pi\,\mathrm{i}. \tag{1.2}$$

$\sigma(z)$ is an entire function which has a unique simple zero at $z=0$ in S_0 with $\sigma'(0)=1$ and has the property

$$\sigma(z+2\omega_j)=-\mathrm{e}^{2\eta_j(z+\omega_j)}\sigma(z), \quad j=1,2. \tag{1.3}$$

A function $\Phi^-(z)$ defined in S^- is called an MQ-function if it is analytic in S^- with the property of double quasi-periodicity

$$\Phi^-(z+2\omega_j)=b_j\,\Phi^-(z), \quad j=1,2, \tag{1.4}$$

where b_1,b_2 are two non-zero constants, called the multipliers.

Define two constants λ and z_0 by the following system of linear equations

$$2\omega_j\lambda-2\eta_j z_0=\log b_j, \quad j=1,2, \tag{1.5}$$

for certain fixed values of $\log b_1$ and $\log b_2$. It is uniquely solvable on account of (1.2). For definiteness, we assume $z_0\in S_0$, which is always possible for suitably chosen $\log b_j$. Note that λ and z_0 are complex in general even if b_1,b_2 are real. It is evident that $\mathrm{e}^{\lambda z}$ is an entire MQ-function with multipliers b_1,b_2 if $z_0=0$ (Case I) and

$$q(z)=\mathrm{e}^{\lambda z}\frac{\sigma(z-z_0)}{\sigma(z)} \tag{1.6}$$

is a meromorphic MQ-function with multipliers b_1,b_2 if $z\neq 0$ (Case II). $q(z)$ has $z=0$ as its unique simple pole and $z=z_0$ as its unique simple zero in S_0 and hence is holomorphic in S^-.

Case I occurs iff

$$\omega_2\log b_1=\omega_1\log b_2 \tag{1.7}$$

is valid for suitably chosen $\log b_1,\log b_2$.

§ 2 Integral Representation for MQ-Functions

We gave an integral representation for doubly periodic analytic functions in S^- (cf. [4]), from which we may easily obtain one for MQ-functions by multiplying a factor $\mathrm{e}^{\lambda z}$ (Case I) or $q(z)$ (Case II). However, such a representation is inconvenient for our purpose. Here we shall give another kind of representation for MQ-functions in S^-.

Given an MQ-function $\Phi^-(z)=u(z)+\mathrm{i}v(z)$ in S^- with multipliers b_1,b_2. Constants λ and $z_0\in S_0$ are defined as in § 1. Assume $\Phi^-(t)=u(t)+\mathrm{i}v(t)\in H$ (Hölder condition) on L.

Case I: $z_0=0$. We shall prove the following lemma.

Lemma 1. *If* (1.7) *is fulfilled for certain* $\log b_1, \log b_2$, *then* $\Phi^-(z)$ *may be represented as*

$$\Phi^-(z)=e^{\lambda z}\left\{\frac{1}{2\pi i}\int_{L_0}\mu(t)e^{-\lambda t}[\zeta(t-z)+\zeta(z)]dt+A\right\}, \quad z\in S^-, \tag{2.1}$$

where $\mu(t)\in H$ *is a real function on* L_0, *uniquely determined up to a term* $\beta_0+\mathrm{Re}\{C\omega(t)\}$ (β_0 *and* C *being respectively arbitrary real and complex constants*), *while* A *is a complex constant uniquely determined by* $\Phi^-(z)$.

Proof. Suppose that there exist $\mu(t)$ on L_0 and constant A such that (2.1) is valid. Denote the function defined by its right-hand member when $z\in S_0^+$ by $\Phi^+(z)$, which has in general a simple pole at $z=0$. Then, by Plemelj's formulas, we have

$$\Phi^\pm(t_0)=\pm\frac{1}{2}\mu(t_0)+\frac{1}{2\pi i}\int_{L_0}\mu(t)e^{-\lambda(t-t_0)}[\zeta(t-t_0)+\zeta(t_0)]dt+Ae^{\lambda t_0}, \quad t_0\in L_0. \tag{2.2}$$

By the same reasoning as in [4], we have

$$\Phi^+(t)=i(Sv)(t)+\beta_0+\mathrm{Re}\{C\omega(t)\}, \tag{2.3}$$

where S is the Schwarz operator of S_0^+ (cf. [7]): $(Sv)(z)$ is holomorphic in S_0^+ with the properties

$$\mathrm{Re}\{(Sv)(t)\}=v(t), \quad (Sv)^+(t)=(Sv)(t), \quad t\in L_0,$$

and β_0, C are constants as described in the lemma. Moreover, we may set

$$\mu(t)=\mu_0(t)+\beta_0+\mathrm{Re}\{C\omega(t)\}, \tag{2.4}$$

where

$$\mu_0(t)=i(Sv)(t)-\Phi^-(t)=-\mathrm{Im}\{(Sv)(t)\}-u(t) \tag{2.5}$$

is uniquely determined by $\Phi^-(t)$. Thus, if the representation exists, then $\mu(t)$ must be of the form (2.4). It is easy to verify the term $\beta_0+\mathrm{Re}\{C\omega(t)\}$ does not effect the value of the integral appeared.

Put

$$\Psi^-(z)=\frac{e^{\lambda z}}{2\pi i}\int_{L_0}\mu_0(t)e^{-\lambda t}[\zeta(t-z)+\zeta(z)]dt, \quad z\in S^-.$$

Substituting (2.5) into it, we readily see that

$$\Psi^-(z)=-\frac{e^{\lambda z}}{2\pi i}\int_{L_0}\Phi^-(t)e^{-\lambda t}\zeta(t-z)dt$$

$$=\Phi^-(z)-\frac{e^{\lambda z}}{2\pi i}\int_\Gamma\Phi^-(\tau)e^{-\lambda\tau}\zeta(\tau-z)d\tau, \quad z\in S_0^-,$$

where Γ is the boundary of S_0 with usual positive sense. We shall show that

$$A=e^{-\lambda z}[\Phi^-(z)-\Psi^-(z)]=\frac{1}{2\pi i}\int_\Gamma\Phi^-(\tau)e^{-\lambda\tau}\zeta(\tau-z)d\tau, \quad z\in S_0^-,$$

is actually a constant. In fact, by using the doubly periodic property of $e^{-\lambda\tau}\Phi^-(\tau)$, we may easily obtain

$$A=\frac{\eta_1}{\pi i}\int_{\gamma_2}e^{-\lambda\tau}\Phi^-(\tau)d\tau-\frac{\eta_2}{\pi i}\int_{\gamma_1}e^{-\lambda\tau}\Phi^-(\tau)d\tau, \tag{2.6}$$

where γ_1 and γ_2 are the directed line-segments from $-\omega_1-\omega_2$ to $\omega_1-\omega_2$ and from $\omega_1-\omega_2$ to $\omega_1+\omega_2$ respectively. The lemma is proved.

Case Ⅱ: $z_0 \neq 0$. In this case, we have the following lemma.

Lemma 2. *If* (1.7) *is not fulfilled for any values of* $\log b_1, \log b_2$ *whatever, then* $\Phi^-(z)$ *may be represented as*

$$\Phi^-(z)=\frac{e^{\lambda z}}{\sigma(z_0)}\frac{1}{2\pi i}\int_{L_0}\mu(t)e^{-\varkappa}\frac{\sigma(t-z+z_0)}{\sigma(t-z)}dt, \quad z\in S^-, \tag{2.7}$$

where $\mu(t)\in H$ *is a real function uniquely determined by* $\Phi^-(z)$ *up to a real constant term* β_0.

Proof. Suppose (2.7) is valid for some real $\mu(t)\in H$. Again define $\Phi^+(z)$ by its right-hand member when $z\in S_0^+$. Then we have

$$\Phi^+(t_0)=\pm\frac{1}{2}\mu(t_0)+\frac{e^{\lambda t_0}}{\sigma(t_0)}\frac{1}{2\pi i}\int_{L_0}\mu(t)e^{-\varkappa}\frac{\sigma(t-t_0+z_0)}{\sigma(t-t_0)}dt, \quad t_0\in L_0. \tag{2.8}$$

Using the same reasoning as above but noting that $\Phi^+(z)$ is now holomorphic in S_0^+, we must have, instead of (2.4),

$$\mu(t)=\mu_0(t)+\beta_0, \tag{2.9}$$

where β_0 is again a real constant and $\mu_0(t)$ is still given by (2.5). The term β_0 does not effect the value of the integral appeared in (2.7).

We have to prove

$$\Phi^-(z)=\frac{e^{\lambda z}}{\sigma(z_0)}\frac{1}{2\pi i}\int_{L_0}\mu_0(t)e^{-\varkappa}\frac{\sigma(t-z+z_0)}{\sigma(t-z)}dt, \quad z\in S^-. \tag{2.10}$$

Substituting (2.5) into it, we see that it is equivalent to

$$\Phi^-(z)=-\frac{e^{\lambda z}}{\sigma(z_0)}\frac{1}{2\pi i}\int_{L_0}\Phi^-(t)e^{-\varkappa}\frac{\sigma(t-z+z_0)}{\sigma(t-z)}dt, \quad z\in S^-.$$

When we arbitrarily fix $z\in S_0^-$, the integrand as a function of t is doubly periodic and analytic in S^- with the single simple pole $t=z$ in S_0^-. Therefore its integral taken along Γ must be equal to zero. So, by using the residue theorem in S_0^-, the above equality is valid for $z\in S_0^-$ and hence for $z\in S^-$, i.e., (2.10) is valid. The lemma is proved.

§3 The Dirichlet Problem of MQ-Functions

Let us now consider the following Dirichlet problem of MQ-functions. Given a real function $f(t)\in H$ on L_0 and two non-zero real constants β_1, β_2, we need to find an MQ-function $\Phi^-(z)$ in S^- with multipliers $b_1=\beta_1, b_2=\beta_2$, satisfying the boundary condition

$$\mathrm{Re}\{\Phi^-(t)\}=f(t), \quad t\in L_0. \tag{3.1}$$

For the case $f(t)=0$, the problem was solved in [8] by the following lemma.

Lemma 3. *The Dirichlet problem of MQ-functions, satisfying*

$$\mathrm{Re}\{\Phi^-(t)\}=0, \quad t\in L_0, \tag{3.1$'$}$$

with given real multipliers β_1, β_2 *has unique non-trivial solution* $\Phi^-(z)$ *(up to a real constant coefficient) if* $\log\beta_1, \log\beta_2$ *satisfy two real conditions provided that their values are suitably chosen, and otherwise* $\Phi^-(z)=0$ *in* S^-.

The mentioned two conditions were given by (3.4) in [8].

For the general case (3.1), two different cases are divided.

Case Ⅰ: $z_0=0$. If the problem (3.1) has a solution $\Phi^-(z)$, then it may be represented as (2.1). Then, by (2.2), we have (in the sequel, $\mu(t)$ is always replaced by $2\mu(t)$).

$$-\mu(t_0)+\mathrm{Re}\left\{\frac{1}{\pi\mathrm{i}}\int_{L_0}\mu(t)\mathrm{e}^{-\lambda(t-t_0)}[\zeta(t-t_0)+\zeta(t_0)]\mathrm{d}t\right\}+\mathrm{Re}\{A\mathrm{e}^{\lambda t_0}\}=f(t_0),\quad t_0\in L_0. \tag{3.2}$$

It is easy to see

$$k(t_0,t)=\mathrm{e}^{-\lambda(t-t_0)}[\zeta(t-t_0)+\zeta(t_0)]-\frac{1}{t-t_0}\in H. \tag{3.3}$$

Thereby (3.2) may be written as

$$\begin{aligned}\boldsymbol{K}_1\mu &\equiv \mu(t_0)-\frac{1}{\pi}\int_{L_0}\mu(t)\frac{\cos(r,n)}{r}\mathrm{d}s+\int_{L_0}k_1(t_0,t)\mu(t)\mathrm{d}s\\ &=-f(t_0)+\mathrm{Re}\{A\mathrm{e}^{\lambda t_0}\},\quad t_0\in L_0,\end{aligned} \tag{3.4}$$

where $k_1(t_0,t)=\mathrm{Re}\{\frac{1}{\pi\mathrm{i}}k(t_0,t)\frac{\mathrm{d}t}{\mathrm{d}s}\}\in H$, and

$$\frac{1}{\pi}\int_{L_0}\mu(t)\frac{\cos(r,n)}{r}\mathrm{d}s=\mathrm{Re}\left\{\frac{1}{\pi\mathrm{i}}\int_{L_0}\frac{\mu(t)}{t-t_0}\mathrm{d}t\right\}, \tag{3.5}$$

where $r=|t-t_0|,(r,n)$ is the angle between the vector $t-t_0$ and the internal normal n of L_0 at t (Cf. [9], § 61).

(3.4) is a Fredholm equation. To consider its solvability, two subcases should be considered.

1) $(3.1)_0$ has only the trivial solution $\Phi^-(z)=0$. Hence $A=0$ by (2.6). This means that $\mathbf{K}_1\mu=0$ has $\beta_0+\mathrm{Re}\{C\omega(t)\}$ as its general solution by Lemma 1.

Thus, $\boldsymbol{K}_1\mu=0$ has exactly three linearly independent (real) solutions

$$1,\ \mathrm{Re}\{\omega(t)\},\ \mathrm{Im}\{\omega(t)\}, \tag{3.6}$$

and so its adjoint equation $\boldsymbol{K}_1{}'\nu=0$ also has three such solutions $\nu_j(t)\,(j=1,2,3)$ exactly and the equation (3.4) is solvable iff the following conditions are satisfied:

$$\mathrm{Re}\left\{A\int_{L_0}\mathrm{e}^{\lambda t}\nu_j(t)\mathrm{d}s\right\}=f_j\left(=\int_{L_0}f(t)\nu_j(t)\mathrm{d}s\right),\quad j=1,2,3. \tag{3.7}$$

In this subcase, we also see that (3.4) is unsolvable if $f(t)=0, A=1$. Denote

$$\int_{L_j}\mathrm{e}^{\lambda t}\nu_j(t)\mathrm{d}s=I_j+\mathrm{i}J_j,\quad j=1,2,3. \tag{3.8}$$

Therefore, $I_j, j=1,2,3$, could not be all zero, say $I_3\neq 0$. Then we may regularize $\nu_j(t)$ such that

$$I_1=I_2=0,\quad I_3=1.$$

Denote $A=\alpha+\mathrm{i}\beta$, then (3.6) becomes

$$\beta J_j=-f_j,\ j=1,2;\quad \alpha=f_3+\beta J_3. \tag{3.9}$$

J_1 and J_2 could not be both zero, since otherwise β may be arbitrary in case $f(t)=0$ so that $f_j=0\,(j=1,2,3)$ and hence $A=\alpha+\mathrm{i}\beta\neq 0$, which is a contradiction. Thus, (3.9) means a single condition of solvability and α,β are uniquely determined.

It is convenient to define the generalized degree of (real) freedom of a non-homogeneous linear problem as the difference $r=l-m$ between the number l of the arbitrary (real) constants in its general solution and the number m of the (real) conditions of solvability. Hence, for the problem discussed here, in this subcase, $r=-1(l=0,m=1)$.

2) $(3.1)_0$ has a unique non-trivial solution $\Phi_0^-(z)$. Let the constant attached to $\Phi_0^-(z)$ be $A=A_0$.

(i) $A_0\neq 0$. Then the equation $\boldsymbol{K}_1\mu=\operatorname{Re}\{Ae^{\varkappa}\}$ is solvable iff $A=A_0(\neq 0)$ and so $\boldsymbol{K}_1\mu=0$ again has exactly three linearly independent solutions (3.6). We may regularize $\nu_j(t)$ as above. The conditions of solvability for (3.4) are still (3.9) for $A=\alpha+\mathrm{i}\beta$. Here we must have $J_1=J_2=0$, for, if on the contrary, $(3.1)_0$ would have a solution $\Phi_0^-(z)$ with $A_0=0$. Hence, (3.9) reduces to two conditions of solvability $f_1=f_2=0$, β may be arbitrary and α is uniquely determined by β:

$$A=f_3+\beta(J_3+\mathrm{i}).$$

The general solution of (3.1) is then

$$\Phi^-(z)=\beta\,\Phi_0^-(z)+\Phi_1^-(z),$$

where β is an arbitrary real constant and $\Phi_1^-(z)$ is a particular solution corresponding to the solution $\mu=\mu_1(t)$ of $\boldsymbol{K}_1\mu=f(t)+\operatorname{Re}\{f_3e^{\varkappa}\}$.

Thus, in this case, $r=-1(l=1,m=2)$.

(ii) $A_0=0$. In this case, besides (3.6), there exists another linearly independent solution $\mu_0(t)$ of $\boldsymbol{K}_1\mu=0$, which gives out a solution $\Phi_0^-(z)\neq 0$ of (3.1). Then $\boldsymbol{K}_1{}'\nu=0$ has four linearly independent solutions $\nu_j(t)(j=0,1,2,3)$. Again define I_j,J_j as in (3.8) but with $j=0$ added. The conditions of solvability of (3.4) are still given by (3.7) but with $j=0,1,2,3$. We show that now I_j, $j=0,1,2,3$, could not be all zero. When $f(t)=0$, (3.7) becomes

$$\alpha I_j-\beta J_j=0,\quad j=0,1,2,3,\tag{3.10}$$

and $\boldsymbol{K}_1\mu=\operatorname{Re}\{Ae^{\varkappa}\}$ is solvable iff $A=0$, i.e., $\alpha=\beta=0$. However, if $I_j=0$, $j=0,1,2,3$, then (3.10) would have solutions $\beta=0$ with α arbitrary, which is a contradiction. Thus, we may regularize $\nu_j(t)$ as before such that

$$I_j=0,\ j=0,1,2;\quad I_3=1,$$

and (3.7) becomes

$$J_j=-f_j,\ j=0,1,2;\quad \alpha_3=f_3+\beta J_3.\tag{3.11}$$

In a similar manner, we also know that J_j, $j=0,1,2$ could not be all zero, then (3.11) reduces to two conditions of solvability and $A=\alpha+\mathrm{i}\beta$ are uniquely determined. When they are satisfied, we may get the general solution of (3.1):

$$\Phi^-(z)=D\Phi_0^-(z)+\Phi_1^-(z),\tag{3.12}$$

where D is an arbitrary real constant and $\Phi_1^-(z)$ is a particular solution of (3.1) corresponding to a particular solution $\mu_1(t)$ of (3.4).

In this case, we still have $r=-1$ $(l=1,m=2)$.

Case II: $\varkappa_0\neq 0$. If the problem is solvable, by using (2.7) and (2.8) ($\mu(t)$ is again

replaced by $2\mu(t)$), we obtain

$$-\mu(t_0)+\mathrm{Re}\left\{\frac{e^{\lambda t_0}}{\sigma(z_0)}\frac{1}{\pi i}\int_{L_0}\mu(t)e^{-\lambda t}\frac{\sigma(t-t_0+z_0)}{\sigma(t-t_0)}dt\right\}=f(t_0),\quad t_0\in L_0. \tag{3.13}$$

Note that we may easily prove

$$\frac{e^{-\lambda(t-t_0)}}{\sigma(z_0)}\frac{\sigma(t-t_0+z_0)}{\sigma(t-t_0)}-\frac{1}{t-t_0}\in H.$$

Hence, as above, (3.13) is a Fredholm equation of the form

$$\begin{aligned}K_2\mu&\equiv\mu(t_0)-\frac{1}{\pi}\int_{L_0}\mu(t)\frac{\cos(r,n)}{r}ds+\int_{L_0}k_2(t_0,t)\mu(t)ds\\&=-f(t_0),\quad t_0\in L_0,\end{aligned} \tag{3.14}$$

where $k_2(t_0,t)\in H$.

On the analogy of the previous case, we know that $K_2\mu=0$ either has only one linearly independent solution 1 or has another such solution $\mu_0(t)$ according as $(3.1)_0$ has only trivial solution or a nontrivial solution $\Phi_0^-(z)$.

In the first case, $K_2'\nu=0$ has also only one solution $\nu(t)$ and (3.14) is solvable iff

$$\int_{L_0}f(t)\nu(t)ds=0. \tag{3.15}$$

If it is satisfied, then $\Phi^-(z)$ is uniquely determined in spite of $\mu(t)$ is determined up to a real constant term β_0. The generalized degree of freedom of the solutions of the problem in this case is also $r=-1$ ($l=0$, $m=1$).

If $K_2\mu=0$ has two solutions 1 and $\mu_0(t)$, then $K_2'\nu=0$ also has two solutions $\nu_1(t)$, $\nu_2(t)$. Then (3.14) is solvable iff

$$f_j=\int_{L_0}f(t)\nu_j(t)ds=0,\quad j=1,2. \tag{3.16}$$

If they are satisfied, then (3.14) has a particular solution $\mu_1(t)$ which corresponds to a solution $\Phi_1(z)$ of (3.1). Its general solution is again given in the form of (3.12). In this case, $r=-1$ ($l=1$, $m=2$).

Hence we obtain the following theorem.

Theorem 1. *The generalized degree of freedom of solutions for the Dirichlet problem of MQ-functions is -1 with at most one arbitrary real constant in its general solution.*

§ 4 Method of MQ-Regularization

Before we study the DH problem, we shall solve the problem of MQ-regularization. That is, given $\gamma(t)=a(t)+ib(t)\in H$, $\neq 0$ on L_0, with the index

$$\kappa=\frac{1}{2\pi}[\arg\gamma(t)]_{L_0}, \tag{4.1}$$

we need to find a real function $p(t)\in H$ on L_0 such that

$$\psi^-(t)=p(t)\gamma(t),\quad t\in L_0, \tag{4.2}$$

is the boundary value of an MQ-function $\psi^-(z)$ in S^- with certain real multipliers β_1,β_2.

But now we allow $\psi^-(z)$ may have some poles in S^-. $p(t)$ is called the factor of MQ-regularization of $\gamma(t)$.

Without loss of generality, we may assume $|\gamma(t)|=1$ and write $\gamma(t)=e^{i\theta(t)}$, where $\theta(t)$ is multi-valued unless $\kappa=\frac{1}{2\pi}[\theta(t)]_{L_0}=0$.

We discuss the following cases for different values of κ:

1° $\kappa=0$. $\theta(t)\in H$ is single-valued in this case. Let us solve the Dirichlet problem of doubly quasi-periodic holomorphic function $\Omega^-(z)$ in addition in S^- satisfying

$$\operatorname{Re}\{-i\,\Omega^-(t)\}=\theta(t), \quad t\in L_0, \tag{4.3}$$

with real addenda α_j:

$$\Omega^-(z+2\omega_j)=\Omega^-(z)+\alpha_j, \quad j=1,2. \tag{4.4}$$

It is known from [5] that this problem has a unique solution $-i\,\Omega^-(z)=U(z)+iV(z)$, where $U(t)=\theta(t)$. Put $\psi^-(z)=e^{\Omega^-(z)}$. Then $\psi^-(z)$ is a holomorphic MQ-function in S^- with real multipliers $\beta_j=e^{\alpha_j}(j=1,2)$ and $\psi^-(t)=e^{-V(t)+i\theta(t)}$ on L_0. The factor of MQ-regularization is $p(t)=e^{-V(t)}\in H$. Here, α_j and hence β_j are uniquely determined.

We note that $\psi^-(z)$ in this case has neither zeros nor poles in S^-.

2° $\kappa\geqslant 2$. Let

$$h_\kappa(z)=\Pi(z)/\sigma^\kappa(z), \quad \Pi(z)=\prod_{j=1}^{\kappa}\sigma(z-z_j), \tag{4.5}$$

where $z_1,\ldots,z_\kappa\in S_0^-$ are fixed and arbitrary but $\sum_{k=1}^{\kappa} z_k\neq 0$. $h_\kappa(z)$ is an elliptic function of order κ with simple zeros $z_1,\ldots,z_\kappa$ and without poles in S_0^-. Its index on L_0 is $-\kappa$. Let $\delta(t)=\arg h_\kappa(t)$. Then

$$\theta_0(t)=\theta(t)+\delta(t) \tag{4.6}$$

is single-valued on L_0. Constructing the factor of MQ-regularization $p_0(t)$ of $e^{i\theta_0(t)}$ as in 1°, we obtain a holomorphic MQ-function $\psi_0^-(z)$ in S^- with real multipliers β_1,β_2, satisfying

$$\psi_0^-(t)=p_0(t)e^{i\theta_0(t)}, \quad t\in L_0. \tag{4.7}$$

Then

$$\psi^-(z)=\psi_0^-(z)/h_\kappa(z) \tag{4.8}$$

is a meromorphic MQ-function with the same multipliers and

$$\psi^-(t)=\frac{p_0(t)e^{i\theta_0(t)}}{h_\kappa(t)}=\frac{p_0(t)}{|h_\kappa(t)|}e^{i\theta(t)}.$$

Hence $p(t)=p_0(t)/h_\kappa(t)$ is a factor of MQ-regularization of $\gamma(t)$.

We note that, in this case, $\psi^-(z)\neq 0$ has exactly κ simple poles $z_1,\ldots,z_\kappa$ in S_0^-.

3° $\kappa=1$. In place of $h_\kappa(t)$ given by (4.5), we take

$$h_1(z)=\frac{\sigma(z-c_1)\sigma(z-z_1)}{\sigma(z)\sigma(z-c_2)}, \tag{4.9}$$

where $c_1,c_2\in S_0^+$ and $z_1=c_2-c_1\in S_0^-$ are fixed and arbitrarily chosen. The remaining discussions are the same as in 2°. Now $\psi^-(z)\neq 0$ has a single simple pole z_1 in S_0^-.

4° $\kappa\leqslant -2$. In this case, we may take $h_\kappa(z)=1/h_{-\kappa}(z)$, where $h_{-\kappa}(z)$ is defined by

(4.5). Then we obtain an MQ-function $\psi^-(z)$, holomorphic in S^-, with simple zeros $z_1, \ldots, z_{-\kappa}$ in S_0^-.

5° $\kappa=-1$. Taking $h_1(z)=1/h_{-1}(z)$ where $h_1(z)$ is given by (4.9), we get an MQ-function $\psi^-(z)$ holomorphic in S^- with a simple zero z_1 in S_0^-.

Thus, we obtain the following theorem.

Theorem 2. *For given $\gamma(t)\in H$, $\neq 0$ on L_0 with index κ, a factor of MQ-regularization $p(t)$ with real multipliers in S^- always exists. The result MQ-function $\psi^-(z)$ has neither zeros nor poles in S_0^- if $\kappa=0$, has κ simple poles and no zeros if $\kappa>0$, has $-\kappa$ simple zeros and no poles if $\kappa<0$.*

Remark. We did not attempt to obtain the general solution of the problem of MQ-regularization which is of no use for our purpose. However, if the locations of the possibly appeared zeros and poles of $\psi^-(z)$ in S_0^- are prescribed, it is not difficult to get it.

§ 5 The DH Problem

It is easy now to solve the DH problem on the basis of the foregoing discussions. Given $\gamma(t)$ as in § 4, we need to find a doubly periodic holomorphic function $\psi^-(z)$ in S^- such that

$$\operatorname{Re}\{\gamma(t)\Psi^-(t)\}=f(t), \quad t\in L_0, \tag{5.1}$$

where $f(t)\in H$ is a given function on L_0. As usual, we call (4.1) the index of the problem (5.1).

According to the results in § 4, we may construct a factor of MQ-regularization $p(t)$ for $\gamma(t)$ such that $\psi^-(t)=p(t)\gamma(t)$ is the boundary value of an MQ-function $\psi^-(z)$ in S^-, having poles or zeros according as $\kappa>0$ or $\kappa<0$ as described in Theorem 2, where

$$\operatorname{Ind}_{L_0}\gamma(t)=\kappa.$$

Put $\Phi^-(z)=\psi^-(z)\Psi^-(z)$. Then the problem (5.1) is reduced to the Dirichlet problem of MQ-function $\Phi^-(z)$ with known real multipliers:

$$\operatorname{Re}\{\Phi^-(t)\}=p(t)f(t), \quad t\in L_0. \tag{5.2}$$

The multipliers β_1, β_2 of $\Phi^-(z)$ are the same as those of $\psi^-(z)$.

Case Ⅰ: $z_0=0$. Consider different cases of κ.

1° $\kappa=0$. Using the corresponding results in § 3, we have the conditions (3.9) or (3.11) of solvability and for determining α, β with

$$f_j=\int_{L_0} p(t)f(t)\nu_j(t)\mathrm{d}s, \tag{5.3}$$

$j=1,2,3$, or $j=0,1,2,3$ for different cases. When they are satisfied, $\psi^-(z)=\Phi^-(z)/\kappa^-(z)$ is the general solution of (5.1), where $\Phi^-(z)$ is the general solution of (5.2). From the discussions made in § 3, the generalized degree of freedom of the DH problem is $r=-1$ and the number of arbitrary constants in its general solution is $l\leqslant 1$.

2° $\kappa<0$. The situation is the same as in 1° except that $\Phi^-(z)$ ought to have $-\kappa$ ze-

ros at $z_1, \dots, z_{-\kappa}$ in S_0^-. Thereby, besides (3.9) or (3.11), the following conditions of solvability

$$\int_{L_0} \mu(t) e^{-\chi} [\zeta(t-z_k)+\zeta(z_k)] dt + A = 0, \quad k=1,\dots,-\kappa, \tag{5.4}$$

must be added, where $\mu(t)$, by (3.4), is any solution of

$$K_1\mu = -p(t_0)f(t_0) + \mathrm{Re}\{Ae^{\chi_0}\}, \quad t_0 \in L_0.$$

(5.4) consists of -2κ real conditions total in number.

By the discussions in § 3, we have, in this case, $r=2\kappa-1$, $l\leqslant 1$.

3° $\kappa>0$. The corresponding MQ-function $\Phi^-(z)$ may have simple poles at $z_1, \dots, z_\kappa$ in S_0^-. Let

$$\chi_k(z) = e^{\lambda z}[\zeta(z-z_k)+\zeta(z)], \quad k=1,\dots,\kappa, \tag{5.5}$$

which is an MQ-function in S^- with single simple pole z_k in S_0^- and with the same multipliers as those of $\Phi^-(z)$. Therefore, in this case, $\Phi^-(z)$ may be represented as

$$\Phi^-(z) = e^{\lambda z}\left\{\frac{1}{\pi i}\int_{L_0} \mu(t) e^{-\chi}[\zeta(t-z)+\zeta(z)]dt + A\right\} + \sum_{k=1}^{\kappa} C_k \chi_k(z), \quad z\in S^-, \tag{5.6}$$

where $C_k = \gamma_k + i\,\delta_k$, $k=1,\dots,\kappa$, are arbitrary complex constants, and $\mu(t)$ is any solution of

$$K_1\mu = -p(t_0)f(t_0) - \mathrm{Re}\{Ae^{\chi_0}\} - \sum_{k=1}^{\kappa}\mathrm{Re}\{C_k\chi_k(t_0)\}, \quad t_0\in L_0. \tag{5.7}$$

Then the conditions (3.9) or (3.11) become now

$$\begin{cases} -\beta J_j + \sum\limits_{k=1}^{\kappa}\mathrm{Re}\left\{C_k\int_{L_0}\chi_k(t)\nu_j(t)ds\right\} = -f_j, \quad j=1,2 \text{ or } j=0,1,2; \\ \alpha+\beta J_3 + \sum\limits_{k=1}^{\kappa}\mathrm{Re}\left\{C_k\int_{L_0}\chi_k(t)\nu_3(t)ds\right\} = -f_3, \end{cases} \tag{5.8}$$

which contain $2\kappa+2$ real constants $\gamma_k, \delta_k (k=1,\dots,\kappa)$ in addition to α, β. Hence, in this case, we have, $r=2\kappa-1$, $l\leqslant 2\kappa+1$.

Case Ⅱ: $z_0\neq 0$. The discussions are only a little different from Casc Ⅰ and will be stated briefly.

1° $\kappa=0$. The conditions of solvability (3.15) or (3.16) now become

$$\int_{L_0} p(t)f(t)\nu(t)ds = 0 \tag{5.9}$$

or

$$f_j = \int_{L_0} p(t)f(t)\nu_j(t)ds = 0, \quad j=1,2, \tag{5.9$'$}$$

respectively. According to the results in § 3, we have, in this case, $r=-1$, $l\leqslant 1$.

2° $\kappa<0$. Besides (5.9) or (5.9)$'$, we have additional conditions of solvability

$$\int_{L_0}\mu(t)e^{-\chi}\frac{\sigma(t-z_k+z_0)}{\sigma(t-z_k)}dt = 0, \quad k=1,\dots,-\kappa, \tag{5.10}$$

where $\mu(t)$ is any solution of $K_2\mu = -p(t_0)f(t_0)$. Here, we have $r=2\kappa-1$, $l\leqslant 1$.

3° $\kappa>0$. Now we should solve the integral equation

$$\boldsymbol{K}_2\mu=-p(t_0)f(t_0)+\sum_{k=1}^{\kappa}\mathrm{Re}\{C_kq(t_0)\chi_k(t_0)\},\quad t_0\in L_0,\tag{5.11}$$

where $q(z)$ is given by (1.6). The conditions of solvability (3.15) or (3.16) become

$$\sum_{k=1}^{\kappa}\mathrm{Re}\left\{\int_{L_0}C_kq(t)\chi_k(t)\mathrm{d}s\right\}=\int_{L_0}p(t)f(t)\nu(t)\mathrm{d}s,\tag{5.12}$$

or

$$\sum_{k=1}^{\kappa}\mathrm{Re}\left\{\int_{L_0}C_kq(t)\chi_k(t)\mathrm{d}s\right\}=\int_{L_0}p(t)f(t)\nu_j(t)\mathrm{d}s,\quad j=1,2.\tag{5.12$'$}$$

Here $r=2\kappa-1$, $l\leqslant 2\kappa+1$.

In conclusion, for any case whatever, we have the following theorem.

Theorem 3. *If the index of the* DH*problem* (5.1) *is* κ, *then its generalized degree of freedom is* $2\kappa-1$, *and when the conditions of its solvability are satisfied, the number of arbitrary (real) constants in its general solution is not greater than* $2\kappa+1$ *if* $\kappa\geqslant 0$ *and 1 if* $\kappa<0$.

We mention that our method of solution is constructive, which reduces the problem to solving certain definite Fredholm integral equation.

Remark. Of course we may formulate the Hilbert problem (5.1) for MQ-functions. Since the process of MQ-regularization is the same as that given in § 4 and the quasi-periodicities may be united together, there is nothing new to be discussed. We may also formulate the Hilbert problem for doubly quasi-periodic analytic functions in addition. However, it may be easily solved by combining the method used here and that in § 5.

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Some Classes of Boundary Value Problems and Singular Integral Equations with a Transformation*

Abstract. In this paper, we consider mainly certain singular integral equations with a transformation. In spite of the classical method of regularization for solution, we are to give a new sectionally jumping method that they may be reduced to simpler Fredholm integral equations which can be systematically solved and ready to be discussed.

Key Words. Riemann boundary value problems; singular integral equations; transformation

0. Introduction

Boundary value problems and singular integral equations with shift for analytic functions were investigated rather completely, when the shift transforms the closed boundary contour or the path of integration homeomorphically to itself. [1] In case of a singular integral equation with translation, when the path of integration is an infinite straight line, some works were made under the restriction that all the given coefficients in the equation turn out to be constants[2,3].

In this paper, singular integral equations with a transformation as given in (2.1) below are mainly considered. Certain kinds of Riemann boundary value problems with a transformation are merely presented and so formulated that they can be reduced as problems to Fredholm integral equations. Though it is known that the given equation may be solved by the method of regularization, yet its corresponding kernel is one with a Cauchy principal value integral, which is complicated and painstaking for further discussions. We are to give a new sectionally jumping method described below that the given equation may be reduced to simpler Fredholm integral equation which can be systematically solved and ready to be discussed.

1. Riemann boundary value problems with a transformation

Let L be a smooth closed contour in the complex plane oriented counter-clockwisely,

* Supported by the NSFC.

and the interior and the exterior regions bounded by L be denoted by S^+ and S^- respectively. Assume $O\in S^+$, transformation $\tau=\alpha(t)$, a continuous function on L, and the image $L'=\alpha(L)$ having no point in common with L. Accordingly, L' belongs entirely to S^+ or S^-. Our problem is to find a sectionally holomorphic function $\Phi(z)$ with jumps on L, satisfying

$$\Phi^+(t)=G(t)\Phi^-(t)+\lambda H(t)\Phi(\alpha(t))+g(t), \quad t\in L, \tag{1.1}$$

where $G(t), H(t), g(t)$ are given functions Hölder continuous on L (class H). For definiteness, $\Phi(\infty)=0$ is required.

Moreover, $G(t)\neq 0$ is assumed all over L. As usual, the problem is then said to be of normal type and

$$\kappa=\mathrm{Ind}_L G(t)=\frac{1}{2\pi}[\arg G(t)]_L,$$

the index.

To solve (1.1) we may set $\varphi(t)=\Phi^+(t)-\Phi^-(t)$ or directly $\Phi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)}{\tau-z}d\tau$, so that we reduce (1.1) to a singular integral equation in $\varphi(t)$ of normal type. With the intention of having a further reduction to a simpler Fredholm integral equation, we proceed as follows.

Construct the canonical function $X(z)$ with respect to $G(t)$[4]:

$$X(z)=\begin{cases} e^{\Gamma(z)}, & z\in S^+, \\ z^{-\kappa}e^{\Gamma(z)}, & z\in S^-, \end{cases}$$

where

$$\Gamma(z)=\frac{1}{2\pi i}\int_L\frac{\log[G(t)t^{-\kappa}]}{t-z}dt, \quad z\notin L,$$

$X(z)$ is not zero everywhere (including its limiting values on either side of L) and is of order κ at infinity, satisfying

$$G(t)=\frac{X^+(t)}{X^-(t)}.$$

Then, (1.1) may be written as

$$\frac{\Phi^+(t)}{X^+(t)}=\frac{\Phi^-(t)}{X^-(t)}+\frac{\lambda H(t)\Phi(\alpha(t))}{X^+(t)}+\frac{g(t)}{X^+(t)}.$$

If we denote $\Psi(z)=\Phi(z)/X(z)$ which is of order $\kappa-1$ at most at ∞, then (1.1) is transferred to an equivalent jump problem with a transformation $\alpha(t)$:

$$\Psi^+(t)=\Psi^-(t)+\lambda H_1(t)\Psi(\alpha(t))+g_1(t), \quad t\in L, \tag{1.2}$$

where

$$H_1(t)=\frac{H(t)X(\alpha(t))}{X^+(t)}, \tag{1.3}$$

$$g_1(t)=\frac{g(t)}{X^+(t)}. \tag{1.4}$$

Let

$$\psi(t)=\Psi^+(t)-\Psi^-(t). \tag{1.5}$$

Then,

$$\Psi(z)=\frac{1}{2\pi i}\int_L\frac{\psi(\tau)}{\tau-z}d\tau+P_{\kappa-1}(z), \tag{1.6}$$

where $P_{\kappa-1}(z)$ is an arbitrary polynomial of degree $\kappa-1$ (identical to zero if $\kappa\leqslant 0$).

Some different cases are separately treated:

1° $\kappa=0$. By (1.2), we have a Fredholm equation in $\psi(t)$:

$$K\psi\equiv\psi(t)-\frac{\lambda H_1(t)}{2\pi i}\int_L\frac{\psi(\tau)}{\tau-\alpha(t)}d\tau=g_1(t), \quad t\in L. \tag{1.7}$$

If (1.2) has a solution, then $\psi(t)$ given by (1.5) is a solution of (1.7). Conversely, if (1.7) has a solution $\psi(t)$, then a solution of (1.2) is obtained through (1.6) (with $P_{\kappa-1}\equiv 0$).

Let λ be an eigenvalue of (1.7). Assume the equation which is adjoint to (1.7)

$$\omega(t)+\frac{\lambda}{2\pi i}\int_L\frac{H_1(\tau)\omega(\tau)}{\alpha(\tau)-t}d\tau=0 \tag{1.8}$$

has l linearly independent solutions $\omega_1(t),\dots,\omega_l(t)$. Then the condition of solvability for (1.7) is

$$\int_L\omega_j(t)g_1(t)dt=0, \quad j=1,\dots,l, \tag{1.9}$$

while the general solution of (1.7) is

$$\psi(t)=\psi_*(t)+\sum_{j=1}^{l}c_j\psi_j(t), \tag{1.10}$$

where $\{\psi_j(t)\}_1^l$ is the complete system of solutions for the homogeneous equation corresponding to (1.7), $c_1,\dots,c_l$ being arbitrary constants, and $\psi_*(t)$ is its particular solution. Therefore, the general solution of (1.1) is

$$\Phi(z)=X(z)\left\{\frac{1}{2\pi i}\int_L\frac{\psi_*(\tau)}{\tau-z}d\tau+\sum_{j=1}^{l}\frac{c_j}{2\pi i}\int_L\frac{\psi_j(\tau)}{\tau-z}d\tau\right\}. \tag{1.11}$$

If λ is not an eigenvalue of (1.7), then it has a unique solution $\psi(t)$ and hence problem (1.2) has also a unique solution

$$\Phi(z)=\frac{X(z)}{2\pi i}\int_L\frac{\psi(\tau)}{\tau-z}d\tau. \tag{1.12}$$

Therefore the generalized degree of freedom of the solution is 0 (for terminology, Cf. [5]).

2° $\kappa<0$. In this case, $\Psi(z)$ has a zero point of order $-\kappa-1$ at least at infinity. Therefore, besides $P_{\kappa-1}\equiv 0$ in (1.6), the following condition must be fulfilled:

$$\int_L\psi(\tau)\tau^k d\tau=0, \quad k=0,\dots,-\kappa-1. \tag{1.13}$$

If λ is not an eigenvalue of (1.7), then it has a unique solution $\psi(t)$. Therefore, the original problem has a unique solution (1.12) if (1.13) is satisfied.

If λ is an eigenvalue, then, when the condition of solvability (1.13) for (1.7) is fulfilled, $\psi(t)$ is found by (1.10). By substitution, (1.13) becomes

$$\sum_{j=1}^{l}c_j\int_L\psi_j(\tau)\tau^k d\tau=-\int_L\psi_*(\tau)\tau^k d\tau, \quad k=0,\dots,-\kappa-1, \tag{1.14}$$

which is a system of $-\kappa$ linear equations in $\{c_j\}_1^l$. Hence, the condition of solvability for the original problem is (1.9) together with the consistency of (1.14). After $\{c_j\}_1^l$ has been obtained, we get the general solution (1.11) for the original problem. If the rank of the coefficient matrix of the consistent system (1.14) is r, then (1.14) actually consists of $-\kappa-r$ equations of constraints and so $l-r$ constants in it may be arbitrarily taken. That is to say, there are $l-\kappa-r$ independent equations in the condition of solvability for (1.1) and its general solution contains $l-r$ arbitrary constants. Therefore, the generalized degree of freedom of the solution is κ.

3° $\kappa>0$. In this case, in place of (1.7), we have the Fredholm equation

$$\boldsymbol{K}\psi=\lambda H_1(t)P_{\kappa-1}(t)+g_1(t),\quad t\in L. \tag{1.15}$$

If λ is not an eigenvalue of $\boldsymbol{K}$, then the general solution of (1.15) is

$$\psi(t)=\psi_*(t)+\sum_{k=0}^{\kappa-1}d_k\chi_k(t) \tag{1.16}$$

which contains κ arbitrary constants, where $\boldsymbol{K}\psi_*=g_1$, $\boldsymbol{K}\chi_k=\lambda H_1(t)t^k$ $(k=0,\ldots,\kappa-1)$. Therefore, the general solution of (1.1) is

$$\Phi(z)=X(z)\left\{\frac{1}{2\pi\mathrm{i}}\int_L\frac{\psi_*(\tau)}{\tau-z}\mathrm{d}\tau+\sum_{k=0}^{\kappa-1}\frac{d_k}{2\pi\mathrm{i}}\int_L\frac{\chi_k(\tau)}{\tau-z}\mathrm{d}\tau\right\}. \tag{1.17}$$

If λ is an eigenvalue of $\boldsymbol{K}$, then the condition of solvability for (1.15) becomes

$$\int_L\omega_j(t)\{\lambda H_1(t)P_{\kappa-1}(t)+g_1(t)\}\mathrm{d}t=0,\quad j=1,\ldots,l,$$

or, what is the same,

$$\lambda\sum_{k=0}^{\kappa-1}d_k\int_L\omega_j(t)H_1(t)t^k\mathrm{d}t=-\int_L\omega_j(t)g_1(t)\mathrm{d}t,\quad j=1,\ldots,l, \tag{1.18}$$

if this is consistent, the general solution of (1.15) is

$$\psi(t)=\psi_*(t)+\sum_{j=1}^{l}c_j\psi_j(t)+\sum_{k=0}^{\kappa-1}d_k\chi_k(t). \tag{1.19}$$

Therefore, the general solution of the original problem is

$$\Phi(z)=X(z)\left\{\frac{1}{2\pi\mathrm{i}}\int_L\frac{\psi_*(\tau)}{\tau-z}\mathrm{d}\tau+\sum_{j=1}^{l}\frac{c_j}{2\pi\mathrm{i}}\int_L\frac{\psi_j(\tau)}{\tau-z}\mathrm{d}\tau+\sum_{k=0}^{\kappa-1}\frac{d_k}{2\pi\mathrm{i}}\int_L\frac{\chi_k(\tau)}{\tau-z}\mathrm{d}\tau\right.$$
$$\left.+d_0+\cdots+d_{\kappa-1}z^{\kappa-1}\right\}, \tag{1.20}$$

where $\{d_k\}_0^{\kappa-1}$ is any system of solutions for (1.18).

If (1.18) is consistent and the rank of its coefficient matrix is r, then there are $\kappa-r$ independent constants in $\{d_k\}$, which means there are actually $l-r$ independent constraints in (1.18). Therefore (1.20) contains $l+\kappa-r$ independent arbitrary constants and hence the generalized freedom of solution is again κ.

2. Application to solution for singular integral equations with a transformation

Using the previous results, we may solve the following equation:

$$a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{\lambda c(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-\alpha(t)}d\tau=f(t),\quad t\in L, \tag{2.1}$$

where $a,b,c,f\in H$ are given on L while $\alpha(t)$ is as before. Assume $a(t)\pm b(t)\neq 0$ on L (normal type).

As usual, let

$$\Phi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)}{\tau-z}d\tau,\quad z\notin L. \tag{2.2}$$

Then $\Phi(\infty)=0$ and equation (2.1) is transferred to problem (1.1), where

$$G(t)=\frac{a(t)-b(t)}{a(t)+b(t)},\quad H(t)=-\frac{c(t)}{a(t)+b(t)},\quad g(t)=\frac{f(t)}{a(t)+b(t)}. \tag{2.3}$$

(2.1) is solvable iff (1.1) is solvable and

$$\varphi(t)=\Phi^+(t)-\Phi^-(t). \tag{2.4}$$

It is easy to write out the concrete expression of $\varphi(t)$ in all different cases. For example, when $\kappa=0$, if λ is not an eigenvalue, then

$$\varphi(t)=\frac{1}{2}[X^+(t)-X^-(t)]\psi(t)+\frac{X^+(t)+X^-(t)}{2\pi i}\int_L\frac{\psi(\tau)}{\tau-t}d\tau; \tag{2.5}$$

if λ is an eigenvalue, then the condition of solvability is

$$\int_L\omega_j(\tau)\frac{f(\tau)d\tau}{[a(\tau)+b(\tau)]X^+(\tau)}=0,\quad j=1,\dots,l, \tag{2.6}$$

and its general solution is

$$\varphi(t)=\frac{1}{2}[X^+(t)-X^-(t)][\psi_*(t)+\sum_{j=1}^{l}c_j\psi_j(t)]$$
$$+\frac{X^+(t)+X^-(t)}{2\pi i}\left\{\int_L\frac{\psi_*(\tau)}{\tau-t}d\tau+\sum_{j=1}^{l}c_j\int_L\frac{\psi_j(\tau)}{\tau-t}d\tau\right\}. \tag{2.7}$$

The condition of solvability and the expression of the solution may be similarly written out when $\kappa>0$ or <0.

3. A particular case

Let us consider a particular case of (1.1). Assume $\alpha(L)\subset S^+$ and $\alpha(t), H(t)$ are respectively boundary values of certain functions $\alpha(z), H(z)$, both holomorphic in S^+: $\alpha(t)=\alpha^+(t), H(t)=H^+(t)$. In this special case, by (1.3), $H_1(t)=H_1^+(t)$ is the boundary value of the function

$$H_1(z)=\frac{H(z)X(\alpha(z))}{X(z)} \tag{3.1}$$

holomorphic in S^+. After (1.2) has been obtained, we may simply proceed as follows. If put

$$\Omega(z)=\begin{cases}\Psi(z)-\lambda H_1(z)\Psi(\alpha(z)), & z\in S^+,\\ \Psi(z), & z\in S^-,\end{cases}$$

then $\Omega^+(t)-\Omega^-(t)=g_1(t)$.

1° Assume $\kappa=0$. Then

$$\Omega(z)=\frac{1}{2\pi\,\mathrm{i}}\int_L\frac{g_1(\tau)}{\tau-z}\mathrm{d}\tau=\frac{1}{2\pi\,\mathrm{i}}\int_L\frac{g(\tau)}{X^+(\tau)(\tau-z)}\mathrm{d}\tau,\quad z\notin L,\tag{3.2}$$

and so

$$\Psi(z)=\frac{1}{2\pi\,\mathrm{i}}\int_L\frac{g(\tau)}{X^+(\tau)(\tau-z)}\mathrm{d}\tau,\quad z\in S^-,\tag{3.3}$$

while

$$\Psi(z)=\Omega(z)+\lambda H_1(z)\Psi(\alpha(z)),\quad z\in S^+,\tag{3.4}$$

or, simply written as

$$\Psi=\Omega+\lambda H_1\Psi^\alpha\quad(z\in S^+).\tag{3.4$'$}$$

By iteration, it gives rise to $\Psi^\alpha=\Omega^\alpha+\lambda H_1^\alpha\Psi^{\alpha^2}$ by which it follows that

$$\Psi=\Omega+\lambda H_1(\Omega^\alpha+\lambda H_1^\alpha\Psi^{\alpha^2})=\Omega+\lambda H_1\Omega^\alpha+\lambda^2H_1H_1^\alpha\Psi^{\alpha^2}.$$

Iterating once more, we have

$$\Psi=\Omega+\lambda H_1\Omega^\alpha+\lambda^2H_1H_1^\alpha\Omega^{\alpha^2}+\lambda^3H_1H_1^\alpha H_1^{\alpha^2}\Psi^{\alpha^3}.$$

Repeating the iteration process again and again, we have in general

$$\Psi=\Omega+\sum_{k=1}^{n}\lambda^kH_1H_1^\alpha\cdots H_1^{\alpha^{k-1}}\Omega^{\alpha^k}+\lambda^{n+1}H_1H_1^\alpha\cdots H_1^{\alpha^n}\Psi^{\alpha^{n+1}},\quad n=1,2,\ldots.\tag{3.5}$$

When $|\lambda H_1|<1$, i.e.,

$$|\lambda|<1/\max_{z\in\bar S^+}\{|H_1(z)|\},$$

or, more generally, for certain k,

$$|\lambda|<1/\max_{z\in\bar S^+}|H_1(\alpha^k(z))|,\tag{3.6}$$

we know that

$$\Psi(z)=\Omega(z)+\sum_{k=1}^{\infty}\lambda^kH_1(z)H_1^\alpha(z)\cdots H_1^{\alpha^{k-1}}(z)\Omega^{\alpha^k}(z)$$

or

$$\Psi(z)=\Omega(z)+\frac{1}{X(z)}\sum_{k=1}^{\infty}\lambda^kH(z)H(\alpha(z))\cdots H(\alpha^{k-1}(z))X(\alpha^k(z))\Omega(\alpha^k(z))$$

is uniformly convergent on $\bar S^+$.

Thus,

$$\Phi(z)=X(z)\Omega(z)+\sum_{k=1}^{\infty}\lambda^kH(z)H(\alpha(z))\cdots H(\alpha^{k-1}(z))X(\alpha^k(z))\Omega(\alpha^k(z))\tag{3.7}$$

is the general solution of (1.1).

For the usually occurring case where

$$\lim_{n\to\infty}H(\alpha^n(z))=z_0\quad(z_0\text{ is a definite finite value}),$$

then, when $z_0\neq0$ and $|\lambda|<1/|z_0|$, (3.7) is really the solution; when $z_0=0$, it is valid for any λ.

2° Assume $\kappa<0$. In this case $\Psi(z)=\Omega(z)$ has a zero point at infinity of order $-\kappa$ at least and so, by (3.3), the condition of solvability for problem (1.1) is

$$\int_L\frac{\tau^jg(\tau)}{X^+(\tau)}\mathrm{d}\tau=0,\quad j=0,1,\ldots,-\kappa+1.\tag{3.8}$$

The rest discussions remain valid.

3° Assume $\kappa>0$. In this case, in place of (3.2), (3.3), we should have respectively

$$\Omega(z)=\frac{1}{2\pi\mathrm{i}}\int_L\frac{g(\tau)}{X^+(\tau)(\tau-z)}\mathrm{d}\tau+P_{\kappa-1}(z),\quad z\notin L,\tag{3.9}$$

$$\Psi(z)=\frac{1}{2\pi\mathrm{i}}\int_L\frac{g(\tau)}{X^+(\tau)(\tau-z)}\mathrm{d}\tau+P_{\kappa-1}(z),\quad z\in S^-;\tag{3.10}$$

the rest discussions remain as above with $\Omega(\alpha^k(z))$ to be understood as

$$\Omega(\alpha^k(z))=\frac{1}{2\pi\mathrm{i}}\int_L\frac{g(\tau)}{X^+(\tau)(\tau-\alpha^k(z))}\mathrm{d}\tau+P_{\kappa-1}(\alpha^k(z)).\tag{3.11}$$

Analogously, we may discuss the problem

$$\Phi^-(t)=G(t)\Phi^+(t)+\lambda H(t)\Phi(\alpha(t))+g(t),\quad t\in L.$$

When $\alpha(L)\subset S^-$ and $\alpha(t),H(t)$ are respectively the boundary values of certain functions holomorphic in S^- (including ∞), simple results analogous to above may be obtained also.

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某些带变换的边值问题和奇异积分方程

摘　要　本文主要考虑某些带变换的奇异积分方程. 尽管用经典的正则化方法可以求解，我们还是给出一种新的分区跳跃方法，使它们化为较简单的可以系统地求解并可迅即讨论的 Fredholm 积分方程.

关键词　Riemann 边值问题；奇异积分方程；变换

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(二)奇异积分和奇异积分方程

沿曲线的积分方程，其解具一阶奇异性

摘　要

本文讨论了沿曲线的 Fredholm 积分方程与 Cauchy 型奇异积分方程当自由项具一阶奇异性时在同样的函数类中求解的问题. 证明了相应的 Fredholm 定理与 Noether 定理(形式有所改变)仍均成立. 对于特征奇异方程及其相联方程还给出了解的积分表示式. 结果也推广到了方程组的情形.

一般讨论沿曲线的积分方程（或组）时，不论是 Fredholm 方程还是 Cauchy 型奇异方程，对于自由项及解至多都假定只有不到一阶的奇异性[1,2,3]. 本文将讨论当自由项在积分曲线上若干固定点处有一阶奇异性，而也容许解在这些点处也有一阶奇异性时，会有怎样的结果；当然，这时曲线积分要理解为 Cauchy 主值意义下的积分. 我们得到，就 Fredholm 方程而言，经典的 Fredholm 定理以及用预解式表示解的公式仍都成立. 而就 Cauchy 型奇异方程而言，Noether 定理稍作修改后也成立. 对于特征方程及其相联方程，并把解及可解条件写成了最终形式. 最后并把以上结果推广到方程组的情形.

§1　Fredholm 方程情况

设 L 是一曲线，由有限个互不相交的简单光滑封闭弧组成；在 L 上取定正向，如[1]，§29 之法. 设 $c_1,\cdots,c_n$ 是 L 上的 n 个不同的点，并记

$$\rho(z)=\prod_{k=1}^{n}(z-c_k). \tag{1.1}$$

定义 L 上的函数类 $H_1^*=H_1^*(c_1,\cdots,c_n)$ 如下：$f(t)\in H_1^*$ 当且仅当

$$f(t)=\frac{f^*(t)}{\rho(t)},\quad f^*(t)\in H \tag{1.2}$$

(H 为满足 Hölder 条件的函数类). 考察 Fredholm 积分方程

$$k_\lambda\varphi\equiv\varphi(t_0)-\lambda\int_L k(t_0,t)\varphi(t)\mathrm{d}t=f(t_0), \tag{1.3}$$

其中 λ 是一参数，而 $k(t_0,t)\in H^*$，即

$$k(t_0,t)=\frac{k^*(t_0,t)}{|t-t_0|^\alpha},\quad 0\leqslant\alpha<1,\ k^*\in H. \tag{1.4}$$

设已知函数 $f(t_0)$ 与未知函数 $\varphi(t_0)$ 均 $\in H_1^*$. 这时(1.3)成立当然指的是 $t_0\neq c_k(k=1,\cdots,n)$.

与(1.2)类似，令

$$\varphi(t)=\frac{\varphi^*(t)}{\rho(t)},\quad \varphi^*(t)\in H, \tag{1.5}$$

则方程(1.3)成为

$$\varphi^*(t_0)-\lambda\rho(t_0)\int_L k(t_0,t)\frac{\varphi^*(t)}{\rho(t)}\mathrm{d}t=f^*(t_0). \tag{1.6}$$

由于 $k(t_0,t)\varphi^*(t)\in H^*$，故

$$\int_L k(t_0,t)\frac{\varphi^*(t)}{\rho(t)}\mathrm{d}t\in H^*,$$

因而它在 $t_0=c_k$ 处有不到一阶的奇异性，这只要把 $\frac{1}{\rho(t)}$ 分项分式后便可看出(参看[1]，§45,3°). 这样，如果(1.3)有解，从(1.6)得知，则必

$$\varphi^*(c_k)=f^*(c_k),\quad k=1,\cdots,n. \tag{1.7}$$

今作代换

$$\omega(t)=\varphi(t)-f(t), \tag{1.8}$$

故 $\omega(t)\in H^*$，则方程(1.3)便可写成

$$k_\lambda\omega\equiv\omega(t_0)-\lambda\int_L k(t_0,t)\omega(t)\mathrm{d}t=\lambda h(t_0), \tag{1.9}$$

其中

$$h(t_0)=\int_L k(t_0,t)f(t)\mathrm{d}t. \tag{1.10}$$

根据与前面相同的理由，可知 $h(t)\in H^*$. 这样，问题就变成对 Fredholm 方程(1.9)在 H^* 类中求解. 且这一转化是等价的：得出(1.9)的解 $\omega(t)$ 后，由(1.8)便可得出(1.3)在 H_1^* 类中的解，这时条件(1.7)无疑成立.

对于齐次方程 $\boldsymbol{k}_\lambda\varphi=0$ 而言，由(1.7)很明显，它在 H_1^* 类中的解必属于 H^* 类，从而属于 H 类(见[1]，§51,2°). 因此对于齐次方程而言，有关的 Fredholm 定理无推广的可能.

如 λ 不是特征值，则(1.9)有唯一解，从而(1.3)也有唯一解. 今设 λ 是特征值. 这时，(1.9)可解的充要条件是(因 $\lambda\neq0$)

$$\int_L h(t)\chi_j(t)\mathrm{d}t=0,\quad j=1,\cdots,l, \tag{1.11}$$

其中$\{\chi_j\}$是相联方程 $\boldsymbol{k}_\lambda'\chi=0$ 的线性无关完全解组(它们均 $\in H$ 已如前述).

以(1.10)代入(1.11)中，并交换积分次序，得

$$\int_L\chi_j(t)\mathrm{d}t\int_L k(t,t_1)f(t_1)\mathrm{d}t_1=0. \tag{1.12}$$

记住 $\boldsymbol{k}_\lambda'\chi_j=0$，(1.12)就可简写成

$$\int_L f(t)\chi_j(t)\mathrm{d}t=0,\quad j=1,\cdots,l. \tag{1.13}$$

这样，下列 Fredholm 定理仍成立：$\boldsymbol{k}_\lambda\varphi=f$ $(f\in H_1^*)$**在** H_1^* **类中可解的充要条件是** f **与** $\boldsymbol{k}_\lambda'\chi=0$ **的所有解正交.**

我们还可用预解核给出方程(1.3)的解的表达式. 设算子 $\boldsymbol{k}_\lambda$ 的预解核为 $\Gamma(t_0,t,\lambda)$. 故

当λ不是特征值时，方程(1.9)的唯一解为

$$\omega(t_0)=\lambda h(t_0)+\lambda^2\int_L \Gamma(t_0,t,\lambda)h(t)\mathrm{d}t. \tag{1.14}$$

以(1.10)代入，注意预解核的熟知性质(见[2]，§5)：

$$\lambda\int_L \Gamma(t_0,t,\lambda)\mathrm{d}t\int_L k(t,t_1)f(t_1)\mathrm{d}t_1=\lambda\int_L f(t_1)\mathrm{d}t_1\int_L \Gamma(t_0,t,\lambda)k(t,t_1)\mathrm{d}t$$

$$=\int_L[\Gamma(t_0,t_1,\lambda)-k(t_0,t_1)]f(t_1)\mathrm{d}t_1,$$

并回到$\varphi(t)$，最后可得

$$\varphi(t_0)=f(t_0)+\lambda\int_L \Gamma(t_0,t,\lambda)f(t)\mathrm{d}t, \tag{1.15}$$

即方程(1.3)在H_1^*类中解的表达式与经典公式一致.

§2 Cauchy 型奇异方程情况

现在考虑方程

$$\boldsymbol{K}\varphi\equiv A(t_0)\varphi(t_0)+\frac{B(t_0)}{\pi\mathrm{i}}\int_L\frac{\varphi(t)}{t-t_0}\mathrm{d}t+\int_L k(t_0,t)\varphi(t)\mathrm{d}t=f(t_0), \tag{2.1}$$

其中$A,B\in H$，$k\in H^*$，$A\pm B\neq 0$，$f\in H_1^*$，未知函数φ也$\in H_1^*$；此外我们还设

$$A(c_k)\neq 0,\quad k=1,\cdots,n. \tag{2.2}$$

在(2.1)中，当然$t_0\neq c_k$，f^*，φ^*意义仍如前.

由于$\dfrac{1}{\pi\mathrm{i}}\displaystyle\int_L\frac{\varphi(t)}{t-t_0}\mathrm{d}t\in H^*$，$\displaystyle\int_L k(t_0,t)\varphi(t)\mathrm{d}t\in H^*$，故若$\varphi(t)$是(2.1)的解，则必

$$A(c_k)\varphi^*(c_k)=f^*(c_k),\quad k=1,\cdots,n. \tag{2.3}$$

现在不能作代换$\omega(t)=A(t)\varphi(t)-f(t)$，因为$A(t)$可能在$L$上有零点. 为克服这一困难，作$n-1$次插值多项式$T(t)$，使

$$T(c_k)=\frac{f^*(c_k)}{A(c_k)},\quad k=1,\cdots,n, \tag{2.4}$$

并令

$$\omega(t)=\varphi(t)-\frac{T(t)}{\rho(t)}, \tag{2.5}$$

于是可知$\omega(t)\in H^*$. 把它代入(2.1)中，则后者成为

$$\boldsymbol{K}\omega\equiv A(t_0)\omega(t_0)+\frac{B(t_0)}{\pi\mathrm{i}}\int_L\frac{\omega(t)}{t-t_0}\mathrm{d}t+\int_L k(t_0,t)\omega(t)\mathrm{d}t=F(t_0), \tag{2.6}$$

其中

$$F(t_0)=f(t_0)-\frac{A(t_0)T(t_0)}{\rho(t_0)}-h(t_0), \tag{2.7}$$

这里

$$h(t_0)=\int_L k(t_0,t)\frac{T(t)}{\rho(t)}\mathrm{d}t. \tag{2.8}$$

在以上的计算中，已注意到

$$\frac{1}{\pi\mathrm{i}}\int_L\frac{T(t)}{\rho(t)}\frac{\mathrm{d}t}{t-t_0}=0,\quad t_0\neq c_k;$$

这只要把$\dfrac{T(t)}{\rho(t)}$进行分项分式便易看出.

对齐次方程 $K\varphi=0$ 来说，易见 $\varphi(t)\in H^*$，从而$\in H$（见[1]，§ 53，附注 3）. 因此有关齐次方程的 Noether 定理这里也不可能推广.

对于非齐次方程(2.1)，现在已转化为等价的方程(2.6)，其中 $F(t_0)\in H^*$. 于是它可解的充要条件为

$$\int_L F(t)\chi_j(t)\mathrm{d}t=0,\quad j=1,\cdots,l',\tag{2.9}$$

其中$\{\chi_j\}$是相联方程 $K'\chi=0$(在 H 类中)的线性无关完全解组①.

以(2.7)代入上式，得

$$\int_L f(t)\chi_j(t)\mathrm{d}t=\int_L \frac{A(t)T(t)}{\rho(t)}\chi_j(t)\mathrm{d}t+\int_L h(t)\chi_j(t)\mathrm{d}t;$$

再以(2.8)代入，并交换积分次序，则又有

$$\begin{aligned}\int_L f(t)\chi_j(t)\mathrm{d}t&=\int_L \frac{T(t)}{\rho(t)}\mathrm{d}t\left\{A(t)\chi_j(t)+\int_L k(t_1,t)\chi_j(t_1)\mathrm{d}t_1\right\}\\&=\frac{1}{\pi\mathrm{i}}\int_L \frac{T(t)}{\rho(t)}\mathrm{d}t\int_L \frac{B(t_1)\chi_j(t_1)}{t_1-t}\mathrm{d}t_1.\end{aligned}$$

注意

$$\frac{T(t)}{\rho(t)}=\sum_{k=1}^{n}\frac{T(c_k)}{\rho'(c_k)(t-c_k)},\tag{2.10}$$

以此代入前式，并用 Poincaré-Bertrand 积分交换次序公式，上式便成为

$$\int_L f(t)\chi_j(t)\mathrm{d}t=\sum_{k=1}^{n}\frac{T(c_k)}{\rho'(c_k)}\left\{\pi\mathrm{i}B(c_k)\chi_j(c_k)+\frac{1}{\pi\mathrm{i}}\int_L B(t_1)\chi_j(t_1)\mathrm{d}t_1\int_L\frac{\mathrm{d}t}{(t-c_k)(t-t_1)}\right\}.$$

上式中最后的积分显然为 0. 又注意(2.3)式，于是可解条件(2.9)最后便成为

$$\frac{1}{\pi\mathrm{i}}\int_L f(t)\chi_j(t)\mathrm{d}t=\sum_{k=1}^{n}\frac{f^*(c_k)B(c_k)\chi_j(c_k)}{A(c_k)\rho'(c_k)},\quad j=1,\cdots,l'.\tag{2.11}$$

于是便得 Noether 定理的推广：**方程(2.1)在所设条件下在 H_1^* 类中可解的充要条件是(2.11)成立，其中$\{\chi_j\}$是 $K'\chi=0$ 的线性无关完全解组.**

§ 3　特征方程及其相联方程的解

对于特征方程

$$K^0\varphi\equiv A(t_0)\varphi(t_0)+\frac{B(t_0)}{\pi\mathrm{i}}\int_L\frac{\varphi(t)}{t-t_0}\mathrm{d}t=f(t_0)\tag{3.1}$$

而言，其解可写成积分形式. 由于这时(2.6)成为

$$K^0\omega=f(t_0)-\frac{A(t_0)T(t_0)}{\rho(t_0)},$$

故其一般解为

$$\omega(t_0)=K^*f-K^*\frac{A(t_0)T(t_0)}{\rho(t_0)}+B^*(t_0)Z(t_0)P_{\kappa-1}(t_0),\tag{3.2}$$

① 把 $c_1,\cdots,c_n$ 当作系数的间断点，则它们都是特异的，因此 Noether 定理对(2.6)成立，见[1]，§ 102，附注 1.

其中

$$K^* f \equiv A^*(t_0)f(t_0)-\frac{B^*(t_0)Z(t_0)}{\pi \mathrm{i}}\int_L \frac{f(t)\mathrm{d}t}{Z(t)(t-t_0)},$$

我们这里已采用[1]，§47 中的记号，κ 为问题的指标，$P_{\kappa-1}$为 $\kappa-1$ 次任意多项式（$\kappa\leqslant 0$ 时，$P_{\kappa-1}\equiv 0$）①.

把 $K^*\frac{A(t_0)T(t_0)}{\rho(t_0)}$ 展开，并用(2.5)回到 $\varphi(t_0)$，则 (3.2)成为

$$\varphi(t_0)=K^* f-B^*(t_0)Z(t_0)\left\{\frac{B(t_0)T(t_0)}{Z(t_0)\rho(t_0)}-\frac{1}{\pi \mathrm{i}}\int_L \frac{A(t)T(t)}{Z(t)\rho(t)}\frac{\mathrm{d}t}{t-t_0}+P_{\kappa-1}(t_0)\right\}. \tag{3.3}$$

这便是特征方程(3.1)的一般解.

为了把一般解(3.3)写成更简明的形式，我们先来证明下一引理. 在下面我们将(部分地)用到这引理，但它本身也有独立的意义.

引理 如果把典则函数的倒数 $1/X(z)$ 在 $z=\infty$ 处的主部记作 $H_\kappa(z)$（它是 κ 次多项式，$\kappa<0$ 时为零)，则我们有

$$\begin{cases}\dfrac{A(t_0)}{Z(t_0)}-\dfrac{1}{\pi \mathrm{i}}\displaystyle\int_L \frac{B(t)\mathrm{d}t}{Z(t)(t-t_0)}=H_\kappa(t_0),\\[2ex] -\dfrac{B(t_0)}{Z(t_0)}+\dfrac{1}{\pi \mathrm{i}}\displaystyle\int_L \frac{A(t)\mathrm{d}t}{Z(t)(t-t_0)}=H_\kappa(t_0);\end{cases} \tag{3.4}$$

此外，还有以下公式成立：

$$\begin{cases}\dfrac{1}{\pi \mathrm{i}}\displaystyle\int_L \frac{A(t)\mathrm{d}t}{Z(t)(t-z)}=\pm\frac{1}{X(z)}+H_\kappa(z), & z\in S^\pm,\\[2ex] \dfrac{1}{\pi \mathrm{i}}\displaystyle\int_L \frac{B(t)\mathrm{d}t}{Z(t)(t-z)}=\frac{1}{X(z)}-H_\kappa(z), & z\in S^\pm.\end{cases} \tag{3.5}$$

证 我们记起

$$Z(t_0)=[A(t_0)+B(t_0)]X^+(t_0)=[A(t_0)-B(t_0)]X^-(t_0),$$
$$X(z)=O(|z|^{-\kappa})\quad (z\to\infty),$$

于是我们有

$$\begin{aligned}&\frac{A(t_0)}{Z(t_0)}-\frac{1}{\pi \mathrm{i}}\int_L \frac{B(t)\mathrm{d}t}{Z(t)(t-t_0)}\\ &\quad=\frac{1}{X^+(t_0)}-\frac{B(t_0)}{Z(t_0)}-\frac{1}{\pi \mathrm{i}}\int_L \frac{B(t)\mathrm{d}t}{Z(t)(t-t_0)}=\lim_{z\to t_0^+}\left\{\frac{1}{X(z)}-\frac{1}{\pi \mathrm{i}}\int_L \frac{B(t)\mathrm{d}t}{Z(t)(t-z)}\right\}\\ &\quad=\frac{1}{X^-(t_0)}+\frac{B(t_0)}{Z(t_0)}-\frac{1}{\pi \mathrm{i}}\int_L \frac{B(t)\mathrm{d}t}{Z(t)(t-t_0)}=\lim_{z\to t_0^-}\left\{\frac{1}{X(z)}-\frac{1}{\pi \mathrm{i}}\int_L \frac{B(t)\mathrm{d}t}{Z(t)(t-z)}\right\},\end{aligned}$$

其中 $z\to t_0^\pm$ 分别表示 z 从 L 的正、负侧趋于 t_0. 由此可见，上式左端是在全平面中某全纯函数的边值；而由 $X(z)$ 在 $z=\infty$ 处的性态，知它是一 κ 次多项式 $H_\kappa(z)$，于是(3.4)中第一式成立. 而且再由上式还可看出，

$$\frac{1}{X(z)}-\frac{1}{\pi \mathrm{i}}\int_L \frac{B(t)\mathrm{d}t}{Z(t)(t-z)}=H_\kappa(z). \tag{3.6}$$

由此立刻知道，$H_\kappa(z)$是$\frac{1}{X(z)}$在 $z=\infty$ 处的主部 .

---

① 当 $\kappa<0$ 时，认为可解条件已满足.

(3.4)中第二式可由第一式反演得来，但也可如下处理：

$$
\begin{aligned}
&-\frac{B(t_0)}{Z(t_0)}+\frac{1}{\pi i}\int_L\frac{A(t)dt}{Z(t)(t-t_0)}\\
&=-\frac{1}{X^+(t_0)}+\frac{A(t_0)}{Z(t_0)}+\frac{1}{\pi i}\int_L\frac{A(t)dt}{Z(t)(t-t_0)}=\lim_{z\to t_0^+}\left\{\frac{-1}{X(z)}+\frac{1}{\pi i}\int_L\frac{A(t)dt}{Z(t)(t-z)}\right\}\\
&=\frac{1}{X^-(t_0)}-\frac{A(t_0)}{Z(t_0)}+\frac{1}{\pi i}\int_L\frac{A(t)dt}{Z(t)(t-t_0)}=\lim_{z\to t_0^-}\left\{\frac{1}{X(z)}+\frac{1}{\pi i}\int_L\frac{A(t)dt}{Z(t)(t-z)}\right\}.
\end{aligned}
$$

根据与前相仿的理由，也可得(3.4)中第二式，且得

$$
\begin{cases}
-\dfrac{1}{X(z)}+\dfrac{1}{\pi i}\displaystyle\int_L\frac{A(t)dt}{Z(t)(t-z)}=H_\kappa(z), & z\in S^+;\\
\dfrac{1}{X(z)}+\dfrac{1}{\pi i}\displaystyle\int_L\frac{A(t)dt}{Z(t)(t-z)}=H_\kappa(z), & z\in S^-;
\end{cases}
\tag{3.7}
$$

从后一式也可看出 $H_\kappa(z)$ 仍是 $\frac{1}{X(z)}$ 在 $z=\infty$ 处的主部，

(3.6)，(3.7)即(3.5)式. 引理得证①.

我们回来化简(3.3)式. 当 $\kappa<0$ 时我们认为可解条件已满足.

用(2.10)代入(3.3)右端最后积分中：

$$
\begin{aligned}
\frac{1}{\pi i}\int_L\frac{A(t)T(t)dt}{Z(t)\rho(t)(t-t_0)}&=\sum_{k=1}^{n}\frac{T(c_k)}{\rho'(c_k)}\frac{1}{\pi i}\int_L\frac{A(t)dt}{Z(t)(t-t_0)(t-c_k)}\\
&=\sum_{k=1}^{n}\frac{T(c_k)}{\rho'(c_k)(t_0-c_k)}\frac{1}{\pi i}\int_L\frac{A(t)}{Z(t)}\left\{\frac{1}{t-t_0}-\frac{1}{t-c_k}\right\}dt.
\end{aligned}
$$

利用(3.4)中第二式代入，再次注意(2.10)，可得

$$
\begin{aligned}
&\frac{1}{\pi i}\int_L\frac{A(t)T(t)dt}{Z(t)\rho(t)(t-t_0)}\\
&=\sum_{k=1}^{n}\frac{T(c_k)}{\rho'(c_k)(t_0-c_k)}\left[\frac{B(t_0)}{Z(t_0)}-\frac{B(c_k)}{Z(c_k)}+H_\kappa(t_0)-H_\kappa(c_k)\right]\\
&=\frac{B(t_0)T(t_0)}{Z(t_0)\rho(t_0)}-\sum_{k=1}^{n}\frac{B(c_k)T(c_k)}{Z(c_k)\rho'(c_k)(t_0-c_k)}+\sum_{k=1}^{n}\frac{T(c_k)}{\rho'(c_k)}\frac{H_\kappa(t_0)-H_\kappa(c_k)}{t_0-c_k}.
\end{aligned}
$$

上式中最后一项显然是 $\kappa-1$ 次多项式($\kappa-1<0$ 时为零)，所以把上式代入(3.3)中时，这一项可并入任意多项式 $P_{\kappa-1}(t_0)$ 中. 此外再注意到(2.4)，最后便得**方程(3.1)在 H_1^* 类中的解的公式**：

$$
\varphi(t_0)=K^*f-B^*(t_0)Z(t_0)\left\{\sum_{k=1}^{n}\frac{B(c_k)f^*(c_k)}{A(c_k)Z(c_k)\rho'(c_k)(t_0-c_k)}-P_{\kappa-1}(t_0)\right\}^{②}. \tag{3.8}
$$

---

① 当 $\kappa<0$ 时，(3.4)中第一式很明显，因为这时 $\frac{1}{Z(t_0)}$ 是 $K^{0'}\chi=0$ 的解. 此外，引理显然还可进一步推广：在各式的 $A(t)$，$B(t)$ 等因子中添一 k 次多项式因子 $P(t)$ 时也可得到类似结果，不过 H_κ 要改为 $H_{\kappa+k}$，它是 $\frac{P(z)}{X(z)}$ 在 $z=\infty$ 处的主部.

② 如果与经典情况一样，把方程(3.1)化为 Riemann 边值问题(注意当密度函数在 $c_1,\cdots,c_n$ 处具有一阶奇异性时，Plemelj 公式仍成立)，也可求得(3.1)的一般解有下形：

$$
\varphi(t_0)=K^*f-B^*(t_0)Z(t_0)\left\{\sum_{k=1}^{n}\frac{C_k}{t_0-c_k}+P_{\kappa-1}(t_0)\right\},
$$

其中 $P_{\kappa-1}$ 仍任意，而 C_k 是一些待定常数. 把它直接代入(3.1)，经过相当复杂的计算，也可得到(3.8)式.

当$\kappa<0$时，由于$\boldsymbol{K}^{0\prime}\chi=0$的一般解是$\dfrac{Q_{-\kappa-1}(t_0)}{Z(t_0)}$（$Q_{-\kappa-1}$为$-\kappa-1$次任意多项式），故可解条件(2.11)现在成为：

$$\int_L \frac{f(t)t^j}{Z(t)}\mathrm{d}t=\sum_{k=1}^{n}\frac{B(c_k)f^*(c_k)c_k^j}{A(c_k)Z(c_k)\rho'(c_k)},\quad j=0,1,\cdots,-\kappa-1. \tag{3.9}$$

最后我们来看相联方程

$$\boldsymbol{K}^{0\prime}\psi\equiv A(t_0)\psi(t_0)-\frac{1}{\pi\mathrm{i}}\int_L\frac{B(t)\psi(t)}{t-t_0}\mathrm{d}t=g(t_0) \tag{3.10}$$

在H_1^*类中的解，其中$g(t_0)=g^*(t_0)/\rho(t_0)\in H_1^*$. 这时不能类似地应用上法. 在(3.10)中两边乘以$B(t_0)$，并令

$$\theta(t)=B(t)\psi(t), \tag{3.11}$$

则(3.10)成为$\theta(t)$的特征方程

$$A(t_0)\theta(t_0)-\frac{B(t_0)}{\pi\mathrm{i}}\int_L\frac{\theta(t)\mathrm{d}t}{t-t_0}=B(t_0)g(t_0). \tag{3.12}$$

这与方程(3.1)所不同的，只是左边$B(t)$改为了$-B(t)$，其指标为$\kappa'=-\kappa$，且这时典则函数是$\dfrac{1}{X(z)}$，故现在相应的标准函数$Z(t)$要改为

$$\frac{A^2(t)-B^2(t)}{Z(t)}=\frac{A(t)-B(t)}{X^+(t)}=\frac{A(t)+B(t)}{X^-(t)}. \tag{3.13}$$

于是根据(3.8)式，(3.12)的一般解（设可解条件满足）为：

$$\theta(t_0)=\boldsymbol{K}^*(Bg)-\frac{B(t_0)}{Z(t_0)}\left\{\sum_{k=1}^{n}\frac{B^{*2}(c_k)Z(c_k)g^*(c_k)}{A^*(c_k)\rho'(c_k)(t_0-c_k)}-P_{-\kappa-1}(t_0)\right\}.$$

记起$\boldsymbol{K}^*$的意义，回到$\psi(t_0)$，由(3.11)，得

$$\psi(t_0)=\boldsymbol{K}^{*\prime}g-\frac{1}{Z(t_0)}\left\{\sum_{k=1}^{n}\frac{B^{*2}(c_k)Z(c_k)g^*(c_k)}{A^*(c_k)\rho'(c_k)(t_0-c_k)}-P_{-\kappa-1}(t_0)\right\}, \tag{3.14}$$

其中$\boldsymbol{K}^{*\prime}$是$\boldsymbol{K}^*$的相联算子. 这便是方程(3.10)在H_1^*类中的一般解.

以上的推理中，严格说来，已假定了$B(t)\neq0$于L上. 但得到(3.14)后可用直接验证法（虽然相当复杂）可以证实：纵然$B(t)$在L上有零点时，(3.14)确是(3.10)的解①.

当$\kappa'<0$即$\kappa>0$时，由(2.11)易见可解条件为：

$$\frac{1}{\pi\mathrm{i}}\int_L g(t)B^*(t)Z(t)t^j\mathrm{d}t=-\sum_{k=1}^{n}\frac{g^*(c_k)B^{*2}(c_k)Z(c_k)c_k^j}{A^*(c_k)\rho'(c_k)},\quad j=0,1,\cdots,\kappa-1. \tag{3.15}$$

注意$\dfrac{B^{*2}}{A^*}=A^*-\dfrac{1}{A}$，所以(3.14)，(3.15)可相应化简.

§4　方程组的情形

以上各节的结论不难推广到方程组上去，即：设已知的$f(t)$与未知的$\varphi(t)$都看作N维向量，而k为N阶矩阵，$\in H^*$. 这时同样地定义N维向量的类H_1^*. 对于Fredholm方程而言，§1中的一切论证和结论都成立，只要注意到(1.12)式中的k现在要改为其转置矩

① 根据前面底注的类似理由，它的确还是一般解.

阵 k'. 至于在证明(1.15)时所用的预解矩阵的性质，现在仍成立(见[1]，§110).

现在来考虑奇异方程(组)(2.1)，这时除以上说明外，现在 A,B 都是 N 阶矩阵，$\in H$，且

$$S=A+B,\quad D=A-B \tag{4.1}$$

在 L 上处处满秩①；此外，还设矩阵

$$A(c_k)\quad (k=1,\cdots,n)$$

也满秩. 解法可比照§2中进行，因而只作简述如下.

这时(2.3)仍成立. 作插值($n-1$ 次)多项式向量 $T(t)$，使

$$T(c_k)=A^{-1}(c_k)f^*(c_k),\quad k=1,\cdots,n,$$

并仍令 $\omega(t)$ 如(2.5)，则仍得(2.6). 现在可解条件(2.9)仍成立，但它可化为

$$\begin{aligned}\int_L f(t)\chi_j(t)\mathrm{d}t&=\int_L\chi_j(t)\frac{A(t)T(t)}{\rho(t)}\mathrm{d}t+\int_L\chi_j(t)\mathrm{d}t\int_L k(t,t_1)\frac{T(t_1)}{\rho(t_1)}\mathrm{d}t_1\\&=\int_L\frac{T(t)}{\rho(t)}\mathrm{d}t\left\{A'(t)\chi_j(t)+\int_L k'(t_1,t)\chi_j(t_1)\mathrm{d}t_1\right\}\\&=\frac{1}{\pi\mathrm{i}}\int_L\frac{T(t)}{\rho(t)}\mathrm{d}t\int_L\frac{B'(t_1)\chi_j(t_1)}{t_1-t}\mathrm{d}t_1.\end{aligned}$$

仍以(2.10)代入，可解条件成为

$$\begin{aligned}\frac{1}{\pi\mathrm{i}}\int_L f(t)\chi_j(t)\mathrm{d}t&=\sum_{k=1}^{n}\frac{A^{-1}(c_k)f^*(c_k)B'(c_k)\chi_j(c_k)}{\rho'(c_k)}\\&=\sum_{k=1}^{n}\frac{\chi_j(c_k)B(c_k)A^{-1}(c_k)f^*(c_k)}{\rho'(c_k)},\quad j=1,\cdots,l',\end{aligned}\tag{4.2}$$

其中$\{\chi_j\}$仍为 $\boldsymbol{K}'\chi=0$ 的线性无关完全解组.

这样，§2中推广的 Noether 定理对方程组仍成立，只要把可解条件(2.11)换作(4.2).

对于特征方程(3.1)的解我们也有较简形式. 这时(3.2)的形式不变，但

$$\boldsymbol{K}^*f=A^*(t_0)f(t_0)-\frac{B^*(t_0)Z(t_0)}{\pi\mathrm{i}}\int_L\frac{Z^{-1}(t)f(t)\mathrm{d}t}{t-t_0}. \tag{4.3}$$

现在(见[1]，§134)

$$\begin{cases}A^*(t)=\dfrac{1}{2}[S^{-1}(t)+D^{-1}(t)],\quad B^*(t)=-\dfrac{1}{2}[S^{-1}(t)-D^{-1}(t)],\\ Z(t)=S(t)X^+(t)=D(t)X^-(t),\end{cases}\tag{4.4}$$

这里 $X(z)$ 是问题相应的典则矩阵. 设 $\kappa_1,\cdots,\kappa_N$ 为方程的各偏指标，$\kappa=\sum\limits_{k=1}^{N}\kappa_k$ 为总指标，则 $X(z)$ 中各(纵)列向量在 $z=\infty$ 处的阶数依次为 $-\kappa_1,\cdots,-\kappa_N$. 又，现在(3.2)中的 $P(t)$ 是一多项式向量：

$$P(t)=(P_{\kappa_1-1},\cdots,P_{\kappa_N-1}).$$

注意到

$$A^*A-B^*B=E$$

(E 为单位矩阵)，回到 $\varphi(t_0)$ 时，(3.2)便成为

① 本节中以后恒将采用[1]第六章中的记号，注意它们与[3]中记号不完全一样.

$$\varphi(t_0)=\boldsymbol{K}^* f-B^*(t_0)Z(t_0)\left\{\frac{Z^{-1}(t_0)B(t_0)T(t_0)}{\rho(t_0)}-\frac{1}{\pi\mathrm{i}}\int_L\frac{Z^{-1}(t)A(t)T(t)\mathrm{d}t}{\rho(t)(t-t_0)}-P(t_0)\right\}. \tag{4.5}$$

为了进一步化简上式,我们也有与§2中相仿的

引理 如果把$X^{-1}(z)$在$z=\infty$处的主部多项式矩阵记作$H(z)$,则我们有

$$\begin{cases}Z^{-1}(t_0)A(t_0)-\dfrac{1}{\pi\mathrm{i}}\displaystyle\int_L\frac{Z^{-1}(t)B(t)\mathrm{d}t}{t-t_0}=H(t_0),\\ -Z^{-1}(t_0)B(t_0)+\dfrac{1}{\pi\mathrm{i}}\displaystyle\int_L\frac{Z^{-1}(t)A(t)\mathrm{d}t}{t-t_0}=H(t_0);\end{cases} \tag{4.6}$$

此外还有:

$$\begin{cases}\dfrac{1}{\pi\mathrm{i}}\displaystyle\int_L\frac{Z^{-1}(t)A(t)\mathrm{d}t}{t-z}=\pm X^{-1}(z)+H(z), \quad z\in S^{\pm},\\ \dfrac{1}{\pi\mathrm{i}}\displaystyle\int_L\frac{Z^{-1}(t)B(t)\mathrm{d}t}{t-z}=X^{-1}(z)-H(z), \quad z\in S^{\pm}.\end{cases}$$

证明方法与§2中相似,从略.

以(4.6)中第二式代入(4.5)式,得

$$\varphi(t_0)=\boldsymbol{K}^* f-B^*(t_0)Z(t_0)\left\{\sum_{k=1}^{n}\frac{Z^{-1}(c_k)B(c_k)T(c_k)}{\rho'(c_k)(t_0-c_k)}+\sum_{k=1}^{n}\frac{H(t_0)-H(c_k)}{t_0-c_k}T(c_k)-P(t_0)\right\}. \tag{4.7}$$

我们知道,$X'^{-1}(z)$在$z=\infty$处的各列的阶数分别为$\kappa_1,\cdots,\kappa_N$(见[1],§131),从而$X^{-1}(z)$的各行的阶数,随之$H(t)$的各行的阶数,也分别为$\kappa_1,\cdots,\kappa_N$. 于是,矩阵

$$\frac{H(t_0)-H(c_k)}{t_0-c_k}$$

的第j行有阶数κ_j-1,接着向量

$$\sum_{k=1}^{n}\frac{H(t_0)-H(c_k)}{t_0-c_k}T(c_k)$$

的第j个分量便是κ_j-1次多项式($\kappa_j-1<0$时为零). (4.7)中相应的项可并入$P(t_0)$中.

注意$T(t)$的作法,最后便得:方程(3.1)的一般解为:

$$\varphi(t_0)=\boldsymbol{K}^* f-B^*(t_0)Z(t_0)\left\{\sum_{k=1}^{n}\frac{Z^{-1}(c_k)B(c_k)A^{-1}(c_k)f^*(c_k)}{\rho'(c_k)(t_0-c_k)}-P(t_0)\right\}. \tag{4.8}$$

由于齐次相联方程$\boldsymbol{K}^{0\prime}\chi=0$的一般解为

$$\chi(t)=Z'^{-1}(t)Q(t),$$

这里$Q(t)$是一多项式向量:

$$Q(t)=(Q_{-\kappa_1-1},\cdots,Q_{-\kappa_N-1}),$$

故(3.1)的可解条件为

$$\frac{1}{\pi\mathrm{i}}\int_L f(t)Z'^{-1}(t)Q(t)\mathrm{d}t=\sum_{k=1}^{n}\frac{B(c_k)A^{-1}(c_k)f^*(c_k)Z'^{-1}(c_k)Q(c_k)}{\rho'(c_k)},$$

或可写成

$$\frac{1}{\pi\mathrm{i}}\int_L Q(t)Z^{-1}(t)f(t)\mathrm{d}t=\sum_{k=1}^{n}\frac{Q(c_k)Z^{-1}(c_k)B(c_k)A^{-1}(c_k)f^*(c_k)}{\rho'(c_k)}. \tag{4.9}$$

现在考虑非齐次相联方程(组)

$$\boldsymbol{K}^{0\prime}\psi\equiv A'(t_0)\psi(t_0)-\frac{1}{\pi\mathrm{i}}\int_L\frac{B'(t)\psi(t)}{t-t_0}\mathrm{d}t=g(t_0),\tag{4.10}$$

$(g(t_0)\in H_1^*)$. 暂设 $B(t)$ 满秩，令

$$\theta(t)=B'(t)\psi(t),\tag{4.11}$$

则方程(4.10)成为

$$A'(t_0)B'^{-1}(t_0)\theta(t_0)-\frac{1}{\pi\mathrm{i}}\int_L\frac{\theta(t)\mathrm{d}t}{t-t_0}=g(t_0).\tag{4.12}$$

它与 (3.1)相类似. 如在相应的记号上加以符号~，则我们有

$$\widetilde{A}=A'B'^{-1},\quad \widetilde{B}=-E;$$

于是

$$\widetilde{S}=A'B'^{-1}-E=D'B'^{-1},\quad \widetilde{D}=A'B'^{-1}+E=S'B'^{-1}.$$

由此容易算得

$$\widetilde{S}^{-1}\widetilde{D}=B'D'^{-1}S'B'^{-1}=\frac{1}{2}(S'-D')D'^{-1}S'B'^{-1}=\frac{1}{2}S'(D'^{-1}S'-E)B'^{-1}=S'D'^{-1}.$$

又因 $X^+=S^{-1}DX^-$，$\widetilde{X}^+=\widetilde{S}^{-1}\widetilde{D}\widetilde{X}^-$，易证

$$\widetilde{X}=X'^{-1}.$$

由于 $Z=SX^+$，从而

$$\widetilde{Z}=\widetilde{S}\widetilde{X}^+=\widetilde{S}X'^{-1}=\widetilde{S}S'Z'^{-1}=B^{*\prime-1}Z'^{-1},$$

这是因为

$$(\widetilde{S}S')^{-1}=S'^{-1}B'D'^{-1}=\frac{1}{2}S'^{-1}(S'-D')D'^{-1}=B^{*\prime}$$

的缘故. 此外，现在

$$\widetilde{A}^*=\frac{1}{2}(\widetilde{S}^{-1}+\widetilde{D}^{-1})=\frac{1}{2}(B'D'^{-1}+B'S'^{-1})=B'A^{*\prime},$$

$$\widetilde{B}^*=-\frac{1}{2}(\widetilde{S}^{-1}-\widetilde{D}^{-1})=-\frac{1}{2}(B'D'^{-1}-B'S'^{-1})=-B'B^{*\prime}.$$

现在的偏指标为 $-\kappa_1,\cdots,-\kappa_N$，总指标为 $-\kappa$.

根据(4.8)，我们有(4.12)的一般解：

$$\theta(t_0)=\widetilde{\boldsymbol{K}}^*g-\widetilde{B}^*(t_0)\widetilde{Z}(t_0)\left\{\sum_{k=1}^n\frac{\widetilde{Z}^{-1}(c_k)\widetilde{B}(c_k)\widetilde{A}^{-1}(c_k)g^*(c_k)}{\rho_i'(c_k)(t_0-c_k)}-Q(t_0)\right\},\tag{4.13}$$

其中

$$\widetilde{\boldsymbol{K}}^*g\equiv\widetilde{A}^*(t_0)g(t_0)-\frac{\widetilde{B}^*(t_0)\widetilde{Z}(t_0)}{\pi\mathrm{i}}\int_L\frac{\widetilde{Z}^{-1}(t)g(t)}{t-t_0}\mathrm{d}t.\tag{4.14}$$

把以上计算所得结果代入(4.13)，(4.14)，并由(4.11)，回到 $\psi(t_0)$ 时，我们有

$$\psi(t_0)=\boldsymbol{K}^{*\prime}g-Z'^{-1}(t_0)\left\{\sum_{k=1}^n\frac{Z(c_k)B^{*\prime}(c_k)B'(c_k)A'^{-1}(c_k)g^*(c_k)}{\rho_i'(c_k)(t_0-c_k)}-Q(t_0)\right\},\tag{4.15}$$

其中 $\boldsymbol{K}^{*\prime}$ 仍是 $\boldsymbol{K}^*$ 的相联算子.

(4.15)便是方程(4.10)的一般解. 以上虽然是在 $B(t_0)$ 满秩的假定下推得的，用直接的验证可以知道没有这假定时(4.15)式仍是解，且为一般解(见§3 中底注).

这时方程(4.10)的可解条件由(4.9)应为

$$\frac{1}{\pi\mathrm{i}}\int_L P(t)\widetilde{Z}^{-1}(t)g(t)\mathrm{d}t=\sum_{k=1}^{n}\frac{P(c_k)\widetilde{Z}^{-1}(c_k)\widetilde{B}(c_k)\widetilde{A}^{-1}(c_k)g^*(c_k)}{\rho_t'(c_k)},$$

或即

$$\frac{1}{\pi\mathrm{i}}\int_L P(t)Z'(t)B^{*\prime}(t)g(t)\mathrm{d}t=\sum_{k=1}^{n}\frac{P(c_k)Z'(c_k)B^{*\prime}(c_k)B'(c_k)A'^{-1}(c_k)g^*(c_k)}{\rho_t'(c_k)}.$$

这又可改写为

$$\frac{1}{\pi\mathrm{i}}\int_L g(t)B^*(t)Z(t)P(t)\mathrm{d}t=\sum_{k=1}^{n}\frac{g^*(c_k)A^{-1}(c_k)B(c_k)B^*(c_k)Z(c_k)P(c_k)}{\rho_t'(c_k)}. \tag{4.16}$$

由于

$$\begin{aligned}A^{-1}BB^*&=-\frac{1}{4}A^{-1}(S-D)(S^{-1}-D^{-1})=\frac{1}{4}A^{-1}(SD^{-1}+DS^{-1}-2E)\\&=\frac{1}{4}A^{-1}[(S+D)D^{-1}+(D+S)S^{-1}-4E]\\&=A^{-1}\left[\frac{1}{2}A(D^{-1}+S^{-1})-E\right]=A^*-A^{-1},\end{aligned}$$

因此(4.16)以及(4.15)中有关部分也可作类似化简.

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ON INTEGRAL EQUATIONS ALONG CURVES, THE SOLUTIONS OF WHICH HAVE SINGULARITIES OF ORDER ONE

Summary

In this paper, the Fredholm integral equations and the Cauchy-type singular integral equations are discussed, the free terms and the unknown solutions of which are assumed to have some fixed singular points of order one on the path of integration. It is proved that the corresponding Fredholm theorem and Noether theorem (with some variation) remain true in such cases. For the characteristic equations and their adjoint equations, the integral expressions of their solutions are obtained. All the above results are also generalized to the case of system of equations.

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关于 Hilbert 核奇异积分方程

前 言

通常谈到 Hilbert 核的奇异积分方程时，都是指的在实数域中的方程：

$$A(s)u(s)-\frac{B(s)}{2\pi}\int_0^{2\pi}u(\sigma)\cot\frac{\sigma-s}{2}\,\mathrm{d}\sigma=f(s),\tag{0.1}$$

或者，除掉奇异积分外，左边再添加一个非奇异积分的项. 但是，把积分路径改为平面曲线的一般情况，即

$$A(t_0)\varphi(t_0)+\frac{B(t_0)}{2\pi\,\mathrm{i}}\int_L\varphi(t)\cot\frac{t-t_0}{2}\,\mathrm{d}t=f(t_0),\tag{0.2}$$

还很少看到. 当然，如果把奇异核写成

$$\cot\frac{t-t_0}{2}=\frac{2}{t-t_0}+\Omega(t,t_0),$$

则(0.2)就化为通常的Cauchy 型奇异方程，因而可用熟知的理论来求解. 然而运用这种方法把它正则化时，不能得出它的有效解答，即便得到的定性结果也不能令人满意，因为这里要解决正则化后方程的求解问题，而后者的显式及解的个数的讨论均较复杂，还要牵涉到相联方程的求解问题.

本文的目的就是要给出方程(0.2)的有效解法，所用方法是把它转化为周期 Riemann 边值问题[1].

当 $A=0,B=1$ 时，(0.2)就成为一般 Hilbert 核积分的反演问题，这在[2]中已讨论过，并获得了完全的答案.

注意，如果令 $z=\mathrm{i}\zeta$，本文的结果就可转化为含复的核 $\operatorname{cth}\frac{\tau-\tau_0}{2}$ 的奇异方程的结果.

§1 封闭曲线与连续系数的情况

我们来讨论方程

$$K\varphi\equiv A(t_0)\varphi(t_0)+\frac{B(t_0)}{a\pi\,\mathrm{i}}\int_L\varphi(t)\cot\frac{t-t_0}{a}\,\mathrm{d}t=f(t_0)^{①},\quad t_0\in L,\tag{1.1}$$

这里 $L=\sum_{j=1}^{p}L_j$ 是由 p 个互不相交的光滑封闭曲线构成，而且把它沿实轴方向平移 $a\pi$ 时，

① 我们这里一般地引进 $a>0$ 而不令 $a=2$，为的是当 $a\to+\infty$ 时便于和 Cauchy 核奇异方程的结果比较.

也互不相交；又 A, B, f 都是给定在 L 上的函数，φ 为未知函数，都属于 H 类(Hölder 条件)，且 $A^2-B^2\neq 0$①. 设各 L_j 已取定一定的正向.

1. 基本引理 如果我们令

$$\Phi(z) = \frac{1}{2a\pi \mathrm{i}}\int_L \varphi(t)\cot\frac{t-z}{a}\,\mathrm{d}t, \tag{1.2}$$

则 $\Phi(z)$ 是一个以 $a\pi$ 为周期的分区全纯函数，其跳跃曲线为 L 及其周期合同曲线. 注意到 $\frac{1}{a}\cot\frac{t-z}{a}$ 的奇异部分和 $\frac{1}{t-z}$ 全同，知 Plemelj 公式仍成立，所以

$$\varphi(t_0) = \Phi^+(t_0) - \Phi^-(t_0); \tag{1.3}$$

而(1.1)成为

$$\Phi^+(t_0) = G(t_0)\,\Phi^-(t_0) + g(t_0),\quad t_0\in L, \tag{1.4}$$

其中

$$G = \frac{A-B}{A+B},\quad g = \frac{f}{A+B}. \tag{1.5}$$

此外，由于

$$\Phi(\pm\infty\mathrm{i}) = \pm\frac{1}{2a\pi}\int_L \varphi(t)\mathrm{d}t,$$

我们还有

$$\Phi(-\infty\mathrm{i}) = -\Phi(+\infty\mathrm{i}). \tag{1.6}$$

反之，如果 $\Phi(z)$ 是一周期分区全纯函数，满足边值条件(1.4)与附加条件(1.6)，则由(1.3)定义的 $\varphi(t_0)$ 必为方程(1.1)的解. 这只要把(1.2)右边的积分暂记为 $\Psi(z)$，则

$$\Psi(-\infty\mathrm{i}) = -\Psi(+\infty\mathrm{i}),\quad \varphi(t_0) = \Psi^+(t_0) - \Psi^-(t_0),$$

$$(A+B)\Psi^+ - (A-B)\Psi^- = K\varphi.$$

由以上各式知，$\Phi(z)-\Psi(z)$ 以 $a\pi$ 为周期且在全平面全纯，在 $z=\pm\infty\mathrm{i}$ 处反号. 把 Liouville 定理应用于周期函数，立刻知道 $\Phi(z)=\Psi(z)$，于是得(1.1)式.

这样我们得到

基本引理 方程(1.1)等价于周期 Riemann 边值问题(1.4)以及附加条件(1.6)；把后一问题的解求出后，(1.1)的解以公式(1.3)给出.

注意，这个基本引理对 §2 所考虑的情况也成立.

2. 指标与典则函数. 不妨取定坐标系，使各 $L_j(j=1,\cdots,p)$ 都不通过 $z=\pm\frac{1}{2}a\pi$ 及其周期合同点. 作一周期带形 S，使所有 L_j 都落在 S 内部，并设 $z=\pm\frac{1}{2}a\pi$ 在带形的两边上(图 1). 不失一般性，假定所有 L_j 都以反时针方向为正向，记 L_j 的内域为 S_j^+，而记 S_j^+ 在带形 S 中的补域为 S_j^-. 注意，对于不同的 j，S 中任意一点可能在 S_j^+ 中或

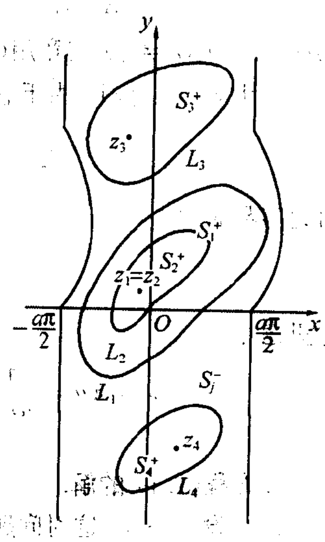

图 1

① 如果容许 A^2-B^2 在 L 上某些点处为零，则可采用[3]，§25 中的方法，并类似地讨论.

S_j^- 中；但 $z=\pm\infty i$ 在一切 $S_j^-(j=1,\cdots,p)$ 中.

令

$$k_j=\frac{1}{2\pi}[\arg G]_{L_j}, \tag{1.7}$$

则

$$k=\sum_{j=1}^{p}k_j \tag{1.8}$$

就是问题(1.4)的指标，也称为方程(1.1)的指标.

在每一 S_j^+ 内取定一点 z_j，于是

$$G(t)\Big/\left(\tan\frac{t}{a}-\tan\frac{z_j}{a}\right)^{k_j}$$

当 t 沿 L_j 环行一周时辐角不变. 根据[1]的§1中的讨论以及[4](§35,4°)中的说明，令

$$\Gamma_j(z)=\frac{1}{2a\pi i}\int_{L_j}\log\frac{G(t)}{(\tan\frac{t}{a}-\tan\frac{z_j}{a})^{k_j}}\cdot\cot\frac{t-z}{a}\,dt, \tag{1.9}$$

$$X_j(z)=\begin{cases}X_j^+(z)=e^{\Gamma_j(z)}, & \text{当 } z\in S_j^+,\\ X_j^-(z)=e^{\Gamma_j(z)}/(\tan\frac{z}{a}-\tan\frac{z_j}{a})^{k_j}, & \text{当 } z\in S_j^-,\end{cases} \tag{1.10}$$

则

$$X(z)=X_1(z)\cdots X_p(z) \tag{1.11}$$

就是问题(1.4)的典则函数. 注意，一般说来，记号 $X^+(z)$ 或 $X^-(z)$ 没有什么意义，因为同一点 z 对于不同的 j 可在 S_j^+ 内或 S_j^- 内. 对于 $t_0\in L_j$ 来说，

$$X^\pm(t_0)=X_1(t_0)\cdots X_{j-1}(t_0)X_j^\pm(t_0)X_{j+1}(t_0)\cdots X_p(t_0),$$

其中 $X_r(t_0)$ $(r\neq j)$ 究竟用(1.10)中哪一式，要看 t_0 属于 S_r^+ 或属于 S_r^- 而定.

对于 $z=\pm\infty i$，由于它们属于一切 S_j^-，所以

$$X(\pm\infty i)=\prod_{j=1}^{p}e^{\Gamma_j(\pm\infty i)}/(\pm i-\tan\frac{z_j}{a})^{k_j}. \tag{1.12}$$

由此容易算出

$$G_\infty=\frac{X(-\infty i)}{X(+\infty i)}=(-1)^k e^{-i\nu}, \tag{1.13}$$

这里已令

$$\nu=\pi\mu=\frac{1}{a\pi i}\sum_{j=1}^{p}\int_{L_j}\log\frac{G(t)}{(\tan\frac{t}{a}-\tan\frac{z_j}{a})^{k_j}}\,dt-\frac{2}{a}\sum_{j=1}^{p}k_jz_j. \tag{1.14}$$

3. 方程(1.1)的解. Ⅰ. 先设 k 为偶数:

1° 设 $k>0$. 这时问题(1.4)的一般解为(见[1])

$$\Phi(z)=\frac{X(z)}{2a\pi i}\int_L\frac{g(t)}{X^+(t)}\cot\frac{t-z}{a}\,dt-\frac{1}{2}X(z)P_k(\tan\frac{z}{a}), \tag{1.15}$$

其中

$$P_k(w)=C_0+C_1w+\cdots+C_kw^k$$

为任意系数的 k 次多项式. 这时补充条件(1.6)成为

$$P_k(\mathrm{i}) + G_\infty P_k(-\mathrm{i}) = \frac{1-G_\infty}{a\pi}\int_L \frac{g(t)}{X^+(t)}\mathrm{d}t,$$

或

$$-\cos\frac{\pi\mu}{2}(C_0 - C_2 + \cdots) + \sin\frac{\pi\mu}{2}(C_1 - C_3 + \cdots) = f_0 \sin\frac{\pi\mu}{2}, \tag{1.16}$$

其中

$$f_0 = \frac{1}{a\pi\,\mathrm{i}}\int_L \frac{g(t)}{X^+(t)}\mathrm{d}t. \tag{1.17}$$

当取定 $C_0,\cdots,C_k$ 满足条件(1.16)，方程(1.1)的一般解为

$$\varphi(t_0) = \frac{X^+(t_0)+X^-(t_0)}{2X^+(t_0)}g(t_0) + \frac{X^+(t_0)-X^-(t_0)}{2a\pi\,\mathrm{i}}\int_L \frac{g(t)}{X^+(t)}\cot\frac{t-t_0}{a}\,\mathrm{d}t$$

$$-\frac{1}{2}[X^+(t_0)-X^-(t_0)]P_k\left(\tan\frac{t_0}{a}\right). \tag{1.18}$$

如果令

$$A^*(t_0) = \frac{A(t_0)}{A^2(t_0)-B^2(t_0)},\quad B^*(t_0) = \frac{B(t_0)}{A^2(t_0)-B^2(t_0)}, \tag{1.19}$$

$$Z(t_0) = [A(t_0)+B(t_0)]X^+(t_0) = [A(t_0)-B(t_0)]X^-(t_0), \tag{1.20}$$

$$\boldsymbol{K}^*f \equiv A^*(t_0)f(t_0) - \frac{B^*(t_0)Z(t_0)}{a\pi\,\mathrm{i}}\int_L \frac{f(t)}{Z(t)}\cot\frac{t-t_0}{a}\,\mathrm{d}t, \tag{1.21}$$

则方程(1.1)的一般解(1.18)可改写为

$$\varphi(t_0) = \boldsymbol{K}^*f + B^*(t_0)Z(t_0)P_k\left(\tan\frac{t_0}{a}\right), \tag{1.18$'$}$$

而 P_k 的系数 $C_0,\cdots,C_k$ 应满足条件(1.16)；这时，(1.17)成为

$$f_0 = \frac{1}{a\pi\,\mathrm{i}}\int_L \frac{f(t)}{Z(t)}\mathrm{d}t. \tag{1.17$'$}$$

于是我们得到：当 $k>0$ 时，方程(1.1)的一般解由(1.18)′式给出，其中 P_k 的系数要满足(1.16)，故共含 k 个独立常数.

2° 设 $k=0$. 这时(1.18)或(1.18)′中的 P_k 成为一常数 C_0，补充条件(1.16)现在成为

$$C_0\cos\frac{\pi\mu}{2} = -f_0\sin\frac{\pi\mu}{2}. \tag{1.22}$$

由此可见，如果 μ 不是奇实数，则应取

$$C_0 = -f_0\tan\frac{\pi\mu}{2}, \tag{1.23}$$

而方程(1.1)这时有唯一解

$$\varphi(t_0) = \boldsymbol{K}^*f - f_0\tan\frac{\pi\mu}{2}B^*(t_0)Z(t_0). \tag{1.24}$$

如果 μ 是一奇实数，则当

$$f_0 = 0 \tag{1.25}$$

时，条件(1.22)恒成立，C_0 可取任何值，这时方程(1.1)的一般解为

$$\varphi(t_0) = \boldsymbol{K}^*f + CB^*(t_0)Z(t_0), \tag{1.26}$$

这里 C 是任意常数；而若 $f_0\neq 0$，则不论 C_0 取何值，(1.22)不能成立，从而(1.1)无解.

总之，当 $k=0$ 时，如果 μ 不是奇实数，则(1.1)有唯一解(1.24)；如果 μ 是一奇实数，且 $f_0=0$，则(1.1)有一般解(1.26)，而若 $f_0\neq 0$，则方程无解.

注意，如果考虑的是齐次方程($f=0$)，则总有 $f_0=0$. 因此，对于方程 $k\varphi=0$，当 $k=0$ 时，如果 μ 不是奇实数，则它只有零解；如果 μ 是奇实数，则它有一个线性无关的解 $B^*(t_0)Z(t_0)$.

3° 设 $k<0$. 这时问题(1.4)当且仅当下列条件满足时才有解(见[1])：

$$\int_L \frac{g(t)}{X^+(t)} \frac{\sin^{j-1}\frac{t}{a}}{\cos^{j+1}\frac{t}{a}}\,\mathrm{d}t=0 \quad (j=1,\cdots,-k-1),$$

亦即

$$\int_L \frac{f(t)}{Z(t)}\left(\tan^{j-1}\frac{t}{a}+\tan^{j+1}\frac{t}{a}\right)\mathrm{d}t=0 \quad (j=1,\cdots,-k-1). \tag{1.27}$$

当这些条件满足时，问题(1.4)有唯一解

$$\Phi(z)=\frac{X(z)}{2a\pi\mathrm{i}}\int_L \frac{g(t)}{X^+(t)}\left(\cot\frac{t-z}{a}+\cot\frac{t}{a}\right)\mathrm{d}t. \tag{1.28}$$

这时，容易算出补充条件(1.6)为

$$\int_L \frac{g(t)}{X^+(t)}\left[(1+G_\infty)\tan\frac{t}{a}+\mathrm{i}(1-G_\infty)\right]\mathrm{d}t=0,$$

或即

$$\int_L \frac{f(t)}{Z(t)} \frac{\sin\left(\frac{t}{a}-\frac{\pi\mu}{2}\right)}{\cos\frac{t}{a}}\,\mathrm{d}t=0. \tag{1.29}$$

如果记

$$f_r=\frac{1}{a\pi\mathrm{i}}\int_L \frac{f(t)}{Z(t)}\tan^r\frac{t}{a}\,\mathrm{d}t, \tag{1.30}$$

则条件(1.27)与(1.29)可合写成($-k$ 个条件)

$$\begin{cases} f_0\sin\dfrac{\pi\mu}{2}=f_1\cos\dfrac{\pi\mu}{2}, \\ f_0=-f_2=f_4=\cdots \\ f_1=-f_3=f_5=\cdots \end{cases} \quad (\text{下标最多到 } -k). \tag{1.31}$$

当条件(1.31)满足时，从(1.28)容易算出方程(1.1)的唯一解为

$$\varphi(t_0)=\boldsymbol{K}^*f-f_1B^*(t_0)Z(t_0). \tag{1.32}$$

注意，如果 μ 不是奇实数，则由(1.31)中的第一个条件，还可把解(1.32)写成

$$\varphi(t_0)=\boldsymbol{K}^*f-f_0\tan\frac{\pi\mu}{2}B^*(t_0)Z(t_0),$$

其形式与(1.24)同.

于是我们得到：当 $k<0$ 时，(1.1)可解的充要条件是(1.31)成立(共 $-k$ 个条件)；当这些条件满足时，它的唯一解由(1.32)给出，特别，如果 μ 不是奇实数，这个解还可写成(1.24)的形式. 对于齐次方程 $K\varphi=0$，当 $k<0$ 时显然只有零解.

Ⅱ. 设 k 为奇数. 这时，只要把Ⅰ中的 ν 改为 $\nu+\pi$，就可得类似结论（当然不存在 $k=0$

的情况). 详细说明从略.

利用本节结果,立刻可得(1.1)的相联方程的一系列结果,只要注意到 Noether 定理现在仍成立.

§2 开口曲线与间断系数情况

1. 一般情况. 现设 L 是在周期带形 S 中带有若干结点的分段光滑曲线,它由 p 个光滑弧段 $L_1,\cdots,L_p$ 组成,各 L_j 取定正向. 仍设各 L_j 不通过 $z=\frac{1}{2}a\pi$ 及其周期合同点. 设 A, $B,f\in H_0$,①并设 $A^2-B^2\neq 0$,这里我们还容许 L 达到 S 的边界,不过 c 与 $c+a\pi$ 要看作同一点.

设 $c_1,\cdots,c_n$ 是 L 上的全部结点(包括各 L_j 的端点以及 A,B 的间断点). 对于特异结点、非特异结点、$h(c_1,\cdots,c_q)$ 类、属于这个类的指标等可与通常对 Cauchy 型奇异积分方程同样地定义. 例如,取整数 $\lambda_j(j=1,\cdots,n)$ 如[4]的 §78,则有指标

$$k=-\sum_{j=1}^{n}\lambda_j. \tag{2.1}$$

这时问题(1.4)的典则函数为

$$X(z)=\Pi(z)\mathrm{e}^{\Gamma(z)}, \tag{2.2}$$

其中

$$\Pi(z)=\prod_{j=1}^{n}\left(\tan\frac{z}{a}-\tan\frac{c_j}{a}\right)^{\lambda_j}, \tag{2.3}$$

$$\Gamma(z)=\frac{1}{2a\pi\mathrm{i}}\int_L\log G(t)\cdot\cot\frac{t-z}{a}\,\mathrm{d}t. \tag{2.4}$$

现在

$$G_\infty=\prod_{j=1}^{n}\left(\frac{\tan\frac{c_j}{a}+\mathrm{i}}{\tan\frac{c_j}{a}-\mathrm{i}}\right)^{\lambda_j}\exp\left\{-\frac{1}{a\pi}\int_L\log G(t)\,\mathrm{d}t\right\}=(-1)^k\mathrm{e}^{-\mathrm{i}\nu}=\mathrm{e}^{-\mathrm{i}\pi\mu}, \tag{2.5}$$

其中

$$\nu=\frac{1}{a\pi\mathrm{i}}\int_L\log G(t)\,\mathrm{d}t+\frac{2}{a}\sum_{j=1}^{n}\lambda_jc_j, \tag{2.6}$$

$$\mu=\frac{\nu}{\pi}-k. \tag{2.7}$$

以后的讨论完全与 §1 同,甚且 §1 中的结果现在全都成立,所不同的只是:这里 $X(z)$ 由(2.2)~(2.4)式给出,μ,ν 由(2.7),(2.6)给出,且解指的是 $h(c_1,\cdots,c_q)$ 类中的解.

在[5]中曾讨论过本节中的问题,但 L 限定在实轴上一周期段的若干区间.

2. 一个重要的特例. 如 $L_j=\widehat{a_jb_j}$ 是连接周期带形 S 左右边界上二合同点的光滑弧段:$b_j=a_j+a\pi$,且设它们彼此不相交(并设 L_j 在 a_j,b_j 处的切线平行),而对 A,B,f,φ 的假设仍同 §1,但在 a_j,b_j 处它们各自有相同的值,这是前述情况的一个特例. 为确定起见,设所

① 我们采用[4]中的记号. 对 $f(t)$ 的要求还可减弱,类似于[4]的 §102 附注 1 中所作.

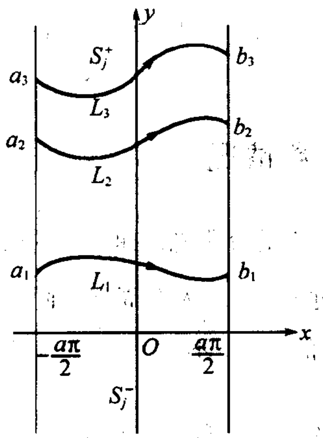

图 2

有 L_j 的正向取定自左至右.

不妨设所有 L_j 都在坐标原点 O 的上方(图 2)，记带形 S 在 L_j 的下侧域为 S_j^-，上侧域为 S_j^+.

这个特例虽是本节前述情况的一个特例，但它却与§1讨论的情况更为相似. 因为经 $w=\tan\frac{z}{a}$ 映射后，L_j 在 w 平面上的像 Γ_j 是包围 $w=i$ 的封闭光滑曲线. 现在所不同的是：$z=-\infty i$ 属于所有 S_j^-，$z=+\infty i$ 属于所有 S_j^+. 注意到这一点后，解法就与§1类似了.

这一特例在 $p=1$ 且 L 位于实轴上时，[6]中曾讨论过.

本文所讨论的函数类还可换作更广泛的类，例如，像 [7] 中讨论Cauchy 型奇异方程时所考虑的函数类完全可用到这里来.

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推广的留数定理及其应用

在 [1] 中我们曾引进高阶奇异积分和推广的留数定理，并作出了它在求解某类奇异积分方程中的应用. 这里我们指出，与通常一样，也可用这推广的留数定理来计算一些较复杂的积分；同时给出用它来求解一类奇异积分方程组的直接方法.

（一） 推广的留数定理

这里有必要先对推广的留数定理作一简单的复述，但条件已稍加放宽.

设 Γ 为 z 平面中一分段光滑的封闭曲线，取定其反时针向为正向，其内域为 D. 设 $f(t)\in H^{n-1}(\Gamma)$，即 $f(t)$ 在 Γ 上有 $n-1$ 阶的导数，后者满足强 Hölder 条件[2]；并设 $n>1$.

当把 Hadamard 关于实轴上发散积分的有限部分概念[3]这一思想运用到 Γ 上来，我们定义

$$\int_\Gamma \frac{f(\tau)}{(\tau-t)^n}d\tau=\frac{1}{(n-1)!}\int_\Gamma \frac{f^{(n-1)}(\tau)}{\tau-t}d\tau,\quad t\in\Gamma. \tag{1.1}$$

其中右边为 Cauchy 主值意义下的积分. 如 $P(\tau)$ 为一多项式，它在 Γ 上每个零点的重数不超过 n，则可定义

$$\int_\Gamma \frac{f(\tau)}{P(\tau)}d\tau,$$

只要把 $1/P(\tau)$ 写成最简分式的和并认为积分运算是线性的.

任取一点 z_0，记 z_0 关于 Γ 的绕数为 $\alpha(z_0)$，即

$$\alpha(z_0)=\frac{1}{2\pi i}\int_\Gamma \frac{d\tau}{\tau-z_0}=\frac{1}{2\pi}[\arg(\tau-z_0)]_\Gamma. \tag{1.2}$$

于是当 $z_0\in D$ 时，$\alpha(z_0)=1$；当 $z_0\,\overline{\in}\,\overline{D}=D+\Gamma$ 时，$\alpha(z_0)=0$；而当 $z_0\in\Gamma$ 时，$\alpha(z_0)=\dfrac{\theta_0}{2\pi}$，其中 θ_0 为 $\overline{D}$ 在 z_0 处的内角. 所以，如果 $t_0\in\Gamma$ 是 Γ 上的一个光滑点，则 $\alpha(t_0)=\dfrac{1}{2}$.

设函数 $\varphi(z)$ 定义在 Γ 上某点 t_0 附近 $\overline{D}$ 的某邻域中，且可写成

$$\varphi(z)=\sum_{k=1}^n \frac{c_k}{(z-t_0)^k}+\varphi_0(z),\quad c_n\neq 0,$$

其中 $\varphi_0(z)$ 在 D 内于 t_0 附近全纯，而在 $\overline{D}$ 上在 t_0 附近(除 t_0 外) 连续，且

$$|\varphi_0(z)|=O\left(\frac{1}{|z-t_0|^\mu}\right),\quad 0\leqslant\mu<1\ (\text{当 } z\to t_0), \tag{1.3}$$

则称 $\varphi(z)$ 关于 $\overline{D}$ 在 t_0 处有一 n 阶极点，并定义留数为 $\operatorname{res}\varphi(t_0)=c_1$；特别，如 $\varphi(z)$ 还在 D 内 t_0 附近有解析性，则有:

$$\operatorname{res}\varphi(t_0)=\lim_{\substack{z\to t_0\\ z\in D}}\frac{d^{n-1}}{dz^{n-1}}[(z-t_0)^n\varphi(z)], \tag{1.4}$$

这和 D 内 n 阶极点留数的定义是一致的.

我们有下面的

推广的留数定理　设 $f(z)$ 在 D 内解析，在 $\overline{D}$ 上有极点 $z_1,\cdots,z_N$(其中有些可能在 Γ 上)，此外则在 $\overline{D}$ 上 $f(z)$ 除去 Γ 上有限个点处有奇异性(1.3)之形外，处处连续，则

$$\frac{1}{2\pi\mathrm{i}}\int_\Gamma f(\zeta)\mathrm{d}\zeta=\sum_{k=1}^{N}\alpha(z_k)\operatorname{res}f(z_k).\tag{1.5}$$

证　由假设，可写

$$f(z)=\sum_{k=1}^{N}\left\{\frac{c_{k1}}{z-z_k}+\cdots+\frac{c_{k,n_k}}{(z-z_k)^{n_k}}\right\}+f_0(z),\quad c_{k,n_k}\neq 0,$$

其中 n_k 为极点 z_k 的阶数，而 $f_0(z)$ 在 $\overline{D}$ 上已无极点，其它性质同 $f(z)$. 所以，由(1.1)立刻知道

$$\frac{1}{2\pi\mathrm{i}}\int_\Gamma f(\zeta)\mathrm{d}\zeta=\sum_{k=1}^{N}\alpha(z_k)\operatorname{res}f(z_k)+\frac{1}{2\pi\mathrm{i}}\int_\Gamma f_0(\zeta)\mathrm{d}\zeta.$$

但由较一般的 Cauchy 定理(参看[4]，第一章引理 2.3)，上式右边积分为零. 于是(1.5)得证.

注　当 $f(z)$ 在 D 内有一些本性奇点时，(1.5)显然也成立.

关于推广的留数定理(1.5)式可作如下的解释. 对于任意一点 z，我们可以认为有"$\alpha(z)$ 个点"在 $\overline{D}$ 上，而另有"$1-\alpha(z)$ 个点"在其外. 当 $z\in D$ 或 $z\notin\overline{D}$ 时，这一说法与通常一致；当 $z\in\Gamma$ 时，如 D 在 z 处的内角为 θ，我们认为 z 的"$\frac{\theta}{2\pi}$个点"在 $\overline{D}$ 上，而"$1-\frac{\theta}{2\pi}$个点"在 D 的补域上. 我们把留数 $\operatorname{res}f(z)$ 也分做两部分，$\alpha(z)\operatorname{res}f(z)$ 就称作 $f(z)$"在 $\overline{D}$ 上"z 处的留数. 因此(1.5)式可解释为$\frac{1}{2\pi\mathrm{i}}\int_\Gamma f(\zeta)\mathrm{d}\zeta$ 恰好等于 $f(\zeta)$"在 $\overline{D}$ 上"的一切极点的留数之和.

如对于 $\overline{D}$ 上的一亚纯函数的零点(极点)的个数也作类似理解，即 $f(z)$ 的一个 r 重(阶)零点(极点)z_0 就认为"在 $\overline{D}$ 上"占有 $r\cdot\alpha(z_0)$ 个点，那么复变函数论中有许多定理可得到相应的推广. 例如，设在 $\overline{D}$ 上，$f(z)$ 有零点 $a_1,\cdots,a_m$，其重数分别为 $\lambda_1,\cdots,\lambda_m$；又有极点 $b_1,\cdots,b_n$，其阶数分别为 $\mu_1,\cdots,\mu_n$，则由推广的留数定理(1.5)，可得

$$\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{f'(z)}{f(z)}\mathrm{d}z=\sum_{k=1}^{m}\lambda_k\alpha_k(a_k)-\sum_{j=1}^{n}\mu_j\alpha_j(b_j).\tag{1.6}$$

这时，(1.6)可解释为：左边的积分等于 $f(z)$"在 $\overline{D}$ 上"零点占有的个数与极点占有的个数之差. 这就和熟知的结果得到了统一的叙述.

(二)　应用于计算积分

推广的留数定理，可以用来计算一些复杂的积分. 下面我们只举一个重要的例子来加以说明. 我们来计算

$$I_n=\int_{-\infty}^{+\infty}\left(\frac{\sin x}{x}\right)^n\mathrm{d}x\quad(n=1,2,\cdots).\tag{2.1}$$

首先我们来考虑

$$\int_{\Gamma}\frac{e^{ikz}}{z^n}dz \quad (k>0), \tag{2.2}$$

其中 Γ 为以原点为中心、以 R 为半径的圆盘的上半个的边界. 由(1.5)知，以上积分等于

$$2\pi i\cdot\frac{1}{2}\operatorname{res}\frac{e^{ikz}}{z^n}\bigg|_{z=0}=\frac{\pi i^n}{(n-1)!}k^{n-1}.$$

但这时通常的 Jordan 引理成立，故令 $R\to+\infty$ 时，可得

$$\int_{-\infty}^{+\infty}\frac{e^{ikx}}{x^n}dx=\frac{\pi i^n}{(n-1)!}k^{n-1},$$

左边事实上就是 Hadamard 的积分有限部分(主值意义下,下同). 由此立刻知道,

$$\int_{-\infty}^{+\infty}\frac{\cos 2kx}{x^{2n}}dx=(-1)^n\pi\frac{(2k)^{2n-1}}{(2n-1)!}, \tag{2.3}$$

$$\int_{-\infty}^{+\infty}\frac{\sin(2k+1)x}{x^{2n+1}}dx=(-1)^n\pi\frac{(2k+1)^{2n}}{(2n)!}. \tag{2.4}$$

但是，我们知道[5]

$$\sin^{2n}x=\frac{1}{4^n}\left[C_n^{2n}+2\sum_{k=1}^{n}(-1)^n C_{n-k}^{2n}\cos 2kx\right],$$

$$\sin^{2n+1}x=\frac{1}{4^n}\sum_{k=0}^{n}(-1)^k C_{n-k}^{2n+1}\sin(2k+1)x,$$

所以由(2.3),(2.4)立刻可得

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\sin^{2n}x}{x^{2n}}dx&=\frac{2\pi}{4^n}\sum_{k=1}^{n}(-1)^{n-k}C_{n-k}^{2n}\frac{(2k)^{2n-1}}{(2n-1)!}\\&=2n\pi\sum_{k=1}^{n}(-1)^{n-k}\frac{k^{2n-1}}{(n-k)!(n+k)!}\\&=\frac{2n\pi}{2^{2n-1}}\sum_{r=0}^{n-1}(-1)^r\frac{(2n-2r)^{2n-1}}{r!(2n-r)!};\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\sin^{2n+1}x}{x^{2n+1}}dx&=\frac{\pi}{4^n}\sum_{k=0}^{n}(-1)^{n-k}C_{n-k}^{2n+1}\frac{(2k+1)^{2n}}{(2n)!}\\&=\frac{(2n+1)\pi}{4^n}\sum_{k=0}^{n}(-1)^{n-k}\frac{(2k+1)^{2n}}{(n-k)!(n+k+1)!}\\&=\frac{(2n+1)\pi}{2^{2n}}\sum_{r=0}^{n}(-1)^r\frac{(2n-2r+1)^{2n}}{r!(2n-r+1)!}.\end{aligned}$$

此二式可以统一成一个式子：

$$I_n=\int_{-\infty}^{+\infty}\left(\frac{\sin x}{x}\right)^n dx=n\pi\sum_{r=0}^{\left[\frac{n-1}{2}\right]}(-1)^r\frac{\left(\frac{n}{2}-r\right)^{n-1}}{r!(n-r)!}. \tag{2.5}$$

用推广的留数定理，也可对一些发散积分在有限部分意义下互相换算，例如在(2.2)中把 Γ_B 理解为前述圆盘在第一象限的边界，就可进行这种换算，这里从略.

(三) 应用于求解奇异积分方程组

我们将应用推广的留数定理，来求解一类具解析的系数和核密度的奇异积分方程组，也就是把[1],[7]中的工作，推广到更一般的情形.

设要求解奇异积分方程组

$$a(t)\varphi(t)+\frac{1}{\pi \mathrm{i}}\int_L \frac{K(t,\tau)}{\tau-t}\varphi(\tau)\mathrm{d}\tau=f(t),\quad t\in L,\tag{3.1}$$

其中 L 为一光滑封闭曲线，围成区域 D；$a(z),K(z,\zeta)$ 是 $\overline{D}$ 上的 $N\times N$ 阶矩阵，$a(z)$ 的各分量在 D 内全纯，在 $\overline{D}$ 上连续，$K(z,\zeta)$ 的元当 $\zeta\in\overline{D}$ 时对 $z\in D$ 全纯，当 $z\in D$ 时对 $\zeta\in D$ 全纯，它们也都在 $\overline{D}$ 上连续；此外并设所涉及的函数当变元在 L 上变动时(包括已知函数 $f(t)$ 在内) 都有足够高阶的 $\in H$ (Hölder 条件) 的导数，未知函数 $\varphi(t)\in H$.

令 $b(z)=K(z,z)$，并记矩阵

$$S(z)=a(z)+b(z),\quad D(z)=a(z)-b(z),$$

而 $\det S(z)$ 在 D 内有零点 $\alpha_1,\cdots,\alpha_m$，在 L 上有零点 $\alpha_{m+1},\cdots,\alpha_{m+p}$，它们的重数依次为 $\lambda_1,\cdots,\lambda_{m+p}$；$\det D(z)$ 在 D 内有零点 $\beta_1,\cdots,\beta_n$，在 L 上有零点 $\beta_{n+1},\cdots,\beta_{n+q}$，它们的重数依次为 $\mu_1,\cdots,\mu_{n+q}$；这里 λ_k,μ_j 都是正整数，且 $\alpha_k\neq\beta_j$. 我们并采用下列记号：

$$\det S(z)=(z-\alpha_k)^{\lambda_k}S_k(z),\quad S_k(\alpha_k)\neq 0,\quad k=1,\cdots,m+p;$$

$$\det D(z)=(z-\beta_j)^{\mu_j}D_j(z),\quad D_j(\beta_j)\neq 0,\quad j=1,\cdots,n+q,$$

$$S^{-1}(z)=S^*(z)/\det S(z),\quad D^{-1}(z)=D^*(z)/\det D(z),$$

其中 $S^*(z),D^*(z)$ 分别为 $S(z),D(z)$ 的伴随矩阵.

用[7] 中类似的方法，令

$$\Phi(z)=\frac{1}{2\pi \mathrm{i}}\int_L \frac{K(z,\tau)}{\tau-z}\varphi(\tau)\mathrm{d}\tau,\tag{3.2}$$

于是由 Plemelj 公式，代入(3.1)，可得

$$\varphi(t)=D^{-1}(t)[f(t)-2\Phi^+(t)].\tag{3.3}$$

为要 $\varphi(t)$ 在 L 上有意义，必须而且只需

$$[D^*(t)f(t)]^{(r)}_{t=\beta_j}=2[D^*(t)\Phi^+(t)]^{(r)}_{t=\beta_j},$$

$$r=0,1,\cdots,\mu_j-1,\ j=n+1,\cdots,n+q.\tag{3.4}$$

以(3.3) 代入(3.2)，根据推广的留数定理(对向量函数而言)，得

$$\begin{aligned}\Phi(z)=&\frac{1}{2\pi \mathrm{i}}\int_L \frac{K(z,\tau)D^{-1}(\tau)f(\tau)}{\tau-z}\mathrm{d}\tau-2b(z)D^{-1}(z)\Phi(z)\\&-2\sum_{j=1}^{n}\mathrm{res}\left\{\frac{K(z,\tau)D^{-1}(\tau)\,\Phi^+(\tau)}{\tau-z}\right\}_{\tau=\beta_j}\\&-\sum_{j=n+1}^{n+q}\mathrm{res}\left\{\frac{K(z,\tau)D^{-1}(\tau)\,\Phi^+(\tau)}{\tau-z}\right\}_{\tau=\beta_j}.\end{aligned}$$

化简后，可得

$$\begin{aligned}\Phi(z)=D(z)S^{-1}(z)\Bigg\{&\frac{1}{2\pi \mathrm{i}}\int_L \frac{K(z,\tau)D^{-1}(\tau)f(\tau)}{\tau-z}\mathrm{d}\tau\\&-2\sum_{j=1}^{n}\mathrm{res}\left[\frac{K(z,\tau)D^*(\tau)\,\Phi^+(\tau)}{(\tau-z)(\tau-\beta_j)^{\mu_j}D_j(\tau)}\right]_{\tau=\beta_j}\\&-\sum_{j=n+1}^{n+q}\mathrm{res}\left[\frac{K(z,\tau)D^*(\tau)\,\Phi^+(\tau)}{(\tau-z)(\tau-\beta_j)^{\mu_j}D_j(\tau)}\right]_{\tau=\beta_j}\Bigg\}.\end{aligned}$$

如记

$$B_{jr}(z)=\frac{1}{r\,!\,(\mu_j-r-1)\,!}\,\frac{\partial^{\mu_j-r-1}}{\partial\tau^{\mu_j-r-1}}\left[\frac{K(z,\tau)D^*(\tau)}{(\tau-z)D_j(\tau)}\right]_{\tau=\beta_j},$$

$$r=0,1,\cdots,\mu_j-1;\quad j=1,\cdots,n+q,\tag{3.5}$$

并令

$$C_{jr}=\Phi^{(r)}(\beta_j),\quad r=0,1,\cdots,\mu_j-1;\ j=1,\cdots,n,$$

则上式可写成

$$\Phi(z)=D(z)S^{-1}(z)\left\{\frac{1}{2\pi\,\mathrm{i}}\int_L\frac{K(z,\tau)D^{-1}(\tau)f(\tau)}{\tau-z}\,\mathrm{d}\tau\right.$$

$$\left.-2\sum_{j=1}^{n}\sum_{r=0}^{\mu_j-1}B_{jr}(z)C_{jr}-\frac{1}{2}\sum_{j=n+1}^{n+q}\sum_{r=0}^{\mu_j-1}B_{jr}(z)f^{(r)}(\beta_j)\right\},\tag{3.6}$$

其中最后一项是这样得来的：当 $j>n$ 时，

$$\operatorname{res}\left[\frac{K(z,\tau)D^*(\tau)\,\Phi^+(\tau)}{(\tau-z)(\tau-\beta_j)^{\mu_j}D_j(\tau)}\right]_{\tau=\beta_j}$$

只牵涉到 $D^*(\tau)\Phi^+(\tau)$ 在 $\tau=\beta_j$ 处直到 μ_j-1 阶的导数，而由(3.4)，它与 $\frac{1}{2}D^*(\tau)f(\tau)$ 在 $\tau=\beta_j$ 处相应导数相同.

为要 $\varphi(t)$ 确实是(3.1)的解（$\varphi(t)$ 由(3.6)取边值代入(3.2)而得），以下三个要求应该满足：

1° 把 $z=\beta_j(j=1,\cdots,n)$ 代入(3.6)及其直到 μ_j-1 阶导数时，两边必须不相矛盾；

2° 在(3.6)中令 $z\to\beta_j(z\in D,\ j=n+1,\cdots,n+q)$ 时，必须保证(3.4)成立；

3° (3.6)右边必须确实是 D 内的全纯函数，且在 $\overline{D}$ 上 $\in H$.

为使 1° 满足，以 $z=\beta_l(l\leqslant n)$ 代入(3.6)及其直到 μ_j-1 阶导数，可得

$$C_{l\rho}=\frac{1}{2\pi\,\mathrm{i}}\int_L\frac{\partial^\rho}{\partial z^\rho}\left[\frac{D(z)S^{-1}(z)K(z,\tau)}{\tau-z}\right]_{z=\beta_l}D^{-1}(\tau)f(\tau)\mathrm{d}\tau$$

$$-2\sum_{j=1}^{n}\sum_{r=0}^{\mu_j-1}\frac{\mathrm{d}^\rho}{\mathrm{d}z^\rho}[D(z)S^{-1}(z)\beta_{jr}(z)]_{z=\beta_l}C_{jr}$$

$$-\frac{1}{2}\sum_{j=n+1}^{n+q}\sum_{r=0}^{\mu_j-1}\frac{\mathrm{d}^\rho}{\mathrm{d}z^\rho}[D(z)S^{-1}(z)B_{jr}(z)]_{z=\beta_l}f(\beta_j),$$

$$\rho=0,1,\cdots,\mu_j-1;\quad l=1,\cdots,n.\tag{3.7}$$

不过右边第二项的和式中 $j=l$ 的那一项，由于 $B_{jr}(z)$ 在 $z=\beta_l$ 处可能出现极点，这时 $z=\beta_l$ 要理解为 $z\to\beta_l$，而如这一极限为 ∞ 时，就应要求 C_{jr} 的相应元为零.

为要讨论要求 2°，首先注意(3.6)中的积分

$$\frac{1}{2\pi\,\mathrm{i}}\int_L\frac{K(z,\tau)D^{-1}(\tau)f(\tau)}{\tau-z}\mathrm{d}\tau=\frac{1}{2\pi\,\mathrm{i}}\int_L\frac{K(z,\tau)D^*(\tau)f(\tau)}{(\tau-\beta_j)^{\mu_j}(\tau-z)}\mathrm{d}\tau,$$

其中

$$\frac{1}{(\tau-\beta_j)^{\mu_j}(\tau-z)}=\frac{1}{(z-\beta_j)^{\mu_j}(\tau-z)}-\frac{1}{(z-\beta_j)^{\mu_j}(\tau-\beta_j)}$$

$$-\frac{1}{(z-\beta_j)^{\mu_j-1}(\tau-\beta_j)^2}-\cdots-\frac{1}{(z-\beta_j)(\tau-\beta_j)^{\mu_j}}.$$

于是(3.6)可改写为

$$D^*(z)\Phi(z)=D_j(z)S^{-1}(z)\frac{1}{2\pi i}\int_L\frac{K(z,\tau)D^*(\tau)f(\tau)}{D_j(\tau)}\times$$
$$\left\{\frac{1}{\tau-z}-\frac{1}{\tau-\beta_j}-\frac{z-\beta_j}{(\tau-\beta_j)^2}-\cdots-\frac{(z-\beta_j)^{\mu_j-1}}{(\tau-\beta_j)^{\mu_j}}\right\}d\tau$$
$$-2(z-\beta_j)^{\mu_j}D_j(z)S^{-1}(z)\sum_{\nu=1}^{n}\sum_{r=0}^{\mu_\nu-1}B_{\nu r}(z)C_{\nu r}$$
$$-\frac{1}{2}D_j(z)S^{-1}(z)\sum_{\nu=n+1}^{n+q}\sum_{r=0}^{\mu_\nu-1}(z-\beta_j)^{\mu_j}B_{\nu r}(z)f^{(r)}(\beta_\nu).$$

令 $z\to\beta_j(j>n)$ 时，得

$$D^*(\beta_j)\Phi(\beta_j)=\frac{1}{2}S^{-1}(\beta_j)b(\beta_j)D^*(\beta_j)f(\beta_j)$$
$$-\frac{1}{2}D_j(\beta_j)S^{-1}(\beta_j)\sum_{r=0}^{\mu_j-1}\lim_{z\to\beta_j}(z-\beta_j)^{\mu_j}B_{jr}(z)\cdot f^{(r)}(\beta_j). \tag{3.8}$$

但由于

$$\lim_{z\to\beta_j}(z-\beta_j)^{\mu_j}B_{jr}(z)$$
$$=\lim_{z\to\beta_j}(z-\beta_j)^{\mu_j}\frac{1}{r!\,(\mu_j-r-1)!}\frac{\partial^{\mu_j-r-1}}{\partial\tau^{\mu_j-r-1}}\left[\frac{K(z,\tau)D^*(\tau)}{D_j(\tau)(\tau-z)}\right]_{\tau=\beta_j}$$
$$=\lim_{z\to\beta_j}(z-\beta_j)^{\mu_j}\frac{1}{(\mu_j-1)!}\left[\frac{K(z,\tau)D^*(\tau)}{D_j(\tau)}\right]_{\tau=\beta_j}\cdot\left[\frac{\partial^{\mu_j-1}}{\partial\tau^{\mu_j-1}}\frac{1}{\tau-z}\right]_{\tau=\beta_j}$$
$$=-\frac{b(\beta_j)D^*(\beta_j)}{D_j(\beta_j)}.$$

以此代入 (3.8)，所以

$$D^*(\beta_j)\,\Phi^+(\beta_j)=S^{-1}(\beta_j)b(\beta_j)D^*(\beta_j)f(\beta_j).$$

因此，要求 2° 就是要求下式成立：

$$[2S^{-1}(\beta_j)b(\beta_j)-E]D^*(\beta_j)f(\beta_j)=0,\quad j=n+1,\cdots,n+q, \tag{3.9}$$

其中 E 为 N 阶单位矩阵.

现在再来看要求 3°. 当 $k\leqslant m$ 时，要求 $\Phi(z)$ 在 $z=\alpha_k$ 处全纯，由(3.6)，就是要求

$$4\sum_{j=1}^{n}\sum_{r=0}^{\mu_j-1}\frac{d^\sigma}{dz^\sigma}[D(z)S^*(z)B_{jr}(z)]_{z=\alpha_k}\cdot C_{jr}$$
$$+\sum_{j=n+1}^{n+q}\sum_{r=0}^{\mu_j-1}\frac{d^\sigma}{dz^\sigma}[D(z)S^*(z)B_{jr}(z)]_{z=\alpha_k}\cdot f^{(r)}(\beta_j)$$
$$=\frac{1}{\pi i}\int_L\frac{\partial^\sigma}{\partial z^\sigma}\left[\frac{D(z)S^*(z)K(z,\tau)}{\tau-z}\right]_{z=\alpha_k}D^{-1}(\tau)f(\tau)d\tau,$$
$$\sigma=0,1,\cdots,\lambda_k-1;\quad k=1,\cdots,m. \tag{3.10}$$

当 $k>m$ 时，为要保证 $\Phi(z)$ 在 $z=\alpha_k$ 处连续，就应要求

$$\frac{D(z)S^*(z)}{S_k(z)}\left\{\frac{1}{2\pi i}\int_L\frac{K(z,\tau)D^{-1}(\tau)f(\tau)}{\tau-z}d\tau\right.$$
$$\left.-2\sum_{j=1}^{n}\sum_{r=0}^{\mu_j-1}B_{jr}(z)C_{jr}-\frac{1}{2}\sum_{j=n+1}^{n+q}\sum_{r=0}^{\mu_j-1}B_{jr}(z)f^{(r)}(\beta_j)\right\}$$

当 $z\to\alpha_k(k>m)$ 时至少要有 λ_k 阶零点，亦即要求在 $z=\alpha_k$ 处上式及其直到 λ_k-1 阶导数

都等于零. 由于 $D(\alpha_k)$ 为满秩矩阵，$S_k(\alpha_k)\neq 0$. 所以上述要求又可写为

$$\frac{d^\rho}{dz^\rho}\left\{S^*(z)\left[\frac{1}{2\pi i}\int_L\frac{K(z,\tau)D^{-1}(\tau)f(\tau)}{\tau-z}d\tau\right.\right.$$
$$\left.\left.-2\sum_{j=1}^{n}\sum_{r=0}^{j-1}B_{jr}(z)C_{jr}-\frac{1}{2}\sum_{j=n+1}^{n+q}\sum_{r=0}^{\mu_j-1}B_{jr}(z)f^{(r)}(\beta_j)\right]\right\}_{z=\alpha_k}=0,$$
$$\rho=0,1,\cdots,\lambda_k-1;\quad k=m+1,\cdots,m+p. \tag{3.11}$$

条件(3.9)成立以及(3.7),(3.10),(3.11)相容就是方程组(3.1)可解的条件(条件的充分性证明以及下面 $\varphi(t)$ 在 L 上 $\in H$ 的证明，与[1]中类似，均从略). 当这些条件满足时，方程组(3.1)的解可在(3.6)中令 $z\to t$ 求极限后代入(3.3)而获得，经化简后，最后得(3.1)的解

$$\varphi(t)=S^{-1}(t)\left\{a(t)D^{-1}(t)f(t)-\frac{1}{\pi i}\int_L\frac{K(t,\tau)D^{-1}(\tau)f(\tau)}{\tau-t}d\tau\right.$$
$$\left.+4\sum_{j=1}^{n}\sum_{r=0}^{\mu_j-1}B_{jr}(t)C_{jr}+\sum_{j=n+1}^{n+q}\sum_{r=0}^{\mu_j-1}B_{jr}(t)f^{(r)}(\beta_j)\right\}, \tag{3.12}$$

这里 C_{jr} 就是由(3.7),(3.10),(3.11)求得的一般解组.

这样，就完全解决了方程组(3.1)的求解问题.

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ON SINGULAR INTEGRALS WITH SINGULARITIES OF HIGH FRACTIONAL ORDER AND THEIR APPLICATIONS

§ 1 Introduction

Singular integrals of high order in complex integration were first introduced in [1] and investigated in [2,3,4]. Using this concept, the classical residue theorem was extended to such cases and was applied to solving certain class of singular integral equations.[3,4] All these discussions were restricted to integrals with singularities of integral order. Here we study the more general case when the orders of singularities appeared may be fractional. A suitable definition of such kind of singular integrals is given and the residue theorem is generalized further.

Using these, we solve directly the following type of singular integral equations:

$$K\varphi \equiv a(t)\varphi(t) + \frac{1}{\pi i}\int_L \frac{K(t,\tau)}{\tau - t}\varphi(\tau)d\tau = f(t), \quad t \in L, \tag{1.1}$$

where L is a closed smooth contour in the complex plane which bounds a finite region D, and $a(z)$, $K(z,\zeta)$ possess certain properties of analyticity in D. There are a series of works about (1.1), either for normal case,[5~8] i.e., $a(t)\pm b(t)\neq 0$ on L where $b(t)=K(t,t)$, or for non-normal case.[3,4,9] Having made the above extensions, we consider in this paper the case when $a(t)\pm b(t)$ may have zero-points of fractional order on L and obtain an effective method of solution of (1,1).

Applying the obtained results, we also solve the following Riemann boundary value problem of non-normal type:

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in L, \tag{1.2}$$

where $G(t)$ may have zero-points and "poles" of fractional order on L. This problem was first solved in [10] (or, cf. [11]). But by the method used here, we obtain explicit forms both for the general solution of (1.2) and for the conditions of its solvability so that it would be very convenient in practice.

§ 2 Singular Integrals of Fractional Order

The concept of singular integrals of high order is actually an extension of that of the

finite part of divergent integrals originated by Hadamard in the real domain. [12] In complex integration, let $L=\widehat{ab}$ be a smooth arc oriented from a to b. We have the following definition in [1]:

Definition 1. Let n be a positive integer. We define

$$\int_L \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau=\frac{1}{n!}\int_L \frac{f^{(n)}(\tau)}{\tau-t}d\tau+\sum_{r=0}^{n-1}\frac{1}{n\cdots(n-r)}\Big[\frac{f^{(r)}(a)}{(a-t)^{n-r}}-\frac{f^{(r)}(b)}{(b-t)^{n-r}}\Big], \tag{2.1}$$

provided that $f(t)\in H_n$ (i.e., $f^{(n)}(t)$ satisfying Hölder condition H), and $f(t)\not\equiv 0$.

If L is closed, i.e. $a=b$, then (2.1) reduces to

$$\int_L \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau=\frac{1}{n!}\int_L \frac{f^{(n)}(\tau)}{\tau-t}d\tau. \tag{2.2}$$

Note that, for any c $(c\neq t)\in L$,

$$\int_L \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau=\int_{\widehat{ac}} \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau+\int_{\widehat{cb}} \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau, \tag{2.3}$$

since it is easy to verify that the right-hand member of (2.3) is actually independent of c.

We may extend Definition 1 to integrals with singularity of fractional order as follows.

Definition 2. Let n be a positive integer and $0<\alpha\leqslant 1$. We define

$$\int_L \frac{f(\tau)}{(\tau-t)^{n+\alpha}}d\tau=\frac{1}{(n+\alpha-1)\cdots\alpha}\int_L \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha}}d\tau$$

$$+\sum_{r=0}^{n-1}\frac{1}{(n+\alpha-1)\cdots(n+\alpha-r-1)}\Big[\frac{f^{(r)}(a)}{(a-t)^{n+\alpha-r-1}}-\frac{f^{(r)}(b)}{(b-t)^{n+\alpha-r-1}}\Big], \tag{2.4}$$

provided that $f(t)\in H_n$, $f(t)\not\equiv 0$.

If L is closed, then (2.4) reduces to

$$\int_L \frac{f(\tau)}{(\tau-t)^{n+\alpha}}d\tau=\frac{1}{(n+\alpha-1)\cdots\alpha}\int_L \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha}}d\tau. \tag{2.5}$$

Note that, in this definition, when $\alpha<1$, $(\tau-t)^{\alpha}$ is understood as a definite branch of the function when the plane is cut by a line joined from t to the point at infinity not intersecting L else where (hence the line is exterior to D when L is closed).

We also note that (2.3) remains valid in this case for the same reason.

Now we extend definition 2 further.

Definition 3. Let $t_1,\dots,t_p$ be different (inner) points of L, ordered in its positive direction. By choosing an arbitrary point c_j interior to each arc $\widehat{t_jt_{j+1}}$ $(j=1,\dots,p-1)$, we define $(c_0=a,\ c_p=b)$

$$\int_L \frac{f(\tau)}{\prod_{j=1}^{p}(\tau-t_j)^{n_j+r_j}}d\tau=\sum_{k=1}^{p}\int_{\widehat{c_{k-1}c_k}} \frac{f(\tau)}{\prod_{j=1}^{p}(\tau-t_j)^{n_j+r_j}}d\tau \tag{2.6}$$

where $n_j\geqslant 0$ is an integer and $0\leqslant\gamma_j<1$, provided that $f(t)\in H_n$, $n=\max n_j$.

If we decompose $1/\Pi(t)=1\Big/\prod_{j=1}^{p}(t-t_j)^{n_j}$ into partial fractions:

$$\frac{1}{\Pi(t)}=\sum_{j=1}^{p}\sum_{r=1}^{n_j}\frac{c_{jr}}{(t-t_j)^{r}}, \tag{2.7}$$

formally we have

$$\int_L \frac{f(\tau)\mathrm{d}\tau}{\prod_{j=1}^{p}(\tau-t_j)^{n_j+r_j}} = \sum_{j=1}^{p}\sum_{r=1}^{n_j} c_{jr}\int_L \frac{f(\tau)\mathrm{d}\tau}{\Pi_j(\tau)(\tau-t_j)^{r+r_j}}, \tag{2.8}$$

where $\Pi_j(t)=\prod_{k\neq j}(\tau-c_k)^{r_k}$, in which the integrals under summation are well defined by (2.6). We point out that (2.8) is actually true. In fact, the definition given in (2.4) is equivalent to

$$\int_L \frac{f(\tau)\mathrm{d}\tau}{(\tau-t)^{n+\alpha}}=\lim_{\varepsilon\to 0^+}\left\{\int_{\widehat{at'}}+\int_{\widehat{t''b}}-\sum_{r=0}^{n-1}\left[\frac{A_r}{(t'-t)^{n+\alpha-r-1}}-\frac{B_r}{(t''-t)^{n+\alpha-r-1}}\right]\right\}, \tag{2.9}$$

where t' and t'' are the two points cut on L with a distance ε from and in different sides of t, while A_r, B_r are certain definite bounded quantities which make the limit involved exists. In (2.6), the situation is similar. Such "divergent parts" to be subtracted in (2.6) could not be changed when the integrand is subjected to arithmetic operations and it follows (2.8).

We shall need some auxiliary results for closed L.

Lemma 1. *If $\varphi(t)\in H$, $c\in L$, $0<\gamma<1$, then in the vicinity of c,*

$$\Phi(z)\equiv\frac{1}{2\pi\mathrm{i}}\int_L\frac{\varphi(\tau)\mathrm{d}\tau}{(\tau-c)^{r}(\tau-z)}=\frac{\varphi(c)}{(z-c)^{r}}+O\left(\frac{1}{|z-c|^{r-\varepsilon}}\right),\quad z\in D; \tag{2.10}$$

where ε is an arbitrary small positive number.

Proof. Taking another point a on L, we decompose L into L_1, L_2, where L_1 is the oriented arc from c to a and L_2 is the other one from a to c. According to [13], we have

$$\frac{1}{2\pi\mathrm{i}}\int_{L_1}=\frac{\mathrm{e}^{\mathrm{i}\gamma\pi}}{2\mathrm{i}\sin\gamma\pi}\,\frac{\varphi(c)}{(z-c)^{r}}+O\left(\frac{1}{|z-c|^{r-\varepsilon}}\right),$$

$$\frac{1}{2\pi\mathrm{i}}\int_{L_2}\equiv-\frac{\mathrm{e}^{-\mathrm{i}\gamma\pi}}{2\mathrm{i}\sin\gamma\pi}\,\frac{\varphi(c)}{(z-c)^{r}}+O\left(\frac{1}{|z-c|^{r-\varepsilon}}\right).$$

Adding together, we obtain (2.10).

Corollary. *With notations as in Lemma 1, We have*

$$\Phi^+(t)=\frac{\varphi(c)}{(t-c)^{r}}+O\left(\frac{1}{|t-c|^{r-\varepsilon}}\right), \tag{2.11}$$

$$\frac{1}{2\pi\mathrm{i}}\int_{L}\frac{\varphi(\tau)\mathrm{d}\tau}{(\tau-c)^{r}(\tau-t)}=\frac{1}{2}\,\frac{\varphi(c)}{(t-c)^{r}}+O\left(\frac{1}{|t-c|^{r-\varepsilon}}\right). \tag{2.12}$$

We then have the following

Lemma 2. *If $f(t)\in H_n$, $c\in L$ and $0<\gamma<1$, then, in the vicinity of c,*

$$\frac{1}{2\pi\mathrm{i}}\int_L\frac{f(\tau)\mathrm{d}\tau}{(\tau-c)^{n-r}(\tau-z)}=\frac{f(c)}{(z-c)^{n+r}}+\frac{f'(c)}{(z-c)^{n+r-1}}+\cdots$$
$$+\frac{f^{(n)}(c)}{n!(z-c)^{r}}+O\left(\frac{1}{|z-c|^{r-1}}\right),\quad z\in D. \tag{2.13}$$

Proof. Let

$$\varphi(\tau)=\begin{cases}\dfrac{f(\tau)-f(c)(\tau-c)-\cdots-\dfrac{f^{(n-1)}(c)}{(n-1)!}(\tau-c)^{n-1}}{(\tau-c)^{n}}, & \tau\neq c,\\[2ex] \dfrac{1}{n!}f^{(n)}(c), & \tau=c,\end{cases}$$

so that $\varphi(t)\in H$. Then, we may rewrite

$$\frac{1}{2\pi i}\int_L \frac{f(\tau)d\tau}{(\tau-c)^{n+r}(\tau-z)}=\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)d\tau}{(\tau-c)^r(\tau-z)}+\frac{f(c)}{2\pi i}\int_L \frac{d\tau}{(\tau-c)^{n+r}(\tau-z)}$$
$$+\frac{f'(c)}{2\pi i}\int_L \frac{d\tau}{(\tau-c)^{n+r-1}(\tau-z)}+\cdots$$
$$+\frac{f^{(n-1)}(c)}{(n-1)!}\frac{1}{2\pi i}\int_L \frac{d\tau}{(\tau-c)^{1+r}(\tau-z)}; \tag{2.14}$$

here, we tacitly assume an identity

$$\int_L \frac{f(\tau)-f(c)(\tau-c)-\cdots-\dfrac{f^{(n-1)}(c)}{(n-1)!}(\tau-c)^{n-1}}{(\tau-c)^{n+r}(\tau-z)}d\tau=\int_L \frac{\varphi(\tau)d\tau}{(\tau-c)^r(\tau-z)}, \tag{2.15}$$

where the right-hand member is an ordinary improper integral. This is actually valid by the following Lemma 3.

Applying Lemma 1 and the equality

$$\frac{1}{2\pi i}\int_L \frac{d\tau}{(\tau-c)^{\beta}(\tau-z)}=\frac{1}{(z-c)^{\beta}},\quad z\in D,\ \beta:\text{fractional}, \tag{2.16}$$

we obtain (2.13). (2.16) follows directly from the extended residue theorem in § 3.

Lemma 3. *Let $f(t)\in H_n$ on $L=\widehat{ab}$, c is an inner point on L, $0<\alpha\leqslant 1$ and $f(c)=f'(c)=\cdots=f^{(n-1)}(c)=0$, then*

$$\int_L \frac{f(\tau)}{(\tau-c)^{n+\alpha}}d\tau=\int_L \frac{\Psi(\tau)}{(\tau-c)^{\alpha}}d\tau, \tag{2.17}$$

where $\Psi(t)=\dfrac{f(t)}{(t-c)^n}$ $(t\neq c)$ and $\Psi(c)=\dfrac{1}{n!}f^{(n)}(c)$.

Proof. Note that, in (2.9), the A_r's and B_r's in fact are linear combinations of $f(t),\ldots,f^{(n-1)}(t)$ with coefficients independent of f.[1] Hence, in our case $(t=c)$, $A_r=B_r=0$ $(r=0,1,\ldots,n-1)$ by supposition. That is, the left-hand integral in (2.17) is actually an ordinary convergent improper integral $(\alpha<1)$ or an integral of Cauchy type $(\alpha=1)$.

When we apply this lemma to prove (2.15), $f(t)$ is replaced by

$$\frac{f(t)-f(c)-\cdots-\dfrac{f^{(n-1)}(c)}{(n-1)!}(t-c)^{n-1}}{t-z}.$$

§ 3 Extension of the Residue Theorem

The classical residue theorem has been generalized in [3,4] to analytic functions with "poles" of integral order on the boundary of a region. Now we extend it further to the case when the integrand may have "poles" of fractional order on it.

In the sequel, L denotes a closed smooth contour which bounds a finite region D. we have

Theorem. *Let $\varphi(z)$ be a function analytic in D with singularities $z_1,\ldots,z_M$. Sup-*

pose $\varphi(z)$ *may be extended continuously to* L *except some points* $t_1,\ldots,t_N$ *on* L, *where the function has a "pole" of order* r_j (r_j *may be fractional*) *at* t_j, *i.e., in the vicinity of* t_j,

$$\varphi(z)=\frac{\varphi_j(z)}{(z-t_j)^{r_j}},\quad \varphi_j(t_j)\neq 0,\quad z\in\overline{D},\ r_j>0,\ j=1,\ldots,N.$$

Moreover, we assume $f(t)=\prod_{j=1}^{N}(t-t_j)^{r_j}\cdot\varphi(t)\in H_r$, $r=\max r_j$. *Then*

$$\frac{1}{2\pi i}\int_L\varphi(t)\mathrm{d}t=\sum_{k=1}^{M}\operatorname{res}\varphi(z_k)+\frac{1}{2}\sum_j{}^*\operatorname{res}\varphi(t_j),\tag{3.1}$$

where $\sum_j{}^*$ *denotes summation with respect to those* j'*s which correspond integral* r_j'*s, while the residue of* $f(t)$ *at* t_j, *as usual, is given by*

$$\operatorname{res}\varphi(t_j)=\frac{1}{(r_j-1)!}\frac{\mathrm{d}^{r_j-1}}{\mathrm{d}t^{r_j-1}}[\varphi(t)(t-t_j)^{r_j}]_{t=t_j}.\tag{3.2}$$

Proof. Obviously, it is sufficient to prove this theorem when $\varphi(z)$ has no singularities in D.

Let $r_j=n_j+\gamma_j$, $0\leqslant\gamma_j<1$, $n_j=[r_j]$. As in (2.8), we have

$$\frac{1}{2\pi i}\int_L\varphi(\tau)\mathrm{d}\tau=\sum_{j=1}^{N}\sum_{r=1}^{n_j}c_{jr}I_{jr},\tag{3.3}$$

where

$$I_{jr}=\frac{1}{2\pi i}\int_L\frac{f(\tau)\mathrm{d}\tau}{\Pi_j(\tau)(\tau-t_j)^{r+r_j}}.\tag{3.4}$$

For a fixed j, we take a small arc $\widehat{ab}$ on L including t_j in its interior but not containing any other t_k. If we surround each $t_k(k\neq j)$ with a sufficiently small circular arc Γ_k in D and replace each arc γ_k cut by Γ_k on L by Γ_k, then, by denoting the obtained contour by L', it is readily seen that

$$I_{jr}=\frac{1}{2\pi i}\int_{L'}\frac{f(\tau)\mathrm{d}\tau}{\Pi_j(\tau)(\tau-t_j)^{r+r_j}},$$

since, by (3.4),

$$I_{jr}=\frac{1}{2\pi i}\left(\int_{L-\widehat{ab}}+\int_{\widehat{ab}}\right),$$

and

$$\int_{L-\widehat{ab}}=\int_{L'-\widehat{ab}}$$

on account of $\int_{\gamma_k+\Gamma_k}=0$.

If $0<\gamma_j<1$, then, by (2.5),

$$I_{jr}=\frac{1}{(r+\gamma_j-1)\cdots\gamma_j}\frac{1}{2\pi i}\int_{L'}\frac{\frac{\mathrm{d}^r}{\mathrm{d}t^r}\frac{f(t)}{\Pi_j(t)}}{(t-t_j)^{\gamma_j}}\mathrm{d}t=0$$

since the integrand is holomorphic in D and has only a singularity of order less than unity on L.

If $\gamma_j=0$, then, by the generalized residue theorem proved in [4],

$$I_{jr}=\frac{1}{2}\operatorname{res}\left(\frac{f(t)}{\Pi_j(t)(t-t_j)^r}\right)_{t=t_j}.$$

But it is evident that

$$\operatorname{res}\left(\frac{f(t)}{\Pi_j(t)(t-t_j)^{r+\gamma_j}}\right)_{t=t_k}=0,\quad k\neq j,$$

therefore, for each j where $\gamma_j=0$,

$$\operatorname{res}\varphi(t_j)=\sum_{k=1}^{n}\sum_{r=1}^{n_k}c_{kr}\operatorname{res}\left(\frac{f(t)}{\Pi_k(t)(t-t_k)^{r+\gamma_k}}\right)_{t=t_j}=2\sum_{r=1}^{n_j}c_{jr}I_{jr}.$$

Hence,

$$\frac{1}{2\pi i}\int_L\varphi(t)dt=\frac{1}{2}\sum_j{}^*\operatorname{res}\varphi(t_j).$$

Remark. This theorem is also valid for multi-connected region D. Moreover, if the boundary of D consists of arc-wise smooth contours, the theorem remains true with the modification that, the last summation in (3.1) should be replaced by

$$\sum_j{}^*\frac{\theta_j}{2\pi}\operatorname{res}\varphi(t_j),$$

where θ_j is the internal angle of $\overline{D}$ at t_j (cf. [4]).

§ 4 Application to Solving Singular Integral Equations

Now we discuss a direct method of solution for equation (1.1), given $a(z)$ holomorphic in D and $\in H$ on $\overline{D}$, $K(z,\zeta)$ holomorphic in $D\times D$ and $\in H$ on $\overline{D}\times\overline{D}$. Denote $b(t)=K(t,t)$. Let $a(z)+b(z)$ have zeros $\alpha_1,\dots,\alpha_m$ in D with multiplicities $\lambda_1,\dots,\lambda_m$ respectively and $\alpha_{m+1},\dots,\alpha_{m+p}$ on L with multiplicities $\lambda_{m+1},\dots,\lambda_{m+p}$ respectively, where λ_k $(k>m)$ is not necessarily integral. Denote

$$a(z)+b(z)=(z-\alpha_k)^{\lambda_k}S_k(z),\ S_k(\alpha_k)\neq 0,\quad z\in\overline{D},$$

where $S_k(t)$ is supposed to be sufficiently smooth on L. Similarly, let $a(z)-b(z)$ have zeros $\beta_1,\dots,\beta_n$ in D with multiplicities $\mu_1,\dots,\mu_n$ respectively and $\beta_{n+1},\dots,\beta_{n+q}$ on L with multiplicities $\mu_{n+1},\dots,\mu_{n+q}$ respectively, where μ_j $(j>n)$ may be fractional. Denote

$$a(z)-b(z)=(z-\beta_j)^{\mu_j}D_j(z),\quad D_j(\beta_j)\neq 0,\ z\in D,$$

where $D_j(t)$ is supposed to be sufficiently smooth on L. For simplicity, we assume $\alpha_k\neq\beta_j$. We also assume $f(t)$ to be sufficiently smooth on L and the required solution $\varphi(t)$ to be continuous and hence $\in H$ on L.

Using the method originated in [7] and developed in [3], we define

$$\Phi(z)=\frac{1}{2\pi i}\int_L\frac{K(z,\tau)}{\tau-z}\varphi(\tau)d\tau,\quad z\in D,\tag{4.1}$$

so that

$$\varphi(t)=\frac{f(t)-2\Phi^+(t)}{a(t)-b(t)}.\tag{4.2}$$

We introduce a symbol $\{r\}$ denoting the greatest integer less than r. In order to guar-

antee the continuity of $\varphi(t)$, we ought to require

$$f^{(r)}(\beta_j)=2\Phi^{+(r)}(\beta_j),\quad r=0,1,\ldots,\{\mu_j\},\ j>n. \tag{4.3}$$

Substituting (4.2) into (4.1) and noting (4.3), we have, by the extended residue theorem in § 3,

$$\Phi(z)=\frac{a(z)-b(z)}{a(z)+b(z)}\left\{\frac{1}{2\pi i}\int_L\frac{K(z,\tau)f(\tau)d\tau}{[a(\tau)-b(\tau)](\tau-z)}\right.$$
$$\left.-2\sum_{j=1}^{n}\sum_{r=0}^{\mu_j-1}B_{jr}(z)\Phi^{(r)}(\beta_j)-\frac{1}{2}\sum_{j>n}{}^{*}\sum_{r=0}^{\mu_j-1}B_{jr}(z)f^{(r)}(\beta_j)\right\}, \tag{4.4}$$

where

$$B_{jr}(z)=\frac{1}{r!(\mu_j-r-1)!}\frac{\partial^{\mu_j-r-1}}{\partial\tau^{\mu_j-r-1}}\left\{\frac{K(z,\tau)}{D_j(\tau)(\tau-z)}\right\}_{\tau=\beta_j}. \tag{4.5}$$

Using similar reasoning as in [3], we know that (1.1) is solvable if and only if the following three conditions are fulfilled:

1° There should not happen any contradiction when we put $z=\beta_j$ ($j\leqslant n$) in (4.4) and its derivatives up to order μ_j-1;

2° The boundary values of (4.4) and its derivatives should satisfy (4.3);

3° $\Phi(z)$ in (4.4) is actually holomorphic in D and $\in H$ on $\bar{D}$.

Condition 1° is fulfilled automatically, which may be proved as in [5]. We show that condition 2° is also fulfilled automatically. We prove (4.3) for each β_j ($j>n$) and omit subscript j for the time being. By (4.4), we need only prove

$$\Psi^{+(r)}(\beta)=\frac{1}{2}f^{(r)}(\beta),\quad r=0,1,\ldots,\{\mu\}, \tag{4.6}$$

where

$$\Psi(z)=\frac{a(z)-b(z)}{a(z)+b(z)}\frac{1}{2\pi i}\int_L\frac{K(z,\tau)f(\tau)}{[a(\tau)-b(\tau)](\tau-z)}d\tau.$$

If μ is an integer, the method of proof is the same as that in [3]. We assume $\mu=s+\gamma$, where $s=[\mu]$, $0<\gamma<1$. Rewrite

$$\Psi(z)=\Psi_1(z)+\frac{a(z)-b(z)}{a(z)+b(z)}\Psi_2(z), \tag{4.7}$$

where

$$\Psi_1(z)=\frac{a(z)-b(z)}{a(z)+b(z)}\frac{b(z)}{2\pi i}\int_L\frac{f(\tau)d\tau}{[a(\tau)-b(\tau)](\tau-z)},$$

$$\Psi_2(z)=\frac{1}{2\pi i}\int_L\frac{K_0(z,\tau)f(\tau)}{D(\tau)(\tau-\beta)^{s+\gamma}}d\tau,$$

$$K_0(z,\tau)=\frac{K(z,\tau)-K(z,z)}{\tau-z}.$$

$\Psi_2(z)$ is holomorphic in D and may be extended continuously to L so that the second term in the right-hand member of (4.7) has zero derivatives up to order s when $z\to\beta$. Thus, in order to prove (4.6), it is sufficient to prove

$$\Psi_1{}^{+(r)}(\beta)=\frac{1}{2}f^{(r)}(\beta),\quad r=0,1,\ldots,s. \tag{4.6'}$$

By (4.7), we have

$$\frac{a(z)+b(z)}{a(z)-b(z)}\Psi_1(z)=\frac{(z-\beta)^{s+\gamma}}{2\pi i}\int_L\frac{f(\tau)d\tau}{D(\tau)(\tau-\beta)^{r+\gamma}(\tau-z)}. \tag{4.8}$$

The left-hand member of (4.8) may be rewritten as

$$(z-\beta)^{s+r}\Psi_1(z)+2\Psi_1(z)/D(z),$$

while its right-hand member by Lemma 2 equals

$$\left(\frac{f(t)}{D(t)}\right)_{t=\beta}+\left(\frac{f(t)}{D(t)}\right)'_{t=\beta}(z-\beta)+\cdots+\left(\frac{f(t)}{D(t)}\right)^{(s)}_{t=\beta}\frac{(t-\beta)^s}{s!}.$$

Therefore

$$2\left(\frac{\Psi_1(z)}{D(z)}\right)^{(r)}_{z=\beta}=\left(\frac{f(t)}{D(t)}\right)^{(r)}_{t=\beta},\quad r=0,1,\ldots,s,$$

which follows (4.6)'.

We turn to condition 3°. For $z=\alpha_k$ $(k\leqslant m)$, similar to [3], we should require

$$\sum_{j=1}^{n}\sum_{r=0}^{\mu_j-1}K_{jr}^{(s)}(\alpha_k)c_{jr}=\frac{1}{\pi i}\int_L\frac{f(\tau)}{a(\tau)-b(\tau)}\frac{\partial^s}{\partial z^s}\left(\frac{K(z,\tau)}{\tau-z}\right)_{z=\alpha_k}d\tau+\sum_{j>n}{}^{*}\sum_{r=0}^{\mu_j-1}B_{jr}^{(s)}(\alpha_k)f^{(r)}(\beta_j),$$

$$s=0,1,\ldots,\lambda_k-1;\ k=1,\ldots,m, \tag{4.9}$$

where

$$K_{jr}(z)=\frac{\partial^r}{\partial\tau^r}\left(\frac{K(z,\tau)}{\tau-z}\right)_{\tau=\beta_j},\quad j=1,\ldots,n.$$

Since

$$\Phi^+(t)=\frac{1}{2}\frac{b(t)f(t)}{a(t)+b(t)}+\frac{a(t)-b(t)}{a(t)+b(t)}\left\{\frac{1}{2\pi i}\int_L\frac{K(t,\tau)f(\tau)}{[a(\tau)-b(\tau)](\tau-t)}d\tau\right.$$
$$\left.-\frac{1}{2}\sum_{j=1}^{n}\sum_{r=0}^{\mu_j-1}K_{jr}(t)c_{jr}-\frac{1}{2}\sum_{j>n}{}^{*}\sum_{r=0}^{\mu_j-1}B_{jr}(t)f^{(r)}(\beta_j)\right\},$$

and $\dfrac{a(t)+b(t)}{a(t)-b(t)}\Phi^+(t)$ should have a zero-point of order $\{\lambda_k\}+1$ at $t=\alpha_k$ $(k>m)$, we must require

$$\sum_{j=1}^{n}\sum_{r=0}^{\mu_j-1}K_{jr}^{(s)}(\alpha_k)c_{jr}=\left(\frac{b(t)f(t)}{a(t)-b(t)}\right)^{(s)}_{t=\alpha_k}+\frac{1}{\pi i}\int_L\frac{f(\tau)}{a(\tau)-b(\tau)}\frac{\partial^s}{\partial t^s}\left(\frac{K(t,\tau)}{\tau-t}\right)_{t=\alpha_k}d\tau$$
$$-\sum_{j>n}{}^{*}\sum_{r=0}^{\mu_j-1}B_{jr}^{(s)}(\alpha_k)f^{(r)}(\beta_j),$$

$$s=0,\ldots,\{\lambda_k\};\ k=m+1,\ldots,m+p. \tag{4.10}$$

(4.9) and (4.10) constitute a system of linear equations in c_{jr}, the consistency of which is the necessary and sufficient condition for the solvability of (1.1). When this is satisfied, its general solution is

$$\varphi(t)=\frac{a(t)f(t)}{a^2(t)-b^2(t)}-\frac{1}{a(t)+b(t)}\frac{1}{\pi i}\int_L\frac{K(t,\tau)f(\tau)}{[a(\tau)-b(\tau)](\tau-t)}d\tau$$
$$+\frac{1}{a(t)+b(t)}\sum_{j=1}^{n}\sum_{r=0}^{\mu_j-1}K_{jr}(t)c_{jr}+\frac{1}{a(t)+b(t)}\sum_{j>n}{}^{*}\sum_{r=0}^{\mu_j-1}B_{jr}(t)f^{(r)}(\beta_j), \tag{4.11}$$

where $\{c_{jr}\}$ is the general system of solutions of (4.9) and (4.10).

Remark 1. If we allow $\varphi(t)\in H^*$ (cf. [13]) at $t=a_k$ $(k>m)$, i. e., it may have a singularity at a_k of order less than unity, the above discussions remain effective with the only modification that $\{\lambda_k\}$ in (4.10) should be replaced by $\{\lambda_k\}-1$.

Remark 2. If, in (1.1), $f(t)\in H^*$, for instance,

$$f(t)=\frac{f^*(t)}{\prod_l (t-c_l)^{\gamma_l}},\ c_l\in L\ (\neq a_k,\ \beta_j),\ 0<\gamma_l<1,\ f^*(t)\in H_n,$$

where n is sufficiently large, the above discussions remain effective too; of course the required solution should also be sought in the same class H^*. In fact, Theorem in § 3 remains valid in such case.

Remark 3. If $a(t)\pm b(t)$ have coincident zeros, similar discussions may be made by using the method given in [9]. This method is also effective even if (1.1) is a system of equations of dimension $N>1$.

§ 5 Method of Solution for a Class of Riemann Boundary Value Problems of Non-normal Type

Now we apply above results to solve the following Riemann boundary value problem: Find a sectionally holomorphic function $\Phi(z)$ satisfying the boundary condition

$$\Phi^+(t)=\frac{\Pi_2(t)}{\Pi_1(t)}G(t)\Phi^-(t)+g(t),\quad t\in L,\ \Phi(\infty)=0, \tag{5.1}$$

where $G(t)\in H$, $\neq 0$ on L, $g(t)$ is sufficiently smooth and

$$\left.\begin{aligned}\Pi_1(t)&=\prod_{k=1}^{p}(t-a_k)^{\lambda_k},\quad a_k\in L,\ \lambda_k>0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad a_k\neq\beta_j,\\ \Pi_2(t)&=\prod_{j=1}^{q}(t-\beta_j)^{\mu_j},\quad \beta_j\in L,\ \mu_j>0,\end{aligned}\right\} \tag{5.2}$$

where λ_k and μ_j may be fractional in general. $(t-a_k)^{\lambda_k}$ and $(t-\beta_j)^{\mu_j}$ are certain fixed branches as stated in § 2. We assume

$$O\in D^-(=D).$$

Denote $\eta=\mathrm{Ind}_L G(t)$. We have the canonical factorization[11]

$$t^{-\eta}G(t)=X^+(t)/X^-(t), \tag{5.3}$$

where

$$X(z)=\exp\left\{\frac{1}{2\pi i}\int_L\frac{\log[\tau^{-\eta}G(\tau)]}{\tau-z}d\tau\right\}. \tag{5.4}$$

Put

$$\Psi(z)=\Phi(z)/X(z),\quad g_0(t)=g(t)/X^+(t). \tag{5.5}$$

Then (5.1) becomes

$$\Psi^+(t)=\frac{\Pi_2(t)}{\Pi_1(t)}t^\eta\Psi^-(t)+g_0(t),\quad t\in L. \tag{5.6}$$

Since $\Psi(\infty)=0$, we may assume

$$\Psi(z)=\frac{1}{2\pi i}\int_L \frac{\psi(\tau)}{\tau-z}d\tau, \quad z\bar{\in}L. \tag{5.7}$$

By Plemelj's formula, (5.6) is equivalent to singular integral equation

$$\frac{1}{2}\Big[1+\frac{\Pi_2(t)}{\Pi_1(t)}t^\eta\Big]\psi(t)+\frac{1}{2}\Big[1-\frac{\Pi_2(t)}{\Pi_1(t)}t^\eta\Big]\frac{1}{\pi i}\int_L \frac{\psi(\tau)}{\tau-t}d\tau=g_0(t), \quad t\in L. \tag{5.8}$$

Consider the following different cases:

1° $\eta>0$. (5.8) may be written as

$$\frac{1}{2}[\Pi_1(t)+\Pi_2(t)t^\eta]\psi(t)+\frac{1}{2}[\Pi_1(t)-\Pi_2(t)t^\eta]\frac{1}{\pi i}\int_L \frac{\psi(t)}{\tau-t}d\tau=g_1(\tau), \quad t\in L. \tag{5.9}$$

where

$$g_1(t)=\Pi_1(t)g_0(t). \tag{5.10}$$

(5.9) is an equation of the type discussed in § 4, where $K(t,\tau)=b(t)$ and

$$a(t)+b(t)=\Pi_1(t), \quad a(t)-b(t)=\Pi_2(t)t^\eta. \tag{5.11}$$

By (4.10), the condition of solvability of (5.9) is the consistency of the system

$$\sum_{r=0}^{\eta-1}K_r^{(s)}(\alpha_k)c_r=\Big[\frac{b(t)g_1(t)}{a(t)-b(t)}\Big]^{(s)}_{t=\alpha_k}+\frac{1}{\pi i}\int_L \frac{g_1(\tau)}{a(\tau)-b(\tau)}\frac{\partial^s}{\partial t^s}\Big[\frac{b(t)}{\tau-t}\Big]_{t=\alpha_k}d\tau$$

$$-\sum_j{}^*\sum_{r=0}^{\mu_j-1}B_{jr}^{(s)}(\alpha_k)g_1^{(s)}(\beta_j), \quad s=0,\dots,\{\lambda_k\};\ k=1,\dots,p, \tag{5.12}$$

where

$$B_{jr}(t)=\frac{1}{r!(\mu_j-r-1)!}\frac{\partial^{\mu_j-r-1}}{\partial\tau^{\mu_j-r-1}}\Big[\frac{b(t)}{D_j(\tau)(\tau-t)}\Big]_{\tau=\beta_j}, \quad r=0,\dots,\mu_j-1, \tag{5.13}$$

in which

$$D_j(\tau)=\tau^\eta\prod_{k\neq j}(\tau-\beta_k)^{\mu_k}\equiv\tau^\eta\Pi_{2,j}(\tau), \quad K_r(t)=\frac{\partial^r}{\partial\tau^r}\Big[\frac{b(t)}{\tau-t}\Big]_{\tau=0}.$$

Substituting (5.11) into (5.13) and notifying the property of $\Pi_1(\tau)$ at $\tau=\alpha_k$, we have

$$B_{jr}^{(s)}(\alpha_k)=\frac{1}{2}\frac{1}{r!(\mu_j-r-1)!}\frac{\partial^{\mu_j-r+s-1}}{\partial\tau^{\mu_j-r-1}\partial t^s}\left\{\frac{t^\eta\Pi_2(t)}{\tau^\eta\Pi_{2,j}(\tau)(\tau-t)}\right\}_{\substack{t=\alpha_k\\ \tau=\beta_j}}, \tag{5.14}$$

$$K_r(t)=-\frac{1}{2}\cdot\frac{1}{r!}\tilde{K}_r(t),$$

where

$$\tilde{K}_r(t)=\frac{\Pi_1(t)-t^\eta\Pi_2(t)}{t^{r+1}}, \tag{5.15}$$

so that

$$\tilde{K}_r^{(s)}(\alpha_k)=-[t^{\eta-r-1}\Pi_2(t)]^{(s)}_{t=\alpha_k}.$$

Denote $C_r=-\frac{1}{r!}\frac{c_r}{2}$. Using (5.10) and (5.5), (5.12) is transferred to

$$\sum_{r=0}^{\eta-1}[t^{\eta-r-1}\Pi_2(t)]^{(s)}_{t=\alpha_k}C_r=\frac{1}{2\pi i}\int_L \frac{\Pi_1(\tau)g(\tau)}{\tau^\eta\Pi_2(\tau)X^+(\tau)}\frac{\partial^s}{\partial t^s}\Big[\frac{t^\eta\Pi_2(t)}{\tau-t}\Big]_{t=\alpha_k}d\tau$$

$$+\sum_j{}^*\sum_{r=0}^{\mu_j-1} B_{jr}^{(s)}(\alpha_k)\left[\frac{\Pi_1(t)g(t)}{X^+(t)}\right]_{t=\beta_j}^{(r)}, \quad s=0,\ldots,\{\lambda_k\},\ k=1,\ldots,p, \tag{5.16}$$

where $B_{jr}^{(s)}(\alpha_k)$ is given by (5.14). When (5.16) is consistent, the general solution of (5.9) is

$$\psi(t)=\frac{\Pi_1(t)+t^\eta\Pi_2(t)}{2t^\eta\Pi_2(t)}\frac{g(t)}{X^+(t)}-\frac{\Pi_1(t)-t^\eta\Pi_2(t)}{2\Pi_1(t)}\frac{1}{\pi i}\int_L\frac{\Pi_1(\tau)g(\tau)d\tau}{\tau^\eta\Pi_2(t)X^+(\tau)(\tau-t)}$$

$$+\frac{1}{\Pi_1(t)}\sum_{r=0}^{\eta-1}C_r\widetilde{K}_r(t)+\frac{1}{\Pi_1(t)}\sum_j{}^*\sum_{r=0}^{\mu_k-1}B_{jr}(t)\left[\frac{\Pi_1(t)g(t)}{X^+(t)}\right]_{t=\beta_j}^{(r)}, \tag{5.17}$$

where $\widetilde{K}_r(t)$ and $B_{jr}(t)$ are given by (5.15) and (5.13) respectively, and C_r is the general system of solutions of (5.16).

After obtaining $\psi(t)$, we get the general solution of (5.1) by (5.7) and (5.5).

2° $\eta=1$. The left-hand member of (5.16) is zero in this case, so that the conditions of solvability become

$$\frac{1}{2\pi i}\int_L\frac{\Pi_1(\tau)g(\tau)}{\Pi_2(\tau)X^+(\tau)}\frac{\partial^s}{\partial t^s}\left[\frac{\Pi_2(t)}{\tau-t}\right]_{t=\alpha_k}d\tau+\sum_j{}^*\sum_{r=0}^{\mu_j-1}B_{jr}^{(s)}(\alpha_k)\left[\frac{\Pi_1(t)g(t)}{X^+(t)}\right]_{t=\beta_j}^{(r)}=0,$$

$$s=0,1,\ldots,\{\lambda_k\},\quad k=1,\ldots,p. \tag{5.18}$$

When they are satisfied, the general solution of (5.9) is again given by (5.17) (with $\eta=0$) and the solution of (5.1) may be obtained through (5.7) and (5.5).

3° $\eta<0$. In this case (5.8) may be rewritten as

$$\frac{1}{2}[t^{-\eta}\Pi_1(t)+\Pi_2(t)]\psi(t)+\frac{1}{2}[t^{-\eta}\Pi_1(t)-\Pi_2(t)]\frac{1}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}d\tau=g_2(t),\quad t\in L, \tag{5.19}$$

where

$$g_2(t)=\frac{t^{-\eta}\Pi_1(t)}{X^+(t)}g(t). \tag{5.20}$$

Thus, we have ($K(t,\tau)=b(t)$)

$$a(t)+b(t)=t^{-\eta}\Pi_1(t),\quad a(t)-b(t)=\Pi_2(t). \tag{5.21}$$

The conditions of solvability of (5.19) is reduced to

$$\frac{1}{\pi i}\int_L\frac{\tau^{-\eta}\Pi_1(\tau)g(\tau)}{\Pi_2(\tau)X^+(\tau)}\frac{\partial^s}{\partial z^s}\left[\frac{\Pi_2(z)}{\tau-z}\right]_{z=0}+\sum_j{}^*\sum_{r=0}^{\mu_j-1}B_{jr}^{(s)}(0)\left[\frac{t^{-\eta}\Pi_1(t)g(t)}{X^+(t)}\right]_{t=\beta_j}^{(r)}=0,$$

$$s=0,1,\ldots,-\eta-1, \tag{5.22}$$

where

$$B_{jr}(z)=\frac{1}{r!(\mu_j-r-1)!}\frac{\partial^{\mu_j-r-1}}{\partial\tau^{\mu_j-r-1}}\left[\frac{z^{-\eta}\Pi_1(z)-\Pi_2(z)}{2\Pi_{2,j}(\tau)(\tau-z)}\right]_{\tau=\beta_j}$$

and hence

$$B_{jr}^{(s)}(0)=-\frac{1}{2}\frac{1}{r!(\mu_j-r-1)!}\frac{\partial^{\mu_j-r+s-1}}{\partial\tau^{\mu_j-r-1}\partial z^s}\left[\frac{\Pi_2(z)}{\Pi_{2,j}(\tau)(\tau-z)}\right]_{\substack{\tau=\beta_j\\ z=0}}.$$

(5.22) then may be simplified to

$$\frac{2}{\pi i}\int_L \frac{\tau^{-\eta-s-1}\Pi_1(\tau)}{\Pi_2(\tau)X^+(\tau)}g(\tau)d\tau$$
$$=\sum_j{}^*\sum_{r=0}^{\mu_j-1}\frac{1}{r!(\mu_j-r-1)!}\left[\frac{1}{\Pi_{2,j}(\tau)\tau^{s+1}}\right]_{\tau=\beta_j}^{(\mu_j-r-1)}\left[\frac{t^{-\eta}\Pi_1(t)g(t)}{X^+(t)}\right]_{t=\beta_j}^{(r)},$$
$$s=0,1,\ldots,-\mu-1. \tag{5.23}$$

When (5.23) is satisfied, equation (5.8) has the unique solution

$$\psi(t)=\frac{t^{-\eta}\Pi_1(t)+\Pi_2(t)}{2\Pi_2(t)X^+(t)}g(t)-\frac{1}{t^{-\eta}\Pi_1(t)}\frac{1}{2\pi i}\int_L\frac{[\Pi_1(\tau)-\tau^{\eta}\Pi_2(\tau)]\Pi_1(\tau)g(\tau)}{\Pi_2(\tau)X^+(\tau)(\tau-t)}d\tau$$
$$+\frac{1}{t^{-\eta}\Pi_1(t)}\sum_j{}^*\sum_{r=0}^{\mu_j-1}B_{jr}(t)\left[\frac{\Pi_1(t)t^{-\eta}g(t)}{X^+(t)}\right]_{t=\beta_j}^{(r)}. \tag{5.24}$$

After getting $\psi(t)$, we may obtain $\Phi(z)$ as above.

The results are rather simple when $\lambda_k,\mu_j<1$ and may be expressed as follows:

1° $\eta>0$. (5.1) has the general solution

$$\Phi(z)=\begin{cases}\dfrac{z^{\eta}\Pi_2(z)X(z)}{\Pi_1(z)}\dfrac{1}{2\pi i}\displaystyle\int_L\frac{\Pi_1(\tau)g(\tau)}{\tau^{\eta}\Pi_2(\tau)X^+(\tau)}\frac{d\tau}{\tau-z}+\frac{\Pi_2(z)X(z)}{\Pi_1(z)}\sum_{r=0}^{\eta-1}C_rz^{\eta-r+1}, \\ \qquad\qquad\text{when } z\in D^+; \\ \dfrac{X(z)}{2\pi i}\displaystyle\int_L\frac{\Pi_1(\tau)g(\tau)}{\tau^{\eta}\Pi_2(\tau)X^+(\tau)}\frac{d\tau}{\tau-z}+\sum_{r=0}^{\eta-1}C_rz^{-r-1}, \quad \text{when } z\in D^-,\end{cases} \tag{5.25}$$

where $C_0,\ldots,C_{\eta-1}$ are constants satisfying the following system of equations:

$$\sum_{r=0}^{\eta-1}\alpha_k^{-r-1}C_r=\frac{1}{2\pi i}\int_L\frac{\Pi_1(\tau)g(\tau)}{\tau^{\eta}\Pi_2(\tau)X^+(\tau)}\frac{d\tau}{\tau-\alpha_k},\quad k=1,\ldots,p; \tag{5.26}$$

if the latter is consistent; otherwise, (5.1) is unsolvable.

Hence, *if $p<\eta$, (5.1) is solvable and there are $\eta-p$ arbitrary constants in the general solution; if $\eta=p$, (5.1) has a unique solution; if $p>\eta$, there are $p-\eta$ conditions of solvability and (5.1) is uniquely solvable if they are fulfilled.*

2° $\eta=0$. We then have

$$\Phi(z)=\begin{cases}\dfrac{\Pi_2(z)X(z)}{\Pi_1(z)}\dfrac{1}{2\pi i}\displaystyle\int_L\frac{\Pi_1(\tau)g(\tau)}{\Pi_2(\tau)X^+(\tau)}\frac{d\tau}{\tau-z}, & \text{when } z\in D^+, \\ \dfrac{X(z)}{2\pi i}\displaystyle\int_L\frac{\Pi_1(\tau)g(\tau)}{\Pi_2(\tau)X^+(\tau)}\frac{d\tau}{\tau-z}, & \text{when } z\in D^-,\end{cases} \tag{5.27}$$

if the following conditions are satisfied:

$$\int_L\frac{\Pi_1(\tau)g(\tau)}{\Pi_2(\tau)X^+(\tau)}\frac{d\tau}{\tau-\alpha_k}=0,\quad k=1,\ldots,p. \tag{5.28}$$

Hence, (5.1) *has a unique solution if and only if these p conditions are satisfied.*

3° $\eta<0$. Besides (5.28), another $-\eta$ conditions of solvability should be fulfilled:

$$\int_L\frac{\tau^s\Pi_1(\tau)g(\tau)}{\Pi_2(\tau)X^+(\tau)}d\tau=0,\quad s=0,1,\ldots,-\eta-1, \tag{5.29}$$

when (5.28) and (5.29) are satisfied, (5.1) has the unique solution

$$\Phi(z)=\begin{cases}\dfrac{\Pi_2(z)X(z)}{\Pi_1(z)}\dfrac{1}{2\pi i}\displaystyle\int_L\dfrac{\Pi_1(\tau)g(\tau)}{\Pi_2(\tau)X^+(\tau)}\dfrac{d\tau}{\tau-z}, & \text{when } z\in D^+,\\ \dfrac{z^{-\eta}X(z)}{2\pi i}\displaystyle\int_L\dfrac{\Pi_1(\tau)g(\tau)}{\Pi_2(\tau)X^+(\tau)}\dfrac{d\tau}{\tau-z}, & \text{when } z\in D^-\end{cases}\tag{5.30}$$

Hence, (5.1) *has a unique solution if and only if* $p-\eta$ *conditions of solvability are satisfied.*

Remark. If we require $\Phi(z)$ has an order of n at infinity instead of $\Phi(\infty)=0$ at the very beginning, the method used here remains in effect. We need only replace (5.7) by

$$\Psi(z)=\frac{1}{2\pi i}\int_L\frac{\psi(\tau)}{\tau-z}d\tau+P_n(z),\quad z\overline{\in}L,$$

where $P_n(z)$ is a polynomial of degree n.

As an illustration, Let us solve the problem

$$\Phi^+(t)=\frac{t(t-1)}{\sqrt{t+1}}\Phi^-(t)+g(t),\quad t\in L,\ \Phi(\infty)=0,\ g(t)\in H,$$

where L is a closed smooth contour passing through $t=\pm1$, $O\in D$ and $\sqrt{z+1}$ is the branch with the value

$$\sqrt{t+1}\,|_{t=1}=\sqrt{2}.$$

Applying the above results ($\eta=1$), we get

$$\varphi(t)=\frac{1}{2}\left(1+\frac{\sqrt{t+1}}{t(t-1)}\right)g(t)-\frac{1}{2}\left(1-\frac{t(t-1)}{\sqrt{t+1}}\right)\frac{1}{\pi i}\int_L\frac{\sqrt{\tau+1}\,g(\tau)}{\tau(\tau-1)(\tau-t)}d\tau$$
$$+\frac{1}{2}C\left(\frac{t-1}{\sqrt{t+1}}-\frac{1}{t}\right)+\frac{g(1)}{\sqrt{2}}\left(\frac{t}{\sqrt{t+1}}-\frac{t}{t-1}\right),$$

where

$$C=\frac{1}{\pi i}\int_L\frac{g(\tau)d\tau}{\tau(\tau+1)(\tau-1)}-\frac{g(1)}{\sqrt{2}}.$$

Finally we obtain

$$\Phi(z)=\begin{cases}\dfrac{z(z-1)}{\sqrt{z+1}}\dfrac{1}{2\pi i}\displaystyle\int_L\dfrac{\sqrt{\tau+1}\,g(\tau)d\tau}{\tau(\tau-1)(\tau-z)}+\dfrac{g(1)}{\sqrt{2}}\dfrac{z}{\sqrt{z+1}}+\dfrac{C}{2}\dfrac{z-1}{\sqrt{z+1}}, & \text{when } z\in D^+,\\ \dfrac{1}{2\pi i}\displaystyle\int_L\dfrac{\sqrt{\tau+1}\,g(\tau)d\tau}{\tau(\tau-1)(\tau-z)}+\dfrac{g(1)}{\sqrt{2}}\dfrac{1}{z-1}+\dfrac{C}{2z}, & \text{when } z\in D^-.\end{cases}$$

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The Approximation of Cauchy-Type Integrals by Some Kinds of Interpolatory Splines

1. INTRODUCTION

There are many references on numerical evaluation of Cauchy-type integrals

$$T_L f(t)=\frac{1}{\pi i}\int_L \frac{f(\tau)}{\tau-t}d\tau, \quad t\in L \tag{1.1}$$

(possibly, with a weight function), by using orthogonal polynomials when L is an interval on the real axis, e.g., [1~4]. If L is the unit circle, it was shown in [5] that such integrals are approximated by interpolatory polynomial splines of any odd degree under the assumption that the density function $f(t)$ is continuous on the boundary of the circle up to a certain order of its derivatives. In [6], the same problem under a similar hypothesis on $f(t)$ was discussed for cubic interpolating splines in case L is an arbitrary smooth closed contour.

Let L be an arbitrary smooth curve, closed or open ($L=\widehat{ab}$), and

$$\Delta:\ a=t_0\prec t_1\prec \cdots \prec t_N=b$$

be a partition of L ($t_N=t_0$ when L is closed), where $t_j\prec t_{j+1}$ means that t_j precedes t_{j+1} when one travels along L in its given direction. If $f(t)$ is a function $\in H^\alpha$ (Hölder condition) on L and $S_\Delta(t)$ linearly interpolates $f(t)$ at the t_j's. Atkinson [7] succeeded in proving the uniform convergence of $T_L S_\Delta$ to $T_L f$ when L is closed:

$$\| T_L f - T_L S_\Delta \|_\infty \leqslant C_\varepsilon \delta^{\alpha-\varepsilon}, \quad \delta = \max|t_{j+1}-t_j|. \tag{1.2}$$

where C_ε is a constant independent of Δ, provided that

$$K_\Delta=\max|t_{j+1}-t_j|/\min|t_{j+1}-t_j| \tag{1.3}$$

is bounded for all the Δ's.

This result was proved by using a theorem (cf. [7], Theorem 2) which itself depends on an interesting but complicated lemma. We find that it may be easily proved by the following simple approach. Let L be a smooth contour and $f(t)\in H^\alpha$ $(0<\alpha<1)$ on it. In the Banach space H^α, we know[8] that

$$\|f\|_{H^\alpha}=\|f\|_\infty+M_\alpha(f), \tag{1.4}$$

where

$$M_\alpha(f)=\sup_{t,t'\in L}\frac{|f(t)-f(t')|}{|t-t'|^\alpha}; \tag{1.5}$$

then the operator T_L is linear with norm $\| T_L \|_a$. Therefore,

$$\| T_L f \|_\infty \leqslant \| T_L f \|_{H^a} \leqslant \| T_L \|_a \| f \|_{H^a} = \| T_L \|_a [\| f \|_\infty + M_a(f)]. \tag{1.6}$$

Now, if $f \in H^\epsilon$ $(0 < \epsilon < 1)$ and $f_n \in H^\epsilon$ is a sequence of functions on L, and if we can estimate $\| e_n(t) \|_\infty = \| f(t) - f_n(t) \|_\infty$ and $M_\epsilon(e_n)$, then, by (1.6), we may estimate $\| T_L e_n \|_\infty$ by

$$\| T_L e_n \|_\infty \leqslant C_\epsilon [\| e_n \|_\infty + M_\epsilon(e_n)]^{①} \tag{1.7}$$

We shall use (1.7) to estimate $\| T_L e_\Delta \|_\infty = \| T_L f - T_L S_\Delta \|_\infty$ both in the cases $f(t) \in H^a$ and $f(t) \in C^1$.

The structure of $T_L S_\Delta(t)$ is very simple because of its linearity on each sub-arc of L. However, $S_\Delta(t)$, as well as $T_L S_\Delta(t)$, is not smooth in general even if $f(t)$ is smooth. We shall establish analogous results for quadratic interpolating splines② in place of linear ones, so that $T_L S_\Delta(t)$ will be smooth. However, if $f'(t) \in H^a$ on L, we could not conclude the convergence of $T_L S_\Delta'(t)$ to $T_L f'(t)$. Using cubic interpolating splines of deficiency 2 which were discussed in [9], we may establish such convergence. We also establish the convergence of $T_L S_\Delta^*(t)$ to $T_L f(t)$ when $f \in H^a$, where S_Δ^* is the modified cubic interpolating spline of deficiency 2 which was also introduced in [9]. Here $T_L S_\Delta^*(t)$ as well as $S_\Delta^*(t)$ is smooth.

The results of this paper are also valid when L is an open smooth arc $\widehat{ab}$.

To show this, we note that (1.7) remains true in this case, provided that the additional requirements

$$f_n(a) = f(a), \quad f_n(b) = f(b) \tag{1.8}$$

are fulfilled. In fact, we may extend L to a smooth contour L^* and simultaneously extend $f(t)$ to $f^*(t)$ on L^* such that $f^*(t) \in H^\epsilon$ on L^*. Since (1.8) is satisfied, we may extend $f_n(t)$ to $f_n^*(t)$ on L^* such that $f_n^*(t) \equiv f^*(t)$ on $L^* - L$, so that $f_n^*(t) \in H^\epsilon$ on L^* too. Let $e_n^* = f^* - f_n^*$, then (1.7) is valid for e_n^*. Now $T_L e_n^* = T_L e_n$, $\| e_n \|_\infty = \| e_n^* \|_\infty$. Let us estimate $M_\epsilon(e_n^*)$. If $t, t' \in L$, then

$$|e_n^*(t) - e_n^*(t')| \leqslant M_\epsilon(e_n) |t - t'|^\epsilon;$$

if $t, t' \in L^* - L$, this is trivial. Let $t \in L$, $t' \in L^* - L$ and let a be situated on the shorter arc of $\widehat{tt'}$. Then

$$|e_n^*(t) - e_n^*(t')| = |e_n(t)| = |e_n(t) - e_n(a)| \leqslant M_\epsilon(e_n) |t - a|^\epsilon$$
$$\leqslant M_\epsilon(e_n) |\widehat{tt'}|^\epsilon \leqslant C_\epsilon M_\epsilon(e_n) |t - t'|^\epsilon,$$

where we have used a well-known inequality

① We use the symbol C_ϵ to represent a constant depending on ϵ which may take different values in different cases. Similarly. C represents an absolute constant taking various values in various cases.

② K. Atkinson also proved such convergence for "quadratic interpolating splines" which are different in meaning from those introduced here. He made interpolations of the values of $f(t)$ at three consecutive knots by a quadratic function. Such splines, in general, are not smooth too.

$$|\widehat{tt'}| \leqslant C|t-t'|. \tag{1.9}$$

Therefore,

$$M_\varepsilon(e_n^*) \leqslant C_\varepsilon M_\varepsilon(e_n)$$

and thence (1.7) remains valid.

2. THE LINEAR INTERPOLATING SPLINES

Let L be a smooth contour and $f(t)\in H^\alpha$ $(0<\alpha<1)$ on it. Denote

$$L_j=\widehat{t_jt_{j+1}},\quad y_j=f(t_j),\quad \Delta y_j=y_{j+1}-y_j,\quad D_j=\Delta y_j/\Delta t_j,$$

(We use the conventions $y_{j+n}=y_j$, $\Delta y_{j+n}=\Delta y_j$, etc.) Then for any linear interpolating spline $S_\Delta(t)$, we have

$$S_\Delta(t)\equiv S_j(t)=y_j+D_j(t-t_j),\quad t\in L_j,\ j=0,1,\dots,N-1. \tag{2.1}$$

It is evident that

$$|e_\Delta(t)|=|f(t)-S_\Delta(t)|\leqslant C\delta^\alpha \tag{2.2}$$

since, by (1.9),

$$|D_j(t-t_j)|\leqslant C\delta^\alpha\frac{|t-t_j|}{|\Delta t_j|}\leqslant C\delta^\alpha\frac{\Delta s_j}{|\Delta t_j|}\leqslant C\delta^\alpha,\quad t\in L_j,$$

where Δs_j is the arc-length of L_j. Similarly, if t,t' belong to the same L_j, then

$$|e_\Delta(t)-e_\Delta(t')|\leqslant C|t-t'|^\alpha\leqslant C\delta^{\alpha-\varepsilon}|t-t'|^\varepsilon. \tag{2.3}$$

If $t\in L_j$, $t'\in L_k$, $j\neq k$, then

$$\begin{aligned}|e_\Delta(t)-e_\Delta(t')| &\leqslant |e_\Delta(t)|+|e_\Delta(t')|=|e_\Delta(t)-e_\Delta(t_j)|+|e_\Delta(t_k)-e_\Delta(t')|\\ &\leqslant C[|t-t_j|^\alpha+|t_k-t'|^\alpha]\leqslant C\delta^{\alpha-\varepsilon}[|t-t_j|^\varepsilon+|t_k-t'|^\varepsilon]\\ &\leqslant C\delta^{\alpha-\varepsilon}(|\widehat{tt_j}|^\varepsilon+|\widehat{t_kt'}|^\varepsilon\leqslant C\delta^{\alpha-\varepsilon}|\widehat{tt'}|^\varepsilon\\ &\leqslant C\delta^{\alpha-\varepsilon}|t-t'|^\varepsilon,\end{aligned} \tag{2.3$'$}$$

i.e., (2.3) remains true. Therefore

$$M_\varepsilon(e_\Delta)\leqslant C\delta^{\alpha-\varepsilon}. \tag{2.4}$$

From (2.2) and (2.4), we obtain, by (1.7),

$$\|T_Le_\Delta\|_\infty\leqslant C_\varepsilon\delta^{\alpha-\varepsilon}. \tag{2.5}$$

Obviously, it is then also true for $\alpha=1$.

Now, let us consider the case $f(t)\in C^1$. We denote the modulus of continuity of $f^{(r)}(t)$ by $\omega_r(\delta)$ $(r\geqslant 0)$ throughout the paper. Then, if $t\in L_j$,

$$\begin{aligned}|f(t)-y_j-D_j(t-t_j)| &= \left|\int_{t_j}^t[f'(\tau)-D_j]\mathrm{d}\tau\right|\\ &= \left|\frac{1}{\Delta t_j}\int_{t_j}^t\mathrm{d}\tau\int_{t_j}^{t_{j+1}}[f'(\tau)-f'(\zeta)]\mathrm{d}\zeta\right|\\ &\leqslant \frac{\omega_1(\delta)}{|\Delta t_j|}\Delta s_j^2\leqslant C\omega_1(\delta)\delta.\end{aligned} \tag{2.6}$$

Similarly, if $t,t'\in L_j$, we have

$$|e_\Delta(t)-e_\Delta(t')|\leqslant C\omega_1(\delta)|t-t'|\leqslant C\omega_1(\delta)\delta^{1-\varepsilon}|t-t'|^\varepsilon; \tag{2.7}$$

if t,t' belong to different L_j's, then as in (2.3)', we also get (2.7). Hence

$$M_\varepsilon(e_\Delta) \leqslant C\omega_1(\delta)\delta^{1-\varepsilon}. \tag{2.8}$$

Again by (1.7), we have from (2.6) and (2.8),

$$\| T_L e_\Delta \|_\infty \leqslant C_\varepsilon \omega_1(\delta)\delta^{1-\varepsilon}. \tag{2.9}$$

If $L=\widehat{ab}$ is an open smooth arc, since

$$S_\Delta(a)=f(a), \quad S_\Delta(b)=f(b), \tag{2.10}$$

(2.5) and (2.9) remain valid.

Thus, we obtain

THEOREM 1. *Let L be a smooth curve, closed or open, and $S_\Delta(t)$ be the linear interpolating spline of $f(t)$. If $f(t)\in H^\alpha$ $(0<\alpha\leqslant 1)$, then*

$$|T_L f - T_L S_\Delta| \leqslant C_\varepsilon \delta^{\alpha-\varepsilon};$$

if $f(t)\in C^1$, then

$$|T_L f - T_L S_\Delta| \leqslant C_\varepsilon \omega_1(\delta)\delta^{1-\varepsilon}.$$

COROLLARY 1. *If $f'(t)\in H^\alpha$ $(0<\alpha\leqslant 1)$, then*

$$|T_L f - T_L S_\Delta| \leqslant C_\varepsilon \delta^{1+\alpha-\varepsilon}. \tag{2.11}$$

COROLLARY 2. *If $|f''(t)|$ is bounded, then*

$$|T_L f - T_L S_\Delta| \leqslant C_\varepsilon \delta^{2-\varepsilon}. \tag{2.12}$$

Corollary 2 is also a result due to Atkinson for closed L.

We note that all the values of the C_ε's in this section do not depend on Δ and so are independent of K_Δ in (1.3); therefore it is not necessary to require K_Δ to be bounded as stated in [7].

3. THE QUADRATIC INTERPOLATING SPLINES

Though there are works describing briefly polynomial interpolating splines on a Jordan curve[10,11] and dealing with quadratic splines on an interval of the real axis[12,13], we shall discuss the latter in the complex domain somewhat in detail, whether L is closed or open.

First, let us consider the case L is closed. A quadratic spline $S_\Delta(t)$ interpolating $f(t)$ at t_j's, if any, may be represented in various ways, for instance,

$$S_\Delta(t)\equiv S_j(t)=y_j+D_j(t-t_j)+\frac{A_j}{\Delta t_j}(t-t_j)(t_{j-1}-t),$$

$$t\in L_j,\ j=0,1,\dots,N-1. \tag{3.1}$$

with the requirements

$$A_{j-1}+A_j=-\Delta D_{j-1},\quad j=1,\dots,N, \tag{3.2}$$

where $\Delta D_{j-1}=D_j-D_{j-1}$, so as to guarantee the continuity of $S_\Delta'(t)$ at $t=t_j$.

If N is odd, we readily see that (3.2) is uniquely solvable:

$$A_j=-\frac{1}{2}(\Delta D_j-\Delta D_{j+1}+\Delta D_{j+2}-\Delta D_{j+3}+\cdots+\Delta D_{j+N-1})$$

$$=\Delta D_{j+1}+\Delta D_{j+3}+\cdots+\Delta D_{j+N-2},\quad j=0,1,\ldots,N-1,\tag{3.3}$$

since $\sum_{j=0}^{N-1}D_j=0$.

If N is even, (3.2) is solvable iff

$$\Delta D_0+\Delta D_2+\cdots+\Delta D_{N-2}=0\tag{3.4}$$

or

$$D_0+D_2+\cdots+D_{N-2}=D_1+D_3+\cdots+D_{N-1}.\tag{3.4$'$}$$

In the case $L=\widehat{ab}$ is an open arc, then, for such a spline, expression (3.1) remains effective, but requirements (3.2) are replaced by

$$A_{j-1}+A_j=-\Delta D_{j-1},\quad j=1,\ldots,N-1.\tag{3.5}$$

Hence, we have a freedom to choose A_0 or A_n. Or, more generally, we may subject them to an additional relation

$$\alpha A_0+\beta A_{N-1}=\gamma,\quad \beta\neq(-1)^N\alpha.\tag{3.6}$$

On solving (3.5) and (3.6), we get

$$A_0=\frac{(-1)^N\gamma+B_{N-2}\beta}{(-1)^N\alpha-\beta},\quad A_j=(-1)^j(A_0+B_{j-1}),\quad j=1,\ldots,N-1,\tag{3.7}$$

where

$$B_j=\Delta D_0-\Delta D_1+\Delta D_2-\Delta D_3+\cdots+(-1)^j\Delta D_j,\quad j=0,1,\ldots,N-2.\tag{3.8}$$

Thus, we have

THEOREM 2. *If L is closed, the quadratic interpolating spline $S_\Delta(t)$ exists uniquely when N is odd and it exists (but not uniquely) iff (3.4) or (3.4)$'$ is fulfilled when N is even; if L is open, it exists uniquely for arbitrary N with additional requirement (3.6).*

Now we turn to the problems of approximation.

Again we consider first the case L is smooth and closed (N: odd). We assume $f(t)\in C^1$. In order to estimate $e_\Delta(t)=f(t)-S_\Delta(t)$, by (3.1), it is necessary to estimate A_j in (3.3). Noting that

$$|\Delta D_{j-1}|\leqslant|D_j-y_j'|+|y_j'-D_{j-1}|\leqslant C\omega_1(\delta)\quad(y_j^{(r)}=f^{(r)}(t_j)),$$

we have

$$|A_j|\leqslant(N-1)C\omega_1(\delta),\tag{3.9}$$

and then

$$\left|\frac{A_j}{\Delta t_j}(t-t_j)(t_{j+1}-t)\right|\leqslant(N-1)C\omega_1(\delta)\delta\leqslant CK_\Delta\omega_1(\delta),\tag{3.10}$$

where K_Δ is given by (1.3). Thus, by (3.1), we obtain from (2.6) and (3.10),

$$\|e_\Delta(t)\|_\infty\leqslant CK_\Delta\omega_1(\delta).\tag{3.11}$$

If L is open, the similar estimate (3.9) is valid for $|B_j|$ by (3.8) and thereby also for $|A_j|$ on account of (3.7), if $\gamma=0$. Hence (3.10) as well as (3.11) remains true.

Therefore, we have

THEOREM 3. *For a quadratic interpolating spline $S_\Delta(t)$, if $f(t)\in C^1$, we have the estimate (with $\gamma=0$)*

$$|f(t)-S_\Delta(t)|\leqslant CK_\Delta\omega_1(\delta),$$

whether L is closed (N: odd) or open (N: arbitrary) with requirement (3. 6).

We could not expect $S_\Delta'(t)$ to tend to $f'(t)$ in this case even if L is closed. In fact, we can only easily obtain the estimate

$$|f'(t)-S_\Delta'(t)| \leqslant CK_\Delta \frac{\omega_1(\delta)}{\delta}. \tag{3.12}$$

Similarly, if $f(t)\in C$, we can only obtain

$$|f(t)-S_\Delta(t)| \leqslant CK_\Delta \frac{\omega(\delta)}{\delta}. \tag{3.13}$$

To estimate $\| T_L e_\Delta(t) \|_\infty$, we assume $f'(t)\in H^\alpha$ $(0 < \alpha < 1)$. Then (3. 11) becomes

$$\| e_\Delta(t) \|_\infty \leqslant CK_\Delta \delta^\alpha. \tag{3.14}$$

If $t, t' \in L_j$, then, by (3. 9),

$$\begin{aligned} |e_\Delta(t)-e_\Delta(t')| &\leqslant |f(t)-f(t')|+|D_j(t-t')|+\left|\frac{A_j}{\Delta t_j}(t+t'-t_j-t_{j+1})(t-t')\right| \\ &\leqslant C|t-t'|^\alpha+(N-1)C\delta^\alpha|t-t'| \\ &\leqslant C\delta^{\alpha-\epsilon}|t-t'|^\epsilon+CK_\Delta\delta^{\alpha-\epsilon}|t-t'|^\epsilon \leqslant CK_\Delta\delta^{\alpha-\epsilon}|t-t'|^\epsilon. \end{aligned} \tag{3.15}$$

If t, t' belong to different L_j's, we may proceed as in (2. 3)' and verify (3. 15) remains true. Therefore,

$$M_\epsilon(e_\Delta) \leqslant CK_\Delta\delta^{\alpha-\epsilon}. \tag{3.16}$$

Together with (3. 14), we have, by (1. 7),

$$\| T_L e_\Delta \|_\infty \leqslant C_\epsilon K_\Delta \delta^{\alpha-\epsilon}. \tag{3.17}$$

Obviously, (3. 17) remains valid then if $\alpha=1$.

For open arc L, since (2. 10) is fulfilled for $S_\Delta(t)$, (3. 17) is also valid.

Thus we obtain

THEOREM 4. *For a quadratic interpolating spline $S_\Delta(t)$, and $f'(t)\in H^\alpha$ $(0 < \alpha \leqslant 1)$, we have the estimate*

$$|T_L f - T_L S_\Delta| \leqslant C_\epsilon K_\Delta \delta^{\alpha-\epsilon},$$

whether L is closed (N: odd), or open (N: arbitrary) with the additional requirement (3. 6).

When $K_\Delta < C$ for a set of $\{\Delta\}$, (3. 11) and (3. 17) mean the corresponding uniform convergence when $\delta=\max|\Delta t_j|\to 0$.

4. THE CUBIC INTERPOLATING SPLINES OF DEFICIENCY 2

Let L be a smooth curve, closed or not. The cubic interpolating spline of deficiency 2 may be represented as

$$\begin{aligned} S_\Delta(t) \equiv S_j(t) = y_j + D_j(t-t_j) &+ \frac{y_j'-D_j}{\Delta t_j^2}(t-t_j)(t-t_{j+1})^2 \\ &+ \frac{y_{j+1}'-D_j}{\Delta t_j^2}(t-t_j)^2(t-t_{j+1}), \quad t\in L_j,\ j=0,\dots,N-1. \end{aligned} \tag{4.1}$$

We proved in [9]: if $f(t)\in C^r$ $(r=1,2,3)$, then

$$|e_\Delta^{(p)}(t)|=|f^{(p)}(t)-S_\Delta^{(p)}(t)|\leqslant C\omega_r(\delta)\delta^{r-p}\quad (0\leqslant p\leqslant r). \tag{4.2}$$

Let us estimate $\| T_L e_\Delta^{(p)}\|_\infty$, $p=0$, 1. We could not expect to estimate it for $p=2,3$, since $T_L S_\Delta^{(p)}(t)$ has unbounded discontinuities at the knots t_j's in these cases.

First, we consider L as closed. We assume $f(t)\in C^1$. If $t,t'\in L_j$, then by (4.1),

$$\begin{aligned} e_\Delta(t)-e_\Delta(t')=&[f(t)-f(t')-D_j(t-t')] \\ &+\frac{y_j'-D_j}{\Delta t_j^2}[(t-t_j)(t-t_{j+1})^2-(t'-t_j)(t'-t_{j+1})^2] \\ &+\frac{y_{j+1}'-D_j}{\Delta t_j^2}[(t-t_j)^2(t-t_{j+1})-(t'-t_j)^2(t'-t_{j+1})] \\ =&I_1+I_2+I_3. \end{aligned} \tag{4.3}$$

Analogous to (2.6), we have

$$|I_1|\leqslant C\omega_1(\delta)|t-t'|.$$

Noting that

$$\begin{aligned} &|(t-t_j)(t-t_{j+1})^2-(t'-t_j)(t'-t_{j+1})^2| \\ &=\left|\int_{t'}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}[(\tau-t_j)(\tau-t_{j+1})^2]\mathrm{d}\tau\right| \\ &=\left|\int_{t'}^{t}(\tau-t_{j+1})(3\tau-2t_j-t_{j+1})\mathrm{d}\tau\right| \\ &\leqslant\left|\int_{t'}^{t}(s_{j+1}-s)(s+s_{j+1}-2s_j)\mathrm{d}s\right|\leqslant C\Delta s_j^2|t-t'|, \end{aligned}$$

where s_j is the arc-length coordinate of t_j, we have

$$\begin{aligned} |I_2|&\leqslant\left|\frac{1}{\Delta t_j^2}\int_{t'}^{t}[f'(t_j)-f'(\tau)]\mathrm{d}\tau\right|\cdot C\Delta s_j^2|t-t'| \\ &\leqslant C\omega_1(\delta)|t-t'| \end{aligned}$$

and a similar estimate for $|I_3|$. Therefore,

$$|e_\Delta(t)-e_\Delta(t')|\leqslant C\omega_1(\delta)|t-t'|\leqslant C\omega_1(\delta)\delta^{1-\epsilon}|t-t'|^\epsilon.$$

If t,t' belong to different L_j's, it is easy to prove this estimate remains true by similar reasoning as before. Hence,

$$M_\epsilon(e_\Delta)\leqslant C\omega_1(\delta)\delta^{1-\epsilon}.$$

Together with (4.2) $(r=1,p=0)$, we obtain, by (1.7),

$$\| T_L e_\Delta\|_\infty\leqslant C_\epsilon\omega_1(\delta)\delta^{1-\epsilon}. \tag{4.4}$$

In order to estimate $\| T_L e_\Delta'\|_\infty$, we assume $f'(t)\in H^\alpha$ $(0<\alpha<1)$. Since

$$\begin{aligned} S_\Delta'(t)=D_j&+\frac{y_j'-D_j}{\Delta t_j^2}(t-t_{j+1})(3t-2t_j-t_{j+1}) \\ &+\frac{y_{j+1}'-D_j}{\Delta t_j^2}(t-t_j)(3t-t_j-2t_{j+1}),\quad t\in L_j, \end{aligned}$$

if $t,t'\in L_j$,

$$\begin{aligned} e_\Delta'(t)-e_\Delta'(t')=&[f'(t)-f'(t')]+\frac{y_j'-D_j}{\Delta t_j^2}\int_{t'}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau-t_{j+1})(3\tau-2t_j-t_{j+1})\mathrm{d}\tau \\ &+\frac{y_{j+1}'-D_j}{\Delta t_j^2}\int_{t'}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau-t_j)(3\tau-t_j-2t_{j+1})\mathrm{d}\tau=J_1+J_2+J_3. \end{aligned}$$

We have

$$
\begin{aligned}
|J_2| &\leqslant C\frac{\delta^\alpha}{|\Delta t_j|^2}\left|\int_{t'}^{t}(3\tau-2t_{j+1}-t_j)\mathrm{d}\tau\right| \\
&\leqslant C\frac{\delta^\alpha}{|\Delta t_j|^2}\left|\int_{t'}^{t}(2s_1-s_0-s)\mathrm{d}s\right| \\
&\leqslant C\frac{\delta^\alpha}{|\Delta t_j|}|t-t'| \leqslant C\delta^{\alpha-\epsilon}|t-t'|^{\epsilon}K_\Delta^{\epsilon}
\end{aligned}
$$

and a similar estimate for $|J_3|$. Obviously, this is also true for $|J_1|$. Therefore,

$$|e_\Delta'(t)-e_\Delta'(t')| \leqslant C\delta^{\alpha-\epsilon}|t-t'|^{\epsilon}K_\Delta^{\epsilon}.$$

which remains true if t,t' belong to different L_j's. Thus,

$$M_\epsilon(e_\Delta') \leqslant C\delta^{\alpha-\epsilon}K_\Delta^{\epsilon}.$$

By virtue of (4.2), we have, by (1.7),

$$\| T_L e_\Delta' \|_\infty \leqslant C_\epsilon \delta^{\alpha-\epsilon}K_\Delta^{\epsilon}. \tag{4.5}$$

Then, obviously, it remains true for $\alpha=1$.

Let us now assume $f(t)\in C^2$. We may obtain a better estimate for $\| T_L e_\Delta \|_\infty$ as well as $\| T_L e_\Delta' \|_\infty$. We rewrite $S_j(t)$ as[9]

$$
\begin{aligned}
S_j(t) &= y_j+y_j'(t-t_j)+\frac{1}{2}y_j''(t-t_j)^2 \\
&-[\Delta y_j-y_j'\Delta t_j-\frac{1}{2}y_j''\Delta t_j^2]\frac{(t-t_j)^2(t+t_j-2t_{j+1})}{\Delta t_j^3} \\
&-[\Delta y_j-y_{j+1}'\Delta t_j+\frac{1}{2}y_{j+1}''\Delta t_j^2]\frac{(t-t_j)^2(t-t_{j+1})}{\Delta t_j^3} \\
&+\frac{y_{j+1}''-y_j''}{2}\frac{(t-t_j)^2(t-t_{j+1})}{\Delta t_j}
\end{aligned} \tag{4.6}
$$

If $t,t'\in L_j$, we have

$$
\begin{aligned}
e_\Delta(t)-e_\Delta(t') &= \left\{f(t)-f(t')-y_j'(t-t')-\frac{1}{2}y_j''[(t-t_j)^2-(t'-t_j)^2]\right\} \\
&-[\Delta y_j-y_j'\Delta t_j-\frac{1}{2}y_j''\Delta t_j^2]\frac{1}{\Delta t_j^3}\int_{t'}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}[(\tau-t_j)^2(\tau+t_j-2t_{j+1})]\mathrm{d}\tau \\
&-[\Delta y_j-y_{j+1}'\Delta t_j+\frac{1}{2}y_{j+1}''\Delta t_j^2]\frac{1}{\Delta t_j^3}\int_{t'}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}|(\tau-t_j)^2(\tau-t_{j+1})|\mathrm{d}\tau \\
&-\frac{y_{j+1}''-y_j''}{2\Delta t_j}\int_{t'}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}[(\tau-t_j)^2(\tau-t_{j+1})]\mathrm{d}\tau \\
&=H_1+H_2+H_3+H_4.
\end{aligned}
$$

We have

$$|H_1| = \left|\int_{t'}^{t}\mathrm{d}\tau\int_{t_j}^{\tau}[f''(\zeta)-y_j'']\mathrm{d}\zeta\right| \leqslant C\omega_2(\delta)\delta|t-t'|,$$

$$|H_2| \leqslant C\frac{\omega_2(\delta)}{|\Delta t_j|}\left|\int_{t'}^{t}(s-s_j)(4s_{j+1}-s_j-3s)\mathrm{d}s\right| \leqslant C\omega_2(\delta)\delta|t-t'|$$

and similar estimates for $|H_3|$ and $|H_4|$. So we obtain

$$|e_\Delta(t)-e_\Delta(t')| \leqslant C\omega_2(\delta)\delta|t-t'| \leqslant C\omega_2(\delta)\delta^{2-\epsilon}|t-t'|^{\epsilon},$$

which may be verified to be valid also if t,t' belong to different L_j's. Thus,

$$M_\varepsilon(e_\Delta) \leqslant C\omega_2(\delta)\delta^{2-\varepsilon}.$$

Together with (4.2), we have, by (1.7),

$$\| T_L e_\Delta \|_\infty \leqslant C_\varepsilon \omega_2(\delta)\delta^{2-\varepsilon}. \tag{4.7}$$

In order to get a better estimate of $\| T_L e_\Delta' \|_\infty$, we differentiate (4.6):

$$S_j'(t) = y_j' + y_j''(t-t_j) - \left[\Delta y_j - y_j'\Delta t_j - \frac{1}{2}y_j''\Delta t_j^2\right]\frac{(t-t_j)(3t+t_j+4t_{j+1})}{\Delta t_j^3}$$
$$-\left[\Delta y_j - y_{j+1}'\Delta t_j + \frac{1}{2}y_{j+1}''\Delta t_j^2\right]\frac{(t-t_j)(3t-t_j-2t_{j+1})}{\Delta t_j^3}$$
$$+\frac{y_{j+1}''-y_j''}{2}\frac{(t-t_j)(3t-t_j-2t_{j+1})}{\Delta t_j}.$$

Hence, if $t, t' \in L_j$, we have

$$e_\Delta'(t) - e_\Delta'(t') = f'(t) - f'(t') - y_j''(t-t')$$
$$-\left[\Delta y_j - y_j'\Delta t_j - \frac{1}{2}y_j''\Delta t_j^2\right]\frac{1}{\Delta t_j^3}\int_{t'}^{t}\frac{d}{d\tau}(\tau-t_j)(3\tau+t_j-4t_{j+1})d\tau$$
$$-\left[\Delta y_j - y_{j+1}'\Delta t_j + \frac{1}{2}y_{j+1}''\Delta t_j^2\right]\frac{1}{\Delta t_j^3}\int_{t'}^{t}\frac{d}{d\tau}(\tau-t_j)(3\tau-t_j-2t_{j+1})d\tau$$
$$+\frac{y_{j+1}''-y_j''}{2\Delta t_j}\int_{t'}^{t}\frac{d}{d\tau}(\tau-t_j)(3\tau-t_j-2t_{j+1})d\tau.$$

Using the same reasoning as before, it is easily seen

$$|e_\Delta'(t) - e_\Delta'(t')| \leqslant C\omega_2(\delta)\delta^{1-\varepsilon}|t-t'|^\varepsilon,$$

which is valid also for t, t' belonging to different L_j's. Thus,

$$M_\varepsilon(e_\Delta') \leqslant C\omega_2(\delta)\delta^{1-\varepsilon},$$

and thereby, by (4.2), we obtain

$$\| T_L e_\Delta' \|_\infty \leqslant C_\varepsilon \omega_2(\delta)\delta^{1-\varepsilon}. \tag{4.8}$$

If $f(t) \in C^3$, after rewriting $S_j(t)$ as[9]

$$S_j(t) = y_j + y_j'(t-t_j) + \frac{y_j''}{2}(t-t_j)^2 + \frac{y_j'''}{6}(t-t_j)^3$$
$$-\left[\Delta y_j - y_j'\Delta t_j - \frac{y_j''}{2}\Delta t_j^2 - \frac{y_j'''}{6}\Delta t_j^3\right]\frac{(t-t_j)^2(t+t_j-2t_{j+1})}{\Delta t_j^3}$$
$$-\left[\Delta y_j - y_{j+1}'\Delta t_j + \frac{y_{j+1}''}{2}\Delta t_j^2 - \frac{y_{j+1}'''}{6}\Delta t_j^3\right]\frac{(t-t_j)^2(t-t_{j+1})}{\Delta t_j^3}$$
$$+\left[y_{j+1}'' - y_j'' - y_j'''\Delta t_j\right]\frac{(t-t_j)^2(t-t_{j+1})}{2\Delta t_j}$$
$$-\frac{y_{j+1}'''-y_j'''}{6}(t-t_j)^2(t-t_{j+1})$$

and proceeding as before, we may get

$$M_\varepsilon(e_\Delta) \leqslant C\omega_3(\delta)\delta^{3-\varepsilon}$$

and hence

$$\| T_L e_\Delta \|_\infty \leqslant C_\varepsilon \omega_3(\delta)\delta^{3-\varepsilon}. \tag{4.9}$$

In the same manner, we may also get

$$\| T_L e_\Delta' \|_\infty \leqslant C_\varepsilon \omega_3(\delta)\delta^{2-\varepsilon}. \tag{4.10}$$

We also note that, if L is open, we have in this case

$$S_\Delta(a)=f(a),\ S_\Delta(b)=f(b),\ S'_\Delta(a)=f'(a),\ S'_\Delta(b)=f'(b),$$

so that all the above results remain true.

Therefore, we obtain

THEOREM 5. *Let $S_\Delta(t)$ be the cubic interpolating spline of deficiency 2. If $f(t)\in C^r$, then*

$$|T_Lf - T_LS_\Delta| \leqslant C_\epsilon\omega_r(\delta)\delta^{r-\epsilon},\quad r=1,2,3, \tag{4.11}$$

$$|T_Lf' - T_LS'_\Delta| \leqslant C_\epsilon\omega_r(\delta)\delta^{r-1-\epsilon},\quad r=2,3, \tag{4.12}$$

and if $f'(t)\in H^\alpha$ $(0<\alpha\leqslant 1)$, then

$$|T_Lf' - T_LS'_\Delta| \leqslant C_\epsilon K^\epsilon_\Delta\delta^{\alpha-\epsilon} \tag{4.13}$$

whether L is a smooth closed contour or an open smooth arc.

5. THE MODIFIED CUBIC INTERPOLATING SPLINES OF DEFICIENCY 2

When $f(t)$ does not possess any derivative but only $\in H^\alpha$, in order to approximate T_Lf by smooth functions, we may use the modified cubic interpolating splines introduced in [9].

Suppose L is closed and $S_\Delta(t)$ is the linear spline as in Section 2. By taking two points t'_j and t''_j, respectively on each L_{j-1} and L_j such that

$$|\widehat{t'_jt_j}|=|\widehat{t_jt''_j}|=\lambda\min\{\Delta s_{j-1},\Delta s_j\},\quad \lambda\leqslant\frac{1}{2}.$$

We interpolate $S_\Delta(t)$ cubically on each $L'_j=\widehat{t'_jt''_j}$ with the values and the first derivatives of $S_\Delta(t)$ at t'_j, t''_j and get $S^*_j(t)$. Then we defined[9]

$$\begin{aligned} S^*_\Delta(t)&=S^*_j(t),\quad \text{when } t\in L'_j,\\ &=S_\Delta(t),\quad \text{otherwise},\end{aligned}$$

and proved that, if $f(t)\in H^\alpha$ $(0<\alpha\leqslant 1)$, then

$$|e^*_\Delta(t)|=|f(t)-S^*_\Delta(t)|\leqslant C\delta^\alpha. \tag{5.1}$$

Let us estimate $M_\epsilon(e^*_\Delta)$. On each arc $\widehat{t''_jt'_{j+1}}$, $S^*_\Delta(t)=S_\Delta(t)$, so, if t,t' belong to it, by (2.3)'.

$$|e^*_\Delta(t)-e^*_\Delta(t')|\leqslant C\delta^{\alpha-\epsilon}|t-t'|^\epsilon. \tag{5.2}$$

If t,t' belong to L'_j, we have, similar to (4.3).

$$\begin{aligned} e^*_\Delta(t)-e^*_\Delta(t')&=[S_\Delta(t)-S_\Delta(t')+D^*_j(t-t')]\\ &\quad+\frac{S'_{j-1}(t'_j)-D^*_j}{\Delta t_j'^2}\int_{t'}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau-t'_j)(\tau-t''_j)^2\mathrm{d}\tau\\ &\quad+\frac{S'_j(t''_j)-D^*_j}{\Delta t_j'^2}\int_{t'}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau-t'_j)^2(\tau-t''_j)\mathrm{d}\tau\\ &=G_1+G_2+G_3,\end{aligned}$$

where

$$D_j^* = \frac{S_j(t_j'') - S_{j-1}(t_j')}{\Delta t_j'}, \quad \Delta t_j' = t_j'' - t_j'.$$

Noting that $S_j(t_j) = S_{j-1}(t_j)$ and

$$S_j(t_j'') - S_j(t_j) = D_j(t_j'' - t_j),$$

we have

$$\begin{aligned} |D_j^*| &\leqslant \frac{1}{|\Delta t_j'|}[|S_j(t_j'') - S_j(t_j)| + |S_{j-1}(t_j) - S_{j-1}(t_j')|] \\ &\leqslant \frac{1}{|\Delta t_j'|}[|D_j||t_j'' - t_j| + |D_{j-1}||t_j' - t_j|] \\ &\leqslant C\Big(\frac{1}{|\Delta t_j|^{1-\alpha}} + \frac{1}{|\Delta t_{j-1}|^{1-\alpha}}\Big), \end{aligned}$$

and thereby

$$|G_1| \leqslant C|t-t'|^\alpha + C\Big(\frac{1}{|\Delta t_j|^{1-\alpha}} + \frac{1}{|\Delta t_{j-1}|^{1-\alpha}}\Big)|t-t'|^1 \leqslant C|t-t'|^\alpha.$$

On the other hand,

$$|S_{j-1}'(t_j') - D_j^*| = |D_{j-1} - D_j|\frac{|t_j'' - t_j|}{|\Delta t_j'|} \leqslant C\Big(\frac{1}{|\Delta t_{j-1}|^{1-\alpha}} + \frac{1}{|\Delta t_j|^{1-\alpha}}\Big),$$

while the modulus of the integral in G_2 is not greater than $C|\widehat{t_j't_j''}|^2|\widehat{tt'}|$, and $|\widehat{t_j't_j''}|/|\Delta t_j'| \leqslant C$, so that we have

$$|G_2| \leqslant C|t-t'|^\alpha$$

and a similar estimate for $|G_3|$. Therefore,

$$|e_\Delta^*(t) - e_\Delta^*(t')| \leqslant C|t-t'|^\alpha \leqslant C\delta^{\alpha-\varepsilon}|t-t'|^\varepsilon. \tag{5.3}$$

If t, t' are situated in neither of the above two cases, we may then always find two points $\tau_1, \tau_2 \in \{t_j', t_j''\}$ (maybe $\tau_1 = \tau_2$) on the shorter arc $\widehat{tt'}$ such that $\widehat{t\tau_1}$ is a sub-arc of some $\widehat{t_j't_j''}$ or $\widehat{t_j''t_{j+1}'}$ and so is $\widehat{\tau_2 t'}$. Then proceeding as before, we may verify (5.3) remains valid. Thus

$$M_\varepsilon(e_\Delta^*) \leqslant C\delta^{\alpha-\varepsilon}.$$

Together with (5.1), we have, by (1.7),

$$\| T_L e_\Delta^* \|_\infty \leqslant C_\varepsilon \delta^{\alpha-\varepsilon} \tag{5.4}$$

if $\alpha < 1$, and so also if $\alpha = 1$.

If L is an open smooth arc. we need not modify $S_\Delta(t)$ near the end-points a and b, so that

$$S_\Delta^*(a) = f(a), \quad S_\Delta^*(b) = f(b),$$

and therefore (5.4) is also valid in this case.

Hence, we obtain:

THEOREM 6. *If L is a smooth curve, closed or not, $f(t) \in H^\alpha$ $(0 < \alpha \leqslant 1)$ and $S_\Delta^*(t)$ is the modified cubic interpolating spline of deficiency 2 described as above, then*

$$|T_L f - T_L S_\Delta^*| \leqslant C_\varepsilon \delta^{\alpha-\varepsilon}.$$

Remark. All the results in this paper are valid when L is a piece-wise smooth curve

without cusps, since inequality (1.9) remains true in this case.

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A Class of Quadrature Formulas of Chebyshev Type for Singular Integrals

1. INTRODUCTION

The extension of quadrature formulas of Chebyshev type to singular integrals had been investigated by several authors [2~4,7]. The Gauss-Chebyshev method of numerical integration was extended to singular integrals by Erdogen and Gupta [3]. Let

$$I(x)=I(x,g)=\frac{1}{\pi}\int_{-1}^{1}\frac{g(t)}{t-x}\frac{\mathrm{d}t}{\sqrt{1-t^2}},\quad -1<x<1;\tag{1.1}$$

$$J(t)=J(t,g)=\frac{1}{\pi}\int_{-1}^{1}\frac{g(x)}{x-t}\sqrt{1-x^2}\mathrm{d}x,\quad -1\leqslant t\leqslant 1.\tag{1.2}$$

They obtained the approximate formulas

$$I(x_j)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)}{t_k-x_j},\quad j=1,\dots,n-1,\tag{1.3}$$

$$J(t_k)\approx\frac{1}{n}\sum_{j=1}^{n-1}\frac{g(x_j)(1-x_j^2)}{x_j-t_k},\quad k=1,\dots,n,\tag{1.4}$$

where

$$t_k=\cos\frac{2k-1}{2n}\pi,\quad x_j=\cos\frac{j}{n}\pi\tag{1.5}$$

are the zeros of Chebyshev polynomials $T_n(x)$ of the first kind of degree n and $U_{n-1}(x)$ of the second kind of degree $n-1$ respectively:

$$T_n(x)=\cos n\theta,\quad U_{n-1}(x)=\sin n\theta/\sin\theta,\quad x=\cos\theta.\tag{1.6}$$

Chawla and Ramakrishnan [2] extended (1.3) and (1.4) to the forms

$$I(x)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)}{t_k-x}+g(x)\frac{U_{n-1}(x)}{T_n(x)},\quad -1<x<1,\ x\neq t_k,\tag{1.7}$$

$$J(t)\approx\frac{1}{n}\sum_{j=1}^{n-1}\frac{g(x_j)(1-x_j^2)}{x_j-t}-g(t)\frac{T_n(t)}{U_{n-1}(t)},\quad -1\leqslant t\leqslant 1,\ t\neq x_j,\tag{1.8}$$

respectively, under certain assumptions of analyticity of $g(x)$, gave some complicated formulas of their remainders and pointed out that they are exact for $g(x)\in\pi_{2n-1}$ and π_{2n-2} respectively (π_m being the class of polynomials of degree not greater than m).

In 1977, the Lobatto-Chebyshev (or trapezoidal Chebyshev) method of numerical integration was extended to evaluate singular integrals by Theocaris and Ioakimidis [7]. They obtained the formula

$$I(t_k)\approx\frac{1}{n}\sum_{j=0}^{n}\lambda_j\frac{g(x_j)}{x_j-t_k},\quad k=1,\dots,n \tag{1.9}$$

$$(x_0=1,x_n=-1,\lambda_0=\lambda_n=\frac{1}{2},\lambda_1=\cdots=\lambda_{n-1}=1),$$

and pointed out that it is exact for $g(x)\in\pi_{2n}$.

The authors of [3] and [7] derived the mentioned formulas for the purpose of solving singular integral equations of the first kind

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(x)}{x-t}\mathrm{d}x+\int_{-1}^{1}K(x,t)\varphi(x)\mathrm{d}x=f(t),\quad -1<t<1, \tag{1.10}$$

numerically by the method of collocation. As pointed out in [7], on this purpose, (1.9) has the advantage over (1.3) in that the values of $g(\pm1)$ may be obtained directly without any complementary procedure such as extrapolation which may give rise to significant errors, while the determination of such values is usually important in practice.

In [2], an approximate formula for the singular integral

$$H(x)=H(x,g)=\frac{1}{\pi}\int_{-1}^{1}\frac{g(t)}{t-x}\sqrt{\frac{1+t}{1-t}}\mathrm{d}t,\quad -1\leqslant x<1, \tag{1.11}$$

was obtained by using Jacobi polynomials $P_n^{(-1/2,1/2)}(x)$ under similar assumptions of analyticity of $g(t)$. In [4], Erdogen et al. obtained the approximate formula

$$H(\sigma_k)\approx\frac{2}{2n+1}\sum_{j=1}^{n}\frac{g(\tau_j)(1+\tau_j)}{\tau_j-\sigma_k},\quad k=1,\dots,n, \tag{1.12}$$

where

$$\tau_k=\cos\frac{2k-1}{2n+1}\pi,\quad \sigma_k=\cos\frac{2k}{2n+1}\pi; \tag{1.13}$$

(τ_k,σ_k) actually are the zeros of $U_{2n}(x)$.

The methods used in [3], [4] and [7] were based on expanding $g(x)$ into Chebyshev series, from which it is hard to estimate the remainders. In [2], the method of complex integration was used due to the assumption of analyticity of $g(t)$ and the formula of estimation for the remainder seems inconvenient to use in applications. In this paper, we shall reprove and extend these formulas systematically by a unified method based on the corresponding formulas for ordinary integrals so that their remainders can be easily estimated. Using the same method, we also establish another set of formulas named Simpson-Chebyshev type which seems more effective when the density function $g(x)$ has less order of smoothness as illustrated by an example at the end of this paper.

Throughout the paper, we assume $g(x)$ to be smooth to certain order which is obvious from the context.

2. SOME LEMMAS

The following lemma will be applied repeatedly.

Lemma 1. *If* $g(x)\in C^{n+1}[a,b]$ *and* $a\leqslant x_0\leqslant b$, *let*

$$G(x)=\begin{cases}\dfrac{g(x)-g(x_0)}{x-x_0}, & \text{when } x\neq x_0,\\ g'(x_0); & \text{when } x=x_0;\end{cases}$$

then

$$G^{(k)}(x)=\begin{cases}\dfrac{g^{(k+1)}(\xi_k)}{k+1}, & \text{when } x\neq x_0,\\ \dfrac{g^{(k+1)}(x_0)}{k+1}, & \text{when } x=x_0,\end{cases}\qquad k=0,1,\dots,n, \tag{2.1}$$

where ξ_k is a value between x and x_0.

Proof. For $x\neq x_0$, it is easy to prove by induction

$$G^{(k)}(x)=k!\,[g(x_0)-g(x)-g'(x)(x_0-x)-\cdots$$
$$-(1/k!)g^{(k)}(x)(x_0-x)^k]/(x_0-x)^{k+1}. \tag{2.2}$$

Equation (2.1) follows immediately from (2.2) by Taylor's theorem, provided $x\neq x_0$. Letting $x\to x_0$ we get (2.1) for $x=x_0$.

Another lemma for integrals of Cauchy principal value is useful.

Lemma 2. *The following formulas are valid:*

$$\frac{1}{\pi}\int_{-1}^{1}\frac{dx}{(x-t)\sqrt{1-x^2}}=0, \tag{2.3}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-x^2}}{x-t}dx=-t, \tag{2.4}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{x-t}\sqrt{\frac{1+x}{1-x}}dx=1. \tag{2.5}$$

Proof. By the residue theorem, we may easily verify

$$\frac{1}{2\pi i}\int_{\gamma^-}\frac{d\zeta}{(\zeta-z)\sqrt{1-\zeta^2}}=\frac{1}{\sqrt{1-z^2}},$$

where γ is a contour surrounding $[-1,1]$ and z is exterior to it, $\sqrt{1-\zeta^2}$ being the analytic continuation of $\sqrt{1-x^2}$ along the upper side of $[-1,1]$. Let γ shrink to $[-1,1]$, and we get

$$\frac{1}{\pi i}\int_{-1}^{1}\frac{dx}{(x-z)\sqrt{1-x^2}}=\frac{1}{\sqrt{1-z^2}},\quad z\notin[-1,1].$$

Applying Plemelj's formulas to it[6], we obtain (2.3).

Equations (2.4) and (2.5) may be obtained in a similar way.

3. FORMULAS OF GAUSS-CHEBYSHEV TYPE

This method is based on the formula[1]

$$I=I(f)=\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt\approx\frac{1}{n}\sum_{k=1}^{n}f(t_k), \tag{3.1}$$

with remainder

$$R_n[I]=\frac{1}{(2n)!2^{2n-1}}f^{(2n)}(\xi),\quad -1<\xi<1. \tag{3.2}$$

If we put $t=\cos\theta$ in (3.1)', we see that it is the same as the formula by the tangential method[5].

By (2.3), we may write (1.1) as

$$I(x,g)=\frac{1}{\pi}\int_{-1}^{1}\frac{g(t)-g(x)}{t-x}\frac{dt}{\sqrt{1-t^2}} \tag{1.1$'$}$$

which is an ordinary integral. Hence by (3.1) we have

$$I(x)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)}{t_k-x}+\frac{g(x)}{n}\sum_{k=1}^{n}\frac{1}{x-t_k},\quad x\neq t_k. \tag{3.3}$$

But it is evident

$$\sum_{k=1}^{n}\frac{1}{x-t_k}=\frac{T_n'(x)}{T_n(x)}=n\frac{U_{n-1}(x)}{T_n(x)} \tag{3.4}$$

since

$$T_n'(x)=nU_{n-1}(x). \tag{3.5}$$

Substituting (3.4) into (3.3), we obtain (1.7).

By Lemma 1 and (3.2), we have the remainder of (1.7)

$$R_n[I(x)]=\frac{1}{(2n)!2^{2n-1}}G^{(2n)}(\xi)=\frac{1}{(2n+1)!2^{2n-1}}g^{(2n+1)}(\xi')$$

or its estimate

$$|R_n[I(x)]|\leqslant\frac{1}{(2n+1)!2^{2n-1}}M_{2n+1}(g), \tag{3.6}$$

where $M_m(g)=\max_{|x|\leqslant 1}|g^{(m)}(x)|$. Then obviously (1.7) is exact for $g(x)\in\pi_{2n}$.

In order to get $I(t_r,g)$, we note that $I(x)$ is continuous in $-1<x<1$. Therefore,

$$I(t_r)\approx\frac{1}{n}\sum_{k=1}^{n}{}'\frac{g(t_k)}{t_k-t_r}+\frac{1}{n}g'(t_r)+\frac{g(t_r)}{n}\lim_{x\to t_r}\frac{nU_{n-1}(x)(x-t_r)-T_n(x)}{(x-t_r)T_n(x)},$$

where $\sum'$ denotes the summation except $k=r$. Noting that

$$(1-x^2)U_{n-1}'(x)=U_{n-2}(x)-(n-1)T_n(x),\quad \frac{U_{n-1}'(t_k)}{U_{n-1}(t_k)}=\frac{t_k}{1-t_k^2}, \tag{3.7}$$

we have

$$\lim_{x\to t_r}\frac{nU_{n-1}(x)(x-t_r)-T_n(x)}{(x-t_r)T_n(x)}=\lim_{x\to t_r}\frac{nU_{n-1}'(x)(x-t_r)}{T_n(x)+(x-x_r)T_n'(x)}$$
$$=\frac{nU_{n-1}'(t_r)}{2T_n'(t_r)}=\frac{t_r}{2(1-t_r^2)}, \tag{3.8}$$

and thereby

$$I(t_r)\approx\frac{1}{n}\sum_{k=1}^{n}{}'\frac{g(t_k)}{t_k-t_r}+\frac{1}{n}g'(t_r)+\frac{t_rg(t_r)}{2n(1-t_r^2)} \tag{3.9}$$

with the same estimate of remainder (3.6), which is also exact for $g(x)\in\pi_{2n}$.

Let $g^*(x)=g(x)(1-x^2)$, then $J(x,g)=I(x,g^*)$, so that from (1.7) we have

$$J(x)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)(1-t_k^2)}{t_k-x}+g(x)\frac{(1-x^2)U_{n-1}(x)}{T_n(x)},\quad x\neq t_k, \tag{3.10}$$

and

$$J(x_j)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)(1-t_k^2)}{t_k-x_j},\quad j=1,\ldots,n-1. \tag{3.11}$$

From (3.10), we know that

$$H=H(g)=\frac{1}{\pi}\int_{-1}^{1}g(t)\sqrt{\frac{1+t}{1-t}}\,dt=-J(+1,g)\approx\frac{1}{n}\sum_{k=1}^{n}g(t_k)(1+t_k), \tag{3.12}$$

$$K=K(g)=\frac{1}{\pi}\int_{-1}^{1}g(t)\sqrt{\frac{1-t}{1+t}}\,dt=J(-1,g)\approx\frac{1}{n}\sum_{k=1}^{n}g(t_k)(1-t_k), \tag{3.12$'$}$$

since $T_n(\pm1)=(\pm1)^n$, $U_{n-1}(\pm1)=(\pm1)^{n-1}n$.

By (3.9), we also have

$$J(t_r)\approx\frac{1}{n}\sum_{k=1}^{n}{}'\frac{g(t_k)(1-t_k^2)}{t_k-t_r}+\frac{1}{n}g'(t_r)(1-t_r^2)-\frac{3}{2n}g(t_r). \tag{3.13}$$

From (3.6), we have the estimate of remainders of these formulas

$$|R_n[J(x)]|\leqslant\frac{1}{(2n+1)!2^{2n-1}}M_{2n+1}(g(x)(1-x^2)), \tag{3.14}$$

which shows they are exact for $g(x)\in\pi_{2n-2}$.

We may also write $H(x,g)=I(x,g(t)(1+t))$ and $K(x,g)=I(x,g(t)(1-t))$, so we have

$$H(x)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)(1+t_k)}{t_k-x}+g(x)\frac{(1+x)U_{n-1}(x)}{T_n(x)},\ -1\leqslant x<1,x\neq t_k, \tag{3.15}$$

$$K(x)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)(1-t_k)}{t_k-x}+g(x)\frac{(1-x)U_{n-1}(x)}{T_n(x)},\ -1<x\leqslant1,x\neq t_k; \tag{3.15$'$}$$

$$H(x_j)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)(1+t_k)}{t_k-x_j},\quad j=1,\ldots,n-1, \tag{3.16}$$

$$K(x_j)\approx\frac{1}{n}\sum_{k=1}^{n}\frac{g(t_k)(1-t_k)}{t_k-x_j},\quad j=1,\ldots,n-1; \tag{3.16$'$}$$

$$H(t_r)\approx\frac{1}{n}\sum_{k=1}^{n}{}'\frac{g(t_k)(1+t_k)}{t_k-t_r}+\frac{1}{n}g'(t_r)(1+t_r)+\frac{g(t_r)}{2n}\frac{2-t_r}{1-t_r},\ r=1,\ldots,n, \tag{3.17}$$

$$K(t_r)\approx\frac{1}{n}\sum_{k=1}^{n}{}'\frac{g(t_k)(1-t_k)}{t_k-t_r}+\frac{1}{n}g'(t_r)(1-t_r)-\frac{g(t_r)}{2n}\frac{2+t_r}{1+t_r},\ r=1,\ldots,n, \tag{3.17$'$}$$

with the estimate of the remainders

$$\left|R_n\left[\begin{matrix}H\\K\end{matrix}(x)\right]\right|\leqslant\frac{1}{(2n+1)!2^{2n-1}}M_{2n+1}(g(t)(1\pm t)), \tag{3.18}$$

which shows that they are exact for $g(t)\in\pi_{2n-1}$.

4. FORMULAS OF LOBATTO-CHEBYSHEV TYPE

This method is based on the approximate formula[1]

$$J=\frac{1}{\pi}\int_{-1}^{1} f(x)\sqrt{1-x^2}\mathrm{d}x\approx\frac{1}{n}\sum_{j=1}^{n-1} f(x_j)(1-x_j^2) \tag{4.1}$$

with remainder

$$R_n[J]=\frac{1}{(2n-2)!2^{2n-1}}f^{(2n-2)}(\xi),\quad -1<\xi<1, \tag{4.2}$$

Using (2.4), we may write

$$J(t)=\frac{1}{\pi}\int_{-1}^{1} G(x)\sqrt{1-x^2}\mathrm{d}x-tg(t),\quad G(x)=\frac{g(x)-g(t)}{x-t},$$

so that, from (4.1), we have

$$J(t)\approx\frac{1}{n}\sum_{j=1}^{n-1}\frac{g(x_j)(1-x_j^2)}{x_j-t}+\frac{g(t)}{n}\sum_{j=1}^{n-1}\frac{1-x_j^2}{t-x_j}-tg(t).$$

But

$$\sum_{j=1}^{n-1}\frac{1-x_j^2}{t-x_j}=(1-t^2)\sum_{j=1}^{n-1}\frac{1}{t-x_j}+\sum_{j=1}^{n-1}(t+x_j)=\frac{(1-t^2)U'_{n-1}(t)}{U_{n-1}(t)}+(n-1)t$$

since $\sum_{j=1}^{n-1} x_j=0$ is the coefficient of t^{n-2} in $U_{n-1}(t)$. Noting that

$$T_n(t)=tU_{n-1}(t)-U_{n-2}(t) \tag{4.3}$$

and thereby, by (3.7),

$$(1-t^2)U'_{n-1}(t)=tU_{n-1}(t)-nT_n(t), \tag{4.4}$$

we readily obtain (1.8).

For $t=x_s$, we have

$$J(x_s)\approx\frac{1}{n}\sum_{j=1}^{n-1}{}'\frac{g(x_j)(1-x_j^2)}{x_j-x_s}+\frac{1}{n}g'(x_s)(1-x_s^2)$$
$$+\frac{g(x_s)}{n}\lim_{t\to x_s}\frac{(1-x_s)^2U_{n-1}(t)-nT_n(t)(x_s-t)}{(x_s-t)U_{n-1}(t)}.$$

Proceeding as in (3.8) and using (3.5), (4.4), we may evaluate the involved limit which is equal to $-\frac{3}{2}t_s$. Hence

$$J(x_s)\approx\frac{1}{n}\sum_{j=1}^{n-1}{}'\frac{g(x_j)(1-x_j^2)}{x_j-x_s}+\frac{1}{n}g'(x_s)(1-x_s^2)-\frac{3}{2n}x_sg(x_s), \tag{4.5}$$

According to (4.2) and Lemma 1, the remainder of (1.8) or (4.5) is

$$R_n[J(t)]=\frac{1}{(2n-1)!2^{2n-1}}g^{(2n-1)}(\xi),\quad -1<\xi<1. \tag{4.6}$$

In order to get formulas for $I(t)$, we first establish

$$I=\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\mathrm{d}x\approx\frac{1}{n}\sum_{j=0}^{n}\lambda_jf(x_j) \tag{4.7}$$

with estimate of the remainder

$$|R_n[I]|\leqslant\frac{1}{(2n-1)!2^{2n-1}}M_{2n}(f). \tag{4.8}$$

To prove this, we define

$$f(x)=F(x)(1-x^2)+Ax+B,\ A=\frac{f(1)-f(-1)}{2},\ B=\frac{f(1)+f(-1)}{2}, \tag{4.9}$$

or

$$F(x)=\frac{1}{2}\left[\frac{f(x)-f(-1)}{x+1}-\frac{f(x)-f(1)}{x-1}\right]. \tag{4.9$'$}$$

Then

$$I=\frac{1}{\pi}\int_{-1}^{1}F(x)\sqrt{1-x^2}\mathrm{d}x+B.$$

By (4.1), we get

$$I\approx\frac{1}{n}\sum_{j=1}^{n-1}[f(x_j)-Ax_j-B]+B=\frac{1}{n}\sum_{j=1}^{n-1}f(x_j)+\frac{1}{n}B$$

which is just (4.7). From (4.2), we have, by Lemma 1,

$$R_n[I]=\frac{1}{(2n-2)!2^{2n-1}}F^{(2n-2)}(\xi)=\frac{1}{(2n-1)!2^{2n}}[f^{(2n-1)}(\xi_1)-f^{(2n-1)}(\xi_2)]$$
$$=\frac{\xi_1-\xi_2}{(2n-1)!2^{2n}}f^{(2n)}(\xi_3),$$

which follows (4.8).

Now, from (1.1)$'$ and (4.7), we get

$$I(t)\approx\frac{1}{n}\sum_{j=0}^{n}\lambda_j\frac{g(x_j)}{x_j-t}-\frac{g(t)}{n}\sum_{j=0}^{n}\frac{\lambda_j}{x_j-t},\quad t\neq x_j.$$

By (4.4), we see that

$$\sum_{j=0}^{n}\frac{\lambda_j}{x_j-t}=\frac{t}{1-t^2}-\sum_{j=1}^{n-1}\frac{1}{t-x_j}=\frac{t}{1-t^2}-\frac{U'_{n-1}(t)}{U_{n-1}(t)}=\frac{nT_n(t)}{(1-t^2)U_{n-1}(t)} \tag{4.10}$$

and therefore

$$I(t)\approx\frac{1}{n}\sum_{j=0}^{n}\lambda_j\frac{g(x_j)}{x_j-t}-g(t)\frac{T_n(t)}{(1-t^2)U_{n-1}(t)},\quad t\neq x_j, \tag{4.11}$$

of which (1.9) is a consequence.

For $t=x_s$, analogous to (3.9), we have

$$I(x_s)\approx\frac{1}{n}\sum_{j=0}^{n}{}'\lambda_j\frac{g(x_j)}{x_j-x_s}+\frac{1}{n}g'(x_s)+\frac{g(x_s)}{2n}\frac{x_s}{1-x_s^2},\quad s=1,\cdots,n-1, \tag{4.12}$$

where we have used the equality

$$\lim_{t\to x_s}\frac{(1-t^2)U_{n-1}(t)+nT_n(t)(t-x_s)}{(t-x_s)U_{n-1}(t)}=-\frac{x_s}{2}. \tag{4.13}$$

From (4.8), we have the estimate of the remainder of (4.11) or (4.12)

$$|R_n[I(t)]|\leqslant\frac{1}{2n+1}\cdot\frac{1}{(2n-1)!2^{2n-1}}M_{2n+1}(g)<\frac{1}{(2n)!2^{2n-1}}M_{2n+1}(g) \tag{4.14}$$

which shows they are exact for $g(x)\in\pi_{2n}$.

Similarly, by defining $f(t)=F(t)(1-t)+f(1)$ or

$$F(t)=-\frac{f(t)-f(1)}{t-1},$$

we may write

$$H=\frac{1}{\pi}\int_{-1}^{1}F(t)\sqrt{1-t^2}\mathrm{d}t+f(1).$$

Then, applying (4.1), (4.2) and Lemma 1, we may get

$$H\approx\frac{1}{n}\sum_{j=0}^{n-1}\lambda_j f(x_j)(1+x_j) \tag{4.15}$$

with remainder

$$R_n[H]=\frac{-1}{(2n-1)!2^{2n-1}}f^{(2n-1)}(\xi),\quad -1<\xi<1. \tag{4.16}$$

In the same way, we may also get

$$K\approx\frac{1}{n}\sum_{j=1}^{n}\lambda_j f(x_j)(1-x_j) \tag{4.15$'$}$$

with remainder

$$R_n[K]=\frac{1}{(2n-1)!2^{2n-1}}f^{(2n-1)}(\xi),\quad -1<\xi<1. \tag{4.16$'$}$$

Now, on account of (2.5), we may write

$$H(t)=\frac{1}{\pi}\int_{-1}^{1}\frac{g(x)-g(t)}{x-t}\sqrt{\frac{1+x}{1-x}}\mathrm{d}x+g(t)$$

and an analogous formula for $K(t)$. Then, by using (4.15)~(4.16)$'$ and Lemma 1, it is easy to obtain

$$H(t)\approx\frac{1}{n}\sum_{j=0}^{n-1}\lambda_j\frac{g(x_j)(1+x_j)}{x_j-t}-g(t)\frac{T_n(t)}{(1-t)U_{n-1}(t)},\quad t\neq x_j; \tag{4.17}$$

$$K(t)\approx\frac{1}{n}\sum_{j=1}^{n}\lambda_j\frac{g(x_j)(1-x_j)}{x_j-t}-g(t)\frac{T_n(t)}{(1+t)U_{n-1}(t)},\quad t\neq x_j; \tag{4.17$'$}$$

$$H(t_k)\approx\frac{1}{n}\sum_{j=0}^{n-1}\lambda_j\frac{g(x_j)(1+x_j)}{x_j-t_k},\quad k=1,\dots,n; \tag{4.18}$$

$$K(t_k)\approx\frac{1}{n}\sum_{j=1}^{n}\lambda_j\frac{g(x_j)(1-x_j)}{x_j-t_k},\quad k=1,\dots,n; \tag{4.18$'$}$$

$$H(x_s)\approx\frac{1}{n}\sum_{j=0}^{n-1}{}'\lambda_j\frac{g(x_j)(1+x_j)}{x_j-x_s}+\frac{1}{n}g'(x_s)(1+x_s)+\frac{g(x_s)}{2n}\frac{2-x_s}{1-x_s}, \tag{4.19}$$

$$K(x_s)\approx\frac{1}{n}\sum_{j=1}^{n}{}'\lambda_j\frac{g(x_j)(1-x_j)}{x_j-x_s}+\frac{1}{n}g'(x_s)(1-x_s)-\frac{g(x_s)}{2n}\frac{2+x_s}{1+x_s}, \tag{4.19$'$}$$

with remainders

$$R_n\left[\begin{matrix}H\\K\end{matrix}(x)\right]=\frac{\mp1}{(2n)!2^{2n-1}}g^{(2n)}(\xi),\quad -1<\xi<1, \tag{4.20}$$

which show they are exact for $g(x)\in\pi_{2n-1}$.

Remark. Formulas for $H(t)$ and $K(t)$ may be also obtained even simpler from those for $I(t)$, but simple forms of remainders could not be obtained.

5. FORMULAS OF JACOBI-CHEBYSHEV TYPE

We may also start from the approximate formulas based on Jacobi polynomials, respectively with respect to weights $\sqrt{(1+x)/(1-x)}$ and $\sqrt{(1-x)/(1+x)}$[1]:

$$H=\frac{1}{\pi}\int_{-1}^{1}f(x)\sqrt{\frac{1+x}{1-x}}\mathrm{d}x\approx\frac{2}{2n+1}\sum_{k=1}^{n}f(\tau_k)(1+\tau_k), \tag{5.1}$$

$$K=\frac{1}{\pi}\int_{-1}^{1} f(x)\sqrt{\frac{1-x}{1+x}}\mathrm{d}x\approx\frac{2}{2n+1}\sum_{k=1}^{n} f(\sigma_k)(1-\sigma_k), \tag{5.1$'$}$$

with remainders

$$R_n\begin{bmatrix}H\\K\end{bmatrix}=\frac{1}{(2n)!2^{4n+1}} f^{(2n)}(\xi), \quad -1<\xi<1, \tag{5.2}$$

where τ_k,σ_k are given by (1.13), being zeros of Jacobi polynomials

$$P_n(x)=P_n^{(-1/2,1/2)}(x), \quad P_n(\tau_k)=0, \tag{5.3}$$

$$Q_n(x)=P_n^{(1/2,-1/2)}(x), \quad Q_n(\sigma_k)=0, \tag{5.3$'$}$$

$P_n(x)$ and $Q_n(x)$ are connected with Chebyshev polynomials by[1]

$$\left.\begin{aligned}P_n(2z^2-1)&=A_nT_{2n+1}(z)/z,\\ Q_n(2z^2-1)&=A_nU_{2n}(z).\end{aligned}\right\}\ (A_n=\text{const}) \tag{5.4}$$

Moreover, we also have

$$U_{2n}(x)=a_n P_n'(x)Q_n(x) \ (a_n=\text{const}) \tag{5.5}$$

and

$$\tau_k=-\sigma_{n-k+1}, \quad P_n(-x)=(-1)^nQ_n(x). \tag{5.6}$$

Then, as in Section 4, by using (5.1), we get

$$H(t)\approx\frac{1}{2n+1}\sum_{k=1}^{n}\frac{g(\tau_k)-g(t)}{\tau_k-t}(1+\tau_k)+g(t), \quad t\neq\tau_k.$$

But from (5.4), we see that

$$P_n(t)+2(1+t)P_n'(t)=(2n+1)Q_n(t), \tag{5.7}$$

so that

$$\sum_{k=1}^{n}\frac{1}{t-\tau_k}=\frac{P_n'(t)}{P_n(t)}=\frac{(2n+1)Q_n(t)-P_n(t)}{2(1+t)P_n(t)} \tag{5.8}$$

and

$$\sum_{k=1}^{n}\frac{1+\tau_k}{\tau_k-t}=\frac{2n+1}{2}\left[1-\frac{Q_n(t)}{P_n(t)}\right].$$

Hence we obtain

$$H(t)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\tau_k)(1+\tau_k)}{\tau_k-t}+g(t)\frac{Q_n(t)}{P_n(t)}, \quad t\neq\tau_k. \tag{5.9}$$

Similarly, we also have

$$K(t)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\sigma_k)(1-\sigma_k)}{\sigma_k-t}-g(t)\frac{P_n(t)}{Q_n(t)}, \quad t\neq\sigma_k, \tag{5.9$'$}$$

which may also be obtained from (5.9) by

$$K(t,g(x))=-H(-t,g(-x)). \tag{5.10}$$

(5.9) and (5.9)$'$ imply (1.12) and

$$K(\tau_k)\approx\frac{2}{2n+1}\sum_{j=1}^{n}\frac{g(\sigma_j)(1-\sigma_j)}{\sigma_j-\tau_k}, \quad k=1,\dots,n. \tag{1.12$'$}$$

Let $t\to\tau_s$ in (5.9), we get

$$H(\tau_s)\approx\frac{2}{2n+1}\sum_{k=1}^{n}{}'\frac{g(\tau_k)(1+\tau_k)}{\tau_k-\tau_s}+\frac{2}{2n+1}g'(\tau_s)(1+\tau_s)$$

$$+\frac{g(\tau_s)}{2n+1}\lim_{t\to\tau_s}\left[\frac{(2n+1)Q_n(t)}{P_n(t)}+\frac{2(1+\tau_s)}{\tau_s-t}\right]. \tag{5.11}$$

We may evaluate the involved limit by (5.7), which equals

$$\frac{3}{2}+(1+\tau_s)\frac{Q_n'(\tau_s)}{Q_n(\tau_s)}.$$

But it is easy to verify

$$\frac{Q_n'(\tau_s)}{Q_n(\tau_s)}=\frac{U_{2n}'(z_s)}{4z_sU_{2n}(z_s)}=\frac{1}{4(1-z_s^2)}=\frac{1}{2(1-\tau_s)},\quad \tau_s=2z_s^2-1, \tag{5.12}$$

by virtue of (5.4). Hence we obtain

$$H(\tau_s)\approx\frac{2}{2n+1}\sum_{k=1}^{n}{}'\frac{g(\tau_k)(1+\tau_k)}{\tau_k-\tau_s}+\frac{2g'(\tau_s)}{2n+1}(1+\tau_s)+\frac{g(\tau_s)}{2n+1}\frac{2-\tau_s}{1-\tau_s},\quad s=1,\dots,n. \tag{5.13}$$

Similarly, by (5.10),

$$K(\sigma_s)\approx\frac{2}{2n+1}\sum_{k=1}^{n}{}'\frac{g(\sigma_k)(1-\sigma_k)}{\sigma_k-\sigma_s}+\frac{2g'(\sigma_s)}{2n+1}(1-\sigma_s)-\frac{g(\sigma_s)}{2n+1}\frac{2+\sigma_s}{1+\sigma_s},\quad s=1,\dots,n. \tag{5.13$'$}$$

By Lemma 1 and (5.2), the remainders of these formulas have the same form

$$R_n\left[\begin{matrix}H\\K\end{matrix}(t)\right]=\frac{1}{(2n+1)!2^{4n+1}}g^{(2n+1)}(\xi),\quad -1<\xi<1, \tag{5.14}$$

which shows they are exact for $g(x)\in\pi_{2n}$.

If we put

$$g(x)=(1+x)g_1(x)+g(-1),$$

we may write

$$I(x,g)=H(x,g_1)$$

on account of (2.3). Then, using (5.9) and (5.8), we obtain

$$I(x)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\tau_k)}{\tau_k-x}-\frac{g(-1)}{(2n+1)(1+x)}+g(x)\frac{Q_n(x)}{(1+x)P_n(x)},\quad x\neq\tau_k. \tag{5.15}$$

Similarly, since $I(x,g(t))=-I(-x,g(-t))$, we also have

$$I(x)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\sigma_k)}{\sigma_k-x}+\frac{g(1)}{(2n+1)(1-x)}-g(x)\frac{P_n(x)}{(1-x)Q_n(x)},\quad x\neq\sigma_k. \tag{5.15$'$}$$

Thus,

$$I(\sigma_j)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\tau_k)}{\tau_k-\sigma_j}-\frac{g(-1)}{(2n+1)(1+\sigma_j)},\quad j=1,\dots,n, \tag{5.16}$$

$$I(\tau_j)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\sigma_k)}{\sigma_k-\tau_j}+\frac{g(1)}{(2n+1)(1-\tau_j)},\quad j=1,\dots,n; \tag{5.16$'$}$$

and we may also get

$$I(\tau_s)\approx\frac{2}{2n+1}\sum_{k=1}^{n}{}'\frac{g(\tau_k)}{\tau_k-\tau_s}+\frac{2}{2n+1}g'(\tau_s)-\frac{g(-1)}{(2n+1)(1+\tau_s)}+\frac{g(\tau_s)}{2n+1}\frac{\tau_s}{1-\tau_s^2}, \tag{5.17}$$

$$I(\sigma_s)\approx\frac{2}{2n+1}\sum_{k=1}^{n}{}'\frac{g(\sigma_k)}{\sigma_k-\sigma_s}+\frac{2}{2n+1}g'(\sigma_s)+\frac{g(1)}{(2n+1)(1-\sigma_s)}+\frac{g(\sigma_s)}{2n+1}\frac{\sigma_s}{1-\sigma_s^2}. \tag{5.17$'$}$$

The remainders of these formulas have the same form

$$R_n[I(x)]=\frac{1}{(2n+2)!2^{4n+1}}g^{(2n+2)}(\xi), \quad -1<\xi<1, \tag{5.18}$$

which shows they are exact for $g(t)\in\pi_{2n+1}$.

Generally, these formulas for $I(x)$ are better than those given in Section 3 or 4 in the sense that all of them are depending on the values of $g(x)$ at n points.

Since

$$J(x,g(t))=H(x,g(t)(1-t))=K(x,g(t)(1+t)),$$

we may also have the following formulas:

$$J(x)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\tau_k)(1-\tau_k^2)}{\tau_k-x}+g(x)\frac{(1-x)Q_n(x)}{P_n(x)}, \tag{5.19}$$

$$J(x)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\sigma_k)(1-\sigma_k^2)}{\sigma_k-x}-g(x)\frac{(1+x)P_n(x)}{Q_n(x)}; \tag{5.19$'$}$$

$$J(\sigma_j)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\tau_k)(1-\tau_k^2)}{\tau_k-\sigma_j}, \quad j=1,\dots,n, \tag{5.20}$$

$$J(\tau_j)\approx\frac{2}{2n+1}\sum_{k=1}^{n}\frac{g(\sigma_k)(1-\sigma_k^2)}{\sigma_k-\tau_j}, \quad j=1,\dots,n; \tag{5.20$'$}$$

$$J(\tau_s)\approx\frac{2}{2n+1}\sum_{k=1}^{n}{}'\frac{g(\tau_k)(1-\tau_k^2)}{\tau_k-\tau_s}+\frac{2g'(\tau_s)}{2n+1}(1-\tau_s^2)-\frac{3}{2n+1}\tau_s g(\tau_s), \quad s=1,\dots,n, \tag{5.21}$$

$$J(\sigma_s)\approx\frac{2}{2n+1}\sum_{k=1}^{n}{}'\frac{g(\sigma_k)(1-\sigma_k^2)}{\sigma_k-\sigma_s}+\frac{2g'(\sigma_s)}{2n+1}(1-\sigma_s^2)-\frac{3}{2n+1}\sigma_s g(\sigma_s), \quad s=1,\dots,n; \tag{5.21$'$}$$

with the estimates of the remainders

$$|R_n[J(x)]|\leqslant\frac{1}{(2n+1)!2^{4n+1}}M_{2n+1}(g(t)(1\mp t)), \tag{5.22}$$

which mean they are exact for $g(t)\in\pi_{2n-1}$.

6. FORMULAS OF SIMPSON-CHEBYSHEV TYPE

If the function $f(x)$ or $g(x)$ does not possess derivatives of sufficiently high order, then the approximate formulas derived based on tangential rule or trapezoidal rule in general are not as accurate as those derived based on Simpson's rule. Thus, in such cases, we may expect formulas for singular integrals based on the latter.

Analogous to Simpson's rule of quadrature for proper integrals, we easily get the Simpson-Chebyshev formula

$$I=\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx\approx\frac{1}{3n}\sum_{k=0}^{2n}\mu_k f(\xi_k), \tag{6.1}$$

where

$$\mu_0=\mu_{2n}=\frac{1}{2}, \quad \mu_{2m}=1, \ m=1,\dots,n-1;$$

$$\mu_{2m+1}=2,\quad m=0,1,\ldots,n-1;\tag{6.2}$$

$$\xi_k=\cos\,(k/2n)\pi,\quad k=0,1,\ldots,2n\ (\xi_{2n-k}=-\xi_k),$$

$\xi_1,\ldots,\xi_{2n-1}$ being the zeros of $U_{2n-1}(t)$. If $f(x)\in C^4[-1,1]$, then we have the estimate of the remainder:

$$|R_{2n}[I]|\leqslant\frac{\pi^4}{2880n^4}M_4(I(\cos\theta)),\quad 0\leqslant\theta\leqslant\pi,\tag{6.3}$$

which is obtained from the corresponding remainder formula for the integral[1]

$$I=\frac{1}{\pi}\int_0^\pi f(\cos\theta)\mathrm{d}\theta.$$

Note that

$$U_{n-1}(\xi_{2m})=0\ (\xi_{2m}=x_m),\quad T_n(\xi_{2m-1})=0\ (\xi_{2m-1}=t_m)\tag{6.4}$$

and

$$U_{2n-1}(t)=2U_{n-1}(t)T_n(t).\tag{6.5}$$

By (1.1)′, we have

$$I(t)\approx\frac{1}{3n}\sum_{k=0}^{2n}\mu_k\frac{g(\xi_k)-g(t)}{\xi_k-t}=\frac{1}{3n}\sum_{k=0}^{2n}\mu_k\frac{g(\xi_k)}{\xi_k-t}+\frac{g(t)}{3n}\left(\sum_{m=1}^{n}\frac{1}{t-\xi_{2m-1}}+\sum_{k=0}^{2n}\frac{\lambda_k}{t-\xi_k}\right)$$
$$=\frac{1}{3n}\sum_{k=0}^{2n}\mu_k\frac{g(\xi_k)}{\xi_k-t}+\frac{g(t)}{3}\left(\frac{U_{n-1}(t)}{T_n(t)}-\frac{2T_{2n}(t)}{(1-t^2)U_{2n-1}(t)}\right),$$

because of (4.10). Since it is easy to prove

$$2(1-t^2)U_{n-1}^2(t)=1-T_{2n}(t),$$

we obtain

$$I(t)\approx\frac{1}{3n}\sum_{k=0}^{2n}\mu_k\frac{g(\xi_k)}{\xi_k-t}+\frac{g(t)}{3}\,\frac{1-3T_{2n}(t)}{(1-t^2)U_{2n-1}(t)},\quad t\neq\xi_k.\tag{6.6}$$

Thus, if we let

$$\left.\begin{aligned}\eta_{\pm j}&=\cos\frac{2j\pi\pm\arccos\frac{1}{3}}{2n},\quad j=1,\ldots,n-1,\\ \eta_{\pm n}&=\pm\cos\frac{\arccos\frac{1}{3}}{2n},\end{aligned}\right\}\tag{6.7}$$

then we have

$$I(\eta_j)\approx\frac{1}{3n}\sum_{k=0}^{2n}\mu_k\frac{g(\xi_k)}{\xi_k-\eta_j},\quad j=\pm1,\ldots,\pm n.\tag{6.8}$$

Equation (6.8) may be used to solve singular integral equation (1.10) numerically in place of (1.8) as well, probably with better accuracy when $k(x,t)$ and $f(t)$ are not smooth enough.

We may get formulas for $I(\xi_r)$ also by letting $t\to\xi_r$ in (6.6). If r is even, we have, by (4.13),

$$I(\xi_r)\approx\frac{1}{3n}\sum_{k=0}^{2n}{}'\mu_k\frac{g(\xi_k)}{\xi_k-\xi_r}+\frac{\mu_r}{3n}g'(\xi_r)+\frac{g(\xi_r)}{3n(1-\xi_r^2)}\lim_{t\to\xi_r}\frac{(1-t^2)U_{2n-1}(t)-2n(\xi_r-t)T_{2n}(t)}{(\xi_r-t)U_{2n-1}(t)}$$

$$=\frac{1}{3n}\sum_{k=0}^{2n}{}'\mu_k\frac{g(\xi_k)}{\xi_k-\xi_r}+\frac{\mu_r}{3n}g'(\xi_r)+\frac{g(\xi_r)}{6n}\frac{\xi_r}{1-\xi_r^2};$$

if r is odd, we then have, by (3.8),

$$I(\xi_r)\approx\frac{1}{3n}\sum_{k=0}^{2n}{}'\mu_k\frac{g(\xi_k)}{\xi_k-\xi_r}+\frac{\mu_r}{3n}g'(\xi_r)+\frac{g(\xi_r)}{6n}\frac{\xi_r}{1-\xi_r^2}+\frac{g(\xi_r)}{3n}\lim_{t\to\xi_r}\frac{T_n(t)+n(\xi_r-t)U_{n-1}(t)}{(\xi_r-t)T_n(t)}$$

$$=\frac{1}{3n}\sum_{k=0}^{2n}{}'\mu_k\frac{g(\xi_k)}{\xi_k-\xi_r}+\frac{\mu_r}{3n}g'(\xi_r)+\frac{g(\xi_r)}{3n}\frac{\xi_r}{1-\xi_r^2}.$$

That is,

$$I(\xi_r)\approx\frac{1}{3n}\sum_{k=0}^{2n}{}'\mu_k\frac{g(\xi_k)}{\xi_k-\xi_r}+\frac{\mu_r}{3n}g'(\xi_r)+\frac{\mu_r g(\xi_r)}{6n}\frac{\xi_r}{1-\xi_r^2},\quad r=1,\dots,2n-1. \tag{6.9}$$

Since $J(t,g)=I(t,g(x)(1-x^2))$, we may easily get

$$J(t)\approx\frac{1}{3n}\sum_{k=1}^{2n-1}\mu_k\frac{g(\xi_k)(1-\xi_k^2)}{\xi_k-t}+\frac{g(t)}{3}\frac{1-3T_{2n}(t)}{U_{2n-1}(t)},\quad t\neq\xi_k, \tag{6.10}$$

$$J(\eta_j)\approx\frac{1}{3n}\sum_{k=1}^{2n-1}\mu_k\frac{g(\xi_k)(1-\xi_k^2)}{\xi_k-\eta_j},\quad j=\pm1,\dots,\pm n, \tag{6.11}$$

$$J(\xi_r)\approx\frac{1}{3n}\sum_{k=1}^{2n-1}{}'\mu_k\frac{g(\xi_k)(1-\xi_k^2)}{\xi_k-\xi_r}+\frac{\mu_r}{3n}g'(\xi_r)(1-\xi_r)-\frac{\mu_r g(\xi_r)}{2n}\xi_r,\quad r=1,\dots,2n-1. \tag{6.12}$$

Since $H(t,g)=I(t,g(x)(1+x)),K(t,g)=I(t,g(x)(1-x))$, we also have

$$H(t)\approx\frac{1}{3n}\sum_{k=0}^{2n-1}\mu_k\frac{g(\xi_k)(1+\xi_k)}{\xi_k-t}+\frac{g(t)}{3}\frac{1-3T_{2n}(t)}{(1-t)U_{2n-1}(t)}, \tag{6.13}$$

$$K(t)\approx\frac{1}{3n}\sum_{k=1}^{2n}\mu_k\frac{g(\xi_k)(1-\xi_k)}{\xi_k-t}+\frac{g(t)}{3}\frac{1-3T_{2n}(t)}{(1+t)U_{2n-1}(t)}; \tag{6.13$'$}$$

$$H(\eta_j)\approx\frac{1}{3n}\sum_{k=0}^{2n-1}\mu_k\frac{g(\xi_k)(1+\xi_k)}{\xi_k-\eta_j},\quad j=\pm1,\dots,\pm n, \tag{6.14}$$

$$K(\eta_j)\approx\frac{1}{3n}\sum_{k=1}^{2n}\mu_k\frac{g(\xi_k)(1-\xi_k)}{\xi_k-\eta_j},\quad j=\pm1,\dots,\pm n; \tag{6.14$'$}$$

$$H(\xi_r)\approx\frac{1}{3n}\sum_{k=0}^{2n-1}{}'\mu_k\frac{g(\xi_k)(1+\xi_k)}{\xi_k-\xi_r}+\frac{\mu_r}{3n}g'(\xi_r)(1+\xi_r)+\frac{\mu_r g(\xi_r)}{6n}\frac{2-\xi_r}{1-\xi_r},\ r=1,\dots,2n-1, \tag{6.15}$$

$$K(\xi_r)\approx\frac{1}{3n}\sum_{k=1}^{2n}{}'\mu_k\frac{g(\xi_k)(1-\xi_k)}{\xi_k-\xi_r}+\frac{\mu_r}{3n}g'(\xi_r)(1-\xi_r)-\frac{\mu_r g(\xi_r)}{6n}\frac{2+\xi_r}{1+\xi_r},\ r=1,\dots,2n-1, \tag{6.15$'$}$$

Remainders of all the formulas obtained in this section may be estimated by means of (6.3) and Lemma 1, provided $g(x)\in C^5[-1,1]$.

Let us consider a numerical example as an illustration.

Consider the singular integral

$$I(t)=\frac{1}{\pi}\int_{-1}^{1}\frac{|x|}{(x-t)\sqrt{1-x^2}}\mathrm{d}x=\frac{2t}{\pi\sqrt{1-t^2}}\ln\frac{1+\sqrt{1-t^2}}{|t|},\quad -1<t<1.$$

For $t=\dfrac{1}{2}$, we find

$$I\left(\frac{1}{2}\right)=0.48405\ldots$$

Here the density function $|x|$ has a corner point at $x=0$. The approximate values of $I\left(\frac{1}{2}\right)$ have been calculated out by three of the methods described above. When the n's are chosen to be odd for Lobatto-Chebyshev method and Jacobi-Chebyshev method, and even for Simpson-Chebyshev method, better approximations are found, which are listed below. It is obvious that the last method is most effective in this example.

n	Lobatto-Chebyshev method	Jacobi-Chebyshev method	n	Simpson-Chebyshev method
3	0.44444	0.49240	2	0.49836
5	0.46667	0.48569		
7	0.47446	0.48429	4	0.48602
9	0.47803	0.48392		
11	0.47993	0.48383	6	0.48457
13	0.48106	0.48381		
15	0.48179	0.48383	8	0.48424
17	0.48228	0.48385		
19	0.48263	0.48387	10	0.48413
21	0.48288	0.48389		
23	0.48307	0.48391	12	0.48409

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有关高阶奇异积分的 Bertrand-Poincaré 型换序公式

内容提要 本文把关于 Cauchy 核奇异积分的 Bertrand-Poincaré 换序公式推广到高阶奇异积分的情况. 主要结果见定理 1 至 3，其中定理 1 已见于文献，但这里的表达方式和证法均是新的且较简.

1957 年，C. Fox[1]将起源于 Hadamard 的实域中发散积分的有限部分概念（参见 [2]）推广到高阶奇异积分. 在 [3], [4] 中我们曾作出关于它们的推广的留数定理以及一些应用. 王传荣[5,6]曾给出有关它们的一些性质，并把著名的 Bertrand-Poincaré 积分换序公式[7]推广到这种高阶奇异积分上来. 本短文将给出一些这种类型的换序公式，其中包括较简地重新表述和证明了 [5] 中的公式（见定理 1).

我们总假定 L 是复平面中一封闭光滑曲线. $f(t,\tau)$ 是定义在 $L\times L$ 上的一函数，具有足够高阶的 Hölder 连续导数.

首先我们注意，对于 L 上的任何两点 t_0, t_1（相同或否，下同），显然有

$$\int_L \frac{d\tau}{\tau-t_0}\int_L \frac{f(t,\tau)}{t-t_1}dt=\int_L \frac{dt}{t-t_1}\int_L \frac{f(t,\tau)}{\tau-t_0}d\tau. \tag{1}$$

由此立刻可得：对任何正整数 m, n，则有[6]

$$\int_L \frac{d\tau}{(\tau-t_0)^n}\int_L \frac{f(t,\tau)}{(t-t_1)^m}dt=\int_L \frac{dt}{(t-t_1)^m}\int_L \frac{f(t,\tau)}{(\tau-\tau_0)^n}d\tau. \tag{2}$$

此可由(1)式以及高阶奇异积分的定义

$$\int_L \frac{\varphi(t)}{(t-t_0)^n}dt=\frac{1}{(n-1)!}\int_L \frac{\varphi^{(n-1)}(t)}{t-t_0}dt, \quad \varphi^{(n-1)}(t)\in H,\ t_0\in L, \tag{3}$$

而得，因为(2)式相当于

$$\int_L \frac{d\tau}{\tau-t_0}\int_L \frac{D_\tau^{n-1}D_t^{m-1}f(t,\tau)}{t-t_1}dt=\int_L \frac{dt}{t-t_1}\int_L \frac{D_\tau^{n-1}D_t^{m-1}f(t,\tau)}{\tau-t_0}d\tau,$$

这里已用记号 $D_\tau\equiv\frac{\partial}{\partial\tau}$, $D_t\equiv\frac{\partial}{\partial t}$.

定理 1 设 m, n 为正整数，$t_0\in L$，则

$$\int_L \frac{d\tau}{(\tau-t_0)^n}\int_L \frac{f(t,\tau)}{(t-\tau)^m}dt=-\frac{\pi^2}{(m-1)!(n-1)!}\mathscr{D}^{n-1}D_t^{m-1}f(t_0,t_0)$$
$$+\int_L dt\int_L \frac{f(t,\tau)d\tau}{(\tau-t_0)^n(t-\tau)^m}, \tag{4}$$

这里已用记号 $\mathscr{D}\equiv D_t+D_\tau$.

证 用熟知的公式

$$\mathrm{D}_t\int_L \frac{f(t,\tau)}{\tau-t}d\tau=\int_L \frac{\mathscr{D}f(t,\tau)}{\tau-t}d\tau$$

以及 Bertrand-Poincaré 公式，可以看到，(4)式左边等于

$$\frac{1}{(m-1)!(n-1)!}\int_L \frac{d\tau}{\tau-t_0}\int_L \frac{\mathscr{D}^{n-1}\mathrm{D}_t^{m-1}f(t,\tau)}{t-\tau}dt$$

$$=\frac{1}{(m-1)!(n-1)!}\left[-\pi^2\mathscr{D}^{n-1}\mathrm{D}_t^{m-1}f(t_0,t_0)+\int_L dt\int_L \frac{\mathscr{D}^{n-1}\mathrm{D}_t^{m-1}f(t,\tau)}{(\tau-t_0)(t-\tau)}d\tau\right].$$

因此，为要证明(4)式，只需证明

$$\frac{1}{(m-1)!(n-1)!}\int_L dt\int_L \frac{\mathscr{D}^{n-1}\mathrm{D}_t^{m-1}f(t,\tau)}{(\tau-t_0)(t-\tau)}d\tau=\int_L dt\int_L \frac{f(t,\tau)d\tau}{(\tau-t_0)^n(t-\tau)^m}. \tag{5}$$

将此式左端写成

$$I_1-I_2=\frac{1}{(m-1)!(n-1)!}\int_L \frac{dt}{t-t_0}\int_L \frac{\mathscr{D}^{n-1}\mathrm{D}_t^{m-1}f(t,\tau)}{\tau-t_0}d\tau$$

$$-\frac{1}{(m-1)!(n-1)!}\int_L \frac{dt}{t-t_0}\int_L \frac{\mathscr{D}^{n-1}\mathrm{D}_t^{m-1}f(t,\tau)}{\tau-t}d\tau.$$

因为

$$\mathscr{D}^k=\sum_{r=0}^{k}C_r^k\mathrm{D}_\tau^{k-r}\mathrm{D}_t^r, \tag{6}$$

故由(3)式知，

$$I_1=\frac{1}{(m-1)!(n-1)!}\sum_{r=0}^{n-1}C_r^{n-1}\int_L \frac{dt}{t-t_0}\int_L \frac{\mathrm{D}_\tau^{n-r-1}\mathrm{D}_t^{m+r-1}f(t,\tau)}{\tau-t_0}d\tau$$

$$=\sum_{r=0}^{n-1}C_{m-1}^{m+r-1}\int_L \frac{dt}{(t-t_0)^{m+r}}\int_L \frac{f(t,\tau)}{(\tau-t_0)^{n-r}}d\tau.$$

将求和指标 r 改为 $n-r-1$，上式又可写为

$$I_1=\sum_{r=0}^{n-1}C_{m-1}^{m+n-r-2}\int_L \frac{dt}{(t-t_0)^{m+n-r-1}}\int_L \frac{f(t,\tau)}{(\tau-t_0)^{r+1}}d\tau.$$

另一方面，又因

$$\mathrm{D}_t^k=\sum_{r=0}^{k}(-1)^rC_r^k\mathscr{D}^{k-r}\mathrm{D}_\tau^r,^{①} \tag{7}$$

故由(3)式与(5)式，并作类似运算，可得

$$I_2=\frac{1}{(m-1)!(n-1)!}\sum_{r=0}^{m-1}(-1)^rC_r^{m-1}\int_L \frac{dt}{t-t_0}\int_L \frac{\mathscr{D}^{m+n-r-2}\mathrm{D}_t^rf(t,\tau)}{\tau-t}d\tau$$

$$=\sum_{r=0}^{m-1}(-1)^rC_{n-1}^{m+n-r-2}\int_L \frac{dt}{(t-t_0)^{m+n-r-1}}\int_L \frac{f(t,\tau)}{(\tau-t_0)^{r+1}}d\tau.$$

容易验证下一恒等式

$$\frac{1}{(\tau-t_0)^n(t-\tau)^m}=\sum_{r=0}^{n-1}C_{m-1}^{m+n-r-2}\frac{1}{(t-t_0)^{m+n-r-1}(\tau-t_0)^{r+1}}$$

$$+\sum_{r=0}^{m-1}C_{n-1}^{m+n-r-2}\frac{1}{(t-t_0)^{m+n-r-1}(t-\tau)^{r+1}}. \tag{8}$$

---

① 注意，将 t 与 τ 互换，(7)式也成立.

将 I_1, I_2 的表达式代入(5)式左边并利用恒等式(8)，便得(5)式. 证毕.

我们还给出下列不同类型的有关高阶奇异积分的换序公式.

定理 2 设 $t_0, t_1 \in L$ 且 n 为一正整数，则

$$\int_L \frac{d\tau}{(\tau-t_0)^n}\int_L \frac{f(t,\tau)dt}{(t-\tau)(t-t_1)}=\begin{cases} -\dfrac{\pi^2}{(t_1-t_0)^n}\Big[f(t_1,t_1)-\sum\limits_{r=0}^{n-1}\dfrac{\mathscr{D}^r f(t_0,t_0)}{r!}(t_1-t_0)^r\Big] \\ \quad +\displaystyle\int_L \frac{dt}{t-t_1}\int_L \frac{f(t,\tau)d\tau}{(\tau-t_0)^n(t-\tau)}, \quad \text{当 } t_1\neq t_0 \text{ 时;} \\ -\dfrac{\pi^2}{n!}\mathscr{D}^n f(t_0,t_0)+\displaystyle\int_L \frac{dt}{t-t_0}\int_L \frac{f(t,\tau)d\tau}{(\tau-t_0)^n(t-\tau)}, \\ \quad \text{当 } t_1=t_0 \text{ 时.} \end{cases} \tag{9}$$

证 先设 $t_1\neq t_0$. 由(8)($n=1$)，(5)以及 Bertrand-Poincaré 公式，(9)的左边等于

$$\begin{aligned}&\int_L\Big[\frac{1}{(t_1-t_0)^n(\tau-t_1)}-\sum_{r=0}^{n-1}\frac{1}{(t_1-t_0)^{n-r}(\tau-t_0)^{r-1}}\Big]d\tau\int_L f(t,\tau)\Big(\frac{1}{t-\tau}-\frac{1}{t-t_1}\Big)dt\\ &\qquad=-\pi^2\Big[\frac{f(t_1,t_1)}{(t_1-t_0)^n}-\sum_{r=0}^{n-1}\frac{1}{(t_1-t_0)^{n-r}}\frac{1}{r!}\mathscr{D}^r f(t_0,t_0)\Big]\\ &\qquad\quad+\int_L\frac{dt}{t-t_1}\int_L\frac{f(t,\tau)d\tau}{(\tau-t_0)^n(t-\tau)},\end{aligned}$$

此即(9)的右边.

用同样的方法可证 $t_1=t_0$ 时(9)式成立. 定理证毕.

对一足够光滑的函数 $F(t)$，我们将引进记号 $P_{n,a}(t;F)$ 表示该函数在 $t=a$ 处的 Taylor 展式的前 $n+1$ 项：

$$P_{n,a}(t;F)\equiv\sum_{r=0}^{n}\frac{F^{(r)}(a)}{r!}(t-a)^r, \tag{10}$$

而把其余项记为

$$R_{n,a}(t;F)=F(t)-P_{n,a}(t,F). \tag{11}$$

于是(9)式中，当 $t_1\neq t_0$ 时右边不带积分的项实际是

$$-\pi^2 R_{n-1,t_0}(t_1;f(t,t))/(t_1-t_0)^n.$$

由此立即看出，当 $t_1\to t_0$ 时，它正好趋于(9)式中当 $t_1=t_0$ 时的相应项.

推论 t_0, t_1, n 如上，则

$$\int_L \frac{d\tau}{\tau-t_0}\int_L \frac{f(t,\tau)dt}{(t-\tau)(t-t_1)^n}=\begin{cases} -\dfrac{\pi^2}{(t_0-t_1)^n}\Big[f(t_0,t_0)-\sum\limits_{r=0}^{n-1}\dfrac{\mathscr{D}^r f(t_1,t_1)}{r!}(t_0-t_1)^r\Big] \\ \quad +\displaystyle\int_L \frac{dt}{(t-t_1)^n}\int_L \frac{f(t,\tau)d\tau}{(t-\tau)(\tau-t_0)}, \quad \text{当 } t_1\neq t_0 \text{ 时;} \\ -\dfrac{\pi^2}{n!}\mathscr{D}^n f(t_0,t_0)\mp\displaystyle\int_L \frac{dt}{(t-t_0)^n}\int_L \frac{f(t,\tau)d\tau}{(t-\tau)(\tau-t_0)}, \\ \quad \text{当 } t_1=t_0 \text{ 时.} \end{cases} \tag{12}$$

在(9)式中交换 t_0, t_1 后即可得出(12)式.

定理 3 t_0, t_1, n 仍如前，则

$$\int_L \frac{d\tau}{\tau-t_0}\int_L \frac{f(t,\tau)dt}{(t-\tau)^n(t-t_1)}=\begin{cases}\dfrac{\pi^2}{(t_1-t_0)^n}[P_{n-1,t_0}(t_1;f(t,t_0))-P_{n-1,t_1}(t_0;f(t_1,\tau))]\\ \quad+\displaystyle\int_L\frac{dt}{t-t_1}\int_L\frac{f(t,\tau)d\tau}{(\tau-t_0)(t-\tau)^n},\quad 当\ t_1\neq t_0\ 时;\\ \dfrac{\pi^2}{n\,!}[(-1)^n D_\tau^n f(t_0,t_0)-D_t^n f(t_0,t_0)]\\ \quad+\displaystyle\int_L\frac{dt}{t-t_0}\int_L\frac{f(t,\tau)d\tau}{(\tau-t_0)(t-\tau)^n},\quad 当\ t_1=t_0\ 时.\end{cases}\tag{13}$$

证 先设 $t_1\neq t_0$. 利用(8)式，可将(13)式左边写成

$$J_1-J_2=\int_L\frac{d\tau}{(\tau-t_0)(t_1-\tau)^n}\int_L\frac{f(t,\tau)}{t-t_1}dt-\sum_{r=0}^{n-1}\int_L\frac{d\tau}{(\tau-t_0)(t_1-\tau)^{n-r}}\int_L\frac{f(t,\tau)}{(t-\tau)^{r+1}}dt.$$

由(2)式知，J_1 中的累次积分可以交换积分次序. 再利用(8)式以及定理1，可写

$$\begin{aligned}J_2&=\sum_{r=0}^{n-1}\frac{1}{(t_1-t_0)^{n-r}}\int_L\Big[\frac{1}{\tau-t_0}-\sum_{s=0}^{n-r-1}\frac{(t_0-t_1)^s}{(\tau-t_1)^{s+1}}\Big]d\tau\int_L\frac{f(t,\tau)}{(t-\tau)^{r+1}}dt\\&=\sum_{r=0}^{n-1}\frac{1}{(t_1-t_0)^{n-r}}\Big\{\int_L\frac{d\tau}{\tau-t_0}\int_L\frac{f(t,\tau)}{(t-\tau)^{r+1}}dt-\sum_{s=0}^{n-r-1}(t_0-t_1)^s\int_L\frac{d\tau}{(\tau-t_1)^{s+1}}\int_L\frac{f(t,\tau)}{(t-\tau)^{r+1}}dt\Big\}\\&=-\pi^2\sum_{r=0}^{n-1}\frac{D_t^r f(t_0,t_0)}{r\,!(t_1-t_0)^{n-r}}+\pi^2\sum_{r=0}^{n-1}\frac{1}{r\,!(t_1-t_0)^{n-r}}\sum_{s=0}^{n-r-1}\frac{(t_0-t_1)^s}{s\,!}\mathscr{D}^s D_t^r f(t_1,t_1)+\cdots\\&=-\frac{\pi^2}{(t_1-t_0)^n}(K_1-K_2)+\cdots,\end{aligned}$$

其中未写出部分为 J_2 中所含累次积分交换积分次序后的结果，而

$$\begin{aligned}K_1&\equiv\sum_{r=0}^{n-1}\frac{D_t^r f(t_0,t_0)}{r\,!}(t_1-t_0)^r=P_{n-1,t_0}(t_1;f(t,t_0)),\\K_2&\equiv\sum_{r=0}^{n-1}\sum_{s=0}^{n-r-1}\frac{(-1)^r(t_0-t_1)^{r+s}}{(r+s)!}C_r^{r+s}\mathscr{D}^s D_t^r f(t_1,t_1)\\&=\sum_{k=0}^{n-1}\sum_{r=0}^{k}\frac{(t_0-t_1)^k}{k\,!}(-1)^r C_r^k\mathscr{D}^{k-r}D_t^r f(t_1,t_1)\\&=\sum_{k=0}^{n-1}\frac{(t_0-t_1)^k}{k\,!}D_\tau^k f(t_1,t_1)=P_{n-1,t_1}(t_0;f(t_1,\tau)).\end{aligned}$$

于是(13)式得证.

再设 $t_1=t_0$. 这时(13)式左边等于

$$(-1)^n\int_L\frac{d\tau}{(\tau-t_0)^{n+1}}\int_L\frac{f(t,\tau)d\tau}{t-t_0}-\sum_{r=0}^{n-1}(-1)^{n-r}\int_L\frac{d\tau}{(\tau-t_0)^{n-r+1}}\int_L\frac{f(t,\tau)dt}{(t-\tau)^{r+1}},$$

其中第一项累次积分可交换积分次序，而后一项交换积分次序时，由定理1，应添加下列和式：

$$\begin{aligned}&\pi^2\sum_{r=0}^{n-1}\frac{(-1)^{n-r}}{(n-r)!r\,!}\mathscr{D}^{n-r}D_t^r f(t_0,t_0)\\&=\frac{\pi^2}{n\,!}\Big[\sum_{r=0}^{n}(-1)^{n-r}C_r^n\mathscr{D}^{n-r}D_t^r f(t_0,t_0)-D_t^n f(t_0,t_0)\Big]\\&=\frac{\pi^2}{n\,!}[(-1)^n D_\tau^n f(t_0,t_0)-D_t^n f(t_0,t_0)].\end{aligned}$$

因此(13)式成立. 定理证毕.

在 $t_1 \neq t_0$ 时的(13)式中令 $t_1 \to t_0$ 时，也正好就是 $t_1 = t_0$ 时的(13)式.

当然我们还可考虑更一般的积分换序公式. 例如，在定理 2 中把(9)式左边分母中的 $(t-t_1)$ 或 $(t-\tau)$ 分别换作 $(t-t_1)^m$ 或 $(t-\tau)^m$，利用(8)式，也可得到一些类似的但较复杂的公式，这里不赘.

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ON METHODS OF SOLUTION FOR SOME KINDS OF SINGULAR INTEGRAL EQUATIONS WITH CONVOLUTION*

Abstract

Methods of solution for some kinds of equations containing Cauchy principal value integral together with convolution are discussed. The general solutions and the conditions of solvability are obtained.

There were rather complete investigations on the method of solution for equations of Cauchy type as well as integral equations of convolution type[1,2]. The invertibility of Wiener-Hopf operators with discontinuous coefficients was considered in [3]. For operators containing both Cauchy principal value integral and convolution, the conditions of their Noethericity were discussed in [4,5] in more general cases. For applications, the problem to find their solutions is very important. In this paper, we give effective methods of solution for certain basic kinds of such equations, including, besides the Cauchy principal value integral, equations with one or two convolution kernels, equations of Wiener-Hopf type and dual equations, in normal cases.

Some special kinds of Riemann boundary value problems with discontinuous coefficients appear in the course of solution, which are solved in the same time. It is necessary for us to introduce certain new classes of functions in advance and to point out some of their properties.

The Fourier transforms used in this paper are understood to be performed in $L_2(-\infty, +\infty)$ and the functions involved certainly belong to this space.

§ 1 Some Classes of Functions and Their Properties

In [2], the concepts of classes $\{0\}$ and $\{\{0\}\}$ were introduced as follows. A function $F(s)$ belongs to $\{\{0\}\}$, if the following two conditions are fulfilled:

1) $F(s)\in\hat{H}$, that is, it satisfies the Hölder condition on the whole real axis, in-

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cluding ∞, i.e., $\pm\infty$ (notation used in [6]);

2) $F(s)\in L_2(-\infty,+\infty)$.

$f(t)\in\{0\}$ if its Fourier transform

$$F(s)=Vf=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)e^{ist}dt,\quad -\infty<s<+\infty, \tag{1.1}$$

belongs to $\{\{0\}\}$. On maintaining condition 1), we strengthen condition 2) slightly to

2)′ $F(s)=O(1/|s|^{\mu})$, $\mu>\frac{1}{2}$, where $|s|$ is sufficiently large.

Then we call $F(s)\in((0))$ or $((0))^{\mu}$ and $f(t)\in(0)$ or $(0)^{\mu}$. From 2)′ it is assured that 2) is valid. If we strengthen 2)′ slightly again to

2)″ $F(s)\in H^{\mu}(N_{\infty})$, $\mu>\frac{1}{2}$, i.e., it belongs to H in the neighborhood N_{∞} of ∞, and $F(\infty)=0$, then we call $F(s)\in\langle\langle 0\rangle\rangle$ or $\langle\langle 0\rangle\rangle^{\mu}$ and $f(t)\in\langle 0\rangle$ or $\langle 0\rangle^{\mu}$. From 2)″ it is assured that 2)′ is valid. Hence

$$\langle\langle 0\rangle\rangle\subset((0))\subset\{\{0\}\},\quad \langle 0\rangle\subset(0)\subset\{0\}.$$

For two functions $k(t)$ and $f(t)$, if we use the notation of convolution

$$k*f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k(t-\tau)f(\tau)d\tau, \tag{1.2}$$

then it is well known that

$$V(k*f)=KF,$$

where K,F are the Fourier transforms of k,f respectively (we always use the capital letter represents the Fourier transform of the corresponding small letter). We know that $k,f\in\{0\}$ implies $k*f\in\{0\}$[2]. Obviously, when at least one of k and $f\in(0)$, then $k*f\in(0)$; when both k and $f\in\langle 0\rangle$, then $k*f\in\langle 0\rangle$.

We also introduce the operator T of Cauchy principal value integral

$$Tf=\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(\tau)}{\tau-t}d\tau,\quad -\infty<t<+\infty. \tag{1.3}$$

From [1,7], T maps $\{\{0\}\}$ and $\langle\langle 0\rangle\rangle$ into themselves respectively and $T^2=I$ (identity).

We also introduce operators N and S:

$$Nf(t)=f(-t),\quad Sf(t)=f(t)\operatorname{sgn}t. \tag{1.4}$$

Obviously $N^2=S^2=I$ and $SN=-NS$.

For the inverse Fourier transform operator V^{-1}:

$$V^{-1}F=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}F(s)e^{-ist}ds,\quad -\infty<t<+\infty, \tag{1.5}$$

it is evident that

$$V^{-1}=NV=VN,\quad V^2=N. \tag{1.6}$$

It was proved in [2], that when applying to functions in $\{0\}$,

$$VS=TV. \tag{1.7}$$

The following lemma plays an important role:

Lemma 1. *When applying to functions in* $\{0\}$,

$$VT=-SV, \tag{1.8}$$

i.e.,

$$V\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{f(\tau)}{\tau-t}d\tau=-F(s)\mathrm{sgn}s. \tag{1.8$'$}$$

Proof. From (1.7), $T=VSV^{-1}$ and so, by (1.6),

$$VT=V^2SV^{-1}=NSNV=-N^2SV=-SV.$$

Note that, from $f\in\{0\},(0)$ or $\langle 0\rangle$, generally we could not assure that Tf belongs to the same class. However, we have

Lemma 2. *If $f\in\{0\},(0)$ or $\langle 0\rangle$ and $F(0)=0$, then Tf belongs to the same class.*

Proof. By supposition, $Vf\in\{\{0\}\}$, $((0))$ or $\langle\langle 0\rangle\rangle$. From Lemma 1

$$VTf=-F(s)\mathrm{sgn}s.$$

Noting that $F(\infty)=F(0)=0$, we know $VTf\in\{\{0\}\}$, $((0))$ or $\langle\langle 0\rangle\rangle$. Therefore $Tf\in\{0\}$, (0) or $\langle 0\rangle$.

Besides, we note that, for the class (0) or $\langle 0\rangle$, the index μ is invariant, provided $\frac{1}{2}<\mu<1$.

Moreover, if $f(t)\in L_1(-\infty,+\infty)$, then $F(0)=0$ is actually

$$\int_{-\infty}^{+\infty}f(t)dt=0.$$

§2 Singular Integral Equations with One Convolution Kernel

Let us solve the following equation

$$a\varphi(t)+\frac{b}{\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k(t-\tau)\varphi(\tau)d\tau=g(t),$$

$$b\neq 0,\quad -\infty<t<+\infty, \tag{2.1}$$

where a and b are constants, $k,g\in\{0\}$ and the unknown function φ is required to be in $\{0\}$ too.

Taking Fourier transforms of both sides of (2.1), by Lemma 1, we get

$$[a-b\mathrm{sgn}s+K(s)]\Phi(s)=G(s),$$

which follows $G(0)=\Phi(0)=0$ since $G(s)$ is continuous at $s=0$.

Restricting ourselves to the normal type, i.e.,

$$K(s)\neq\begin{cases}-(a-b), & 0\leqslant s\leqslant+\infty,\\ -(a+b), & -\infty\leqslant s\leqslant 0,\end{cases} \tag{2.2}$$

we obtain

$$\Phi(s)=\frac{G(s)}{a-b\mathrm{sgn}s+K(s)}. \tag{2.3}$$

Since $G(0)=0$ and $G(s)\in\{\{0\}\}$, we conclude $\Phi(s)\in\{\{0\}\}$ and hence $\varphi=V^{-1}\Phi$ is truly the unique solution of (2.1) in $\{0\}$.

We also see that $\varphi\in(0)^\mu$ if $g\in(0)^\mu$ and $\varphi\in\langle 0\rangle^\mu$ if $k,g\in\langle 0\rangle^\mu$, provided $\mu<1$.

Thus, we obtain

Theorem 1. *If $k,g\in\{0\}$, in case of normal type, i.e., (2.2) to be valid, then (2.1) is solvable if and only if $G(0)=0$ and has the unique solution $\varphi=V^{-1}\Phi$ in $\{0\}$, where Φ is given by (2.3). Moreover, $g\in(0)$ implies $\varphi\in(0)$ and $k,g\in\langle 0\rangle$ implies $\varphi\in\langle 0\rangle$.*

After simplification, φ may be written as

$$\varphi(t)=g_0(t)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_0(t-\tau)g_0(\tau)\mathrm{d}\tau, \tag{2.4}$$

where $g_0=V^{-1}G_0$, $k_0=V^{-1}K_0$, in which

$$G_0(s)=\begin{cases}\dfrac{G(s)}{a-b}, & s\geqslant 0,\\ \dfrac{G(s)}{a+b}, & s\leqslant 0;\end{cases}\qquad K_0(s)=\begin{cases}\dfrac{K(s)}{a-b+K(s)}, & s>0,\\ \dfrac{K(s)}{a+b+K(s)}, & s<0.\end{cases} \tag{2.5}$$

Noting that, although $K_0(s)$ is discontinuous at $s=0$, it would not influence the property $k_0*g_0\in\{0\}$ since $G(0)=0$.

§ 3 Singular Integral Equations with Two Convolution Kernels

Let us solve the equation

$$a\varphi(t)+\frac{b}{\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\varphi(\tau)}{\tau-t}\mathrm{d}\tau+\frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty}k_1(t-\tau)\varphi(\tau)\mathrm{d}\tau+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}k_2(t-\tau)\varphi(\tau)\mathrm{d}\tau=g(t),$$

$$b\neq 0,\quad -\infty<t<+\infty, \tag{3.1}$$

where a,b are again constants, $k_1,k_2,g\in\{0\}$ and the unknown functions $\varphi_\pm$ and hence φ are required to be in $\{0\}$. Here we have denoted

$$\varphi_+(t)=\begin{cases}\varphi(t), & t\geqslant 0,\\ 0, & t<0;\end{cases}\qquad \varphi_-(t)=\begin{cases}0, & t\geqslant 0,\\ -\varphi(t), & t<0.\end{cases}$$

Assume that equation (3.1) has a solution. Taking Fourier transform, by Lemma 1, we get

$$(a-b\operatorname{sgn}s)\Phi(s)+K_1(s)\Phi^+(s)-K_2(s)\Phi^-(s)=G(s), \tag{3.2}$$

where $\Phi^\pm(s)$ are respectively the Fourier transforms of $\varphi_\pm(t)$, which are the boundary values of the (sectionally) holomorphic function $\Phi(z)$ in the upper and the lower half-planes respectively, and $\Phi(s)=\Phi^+(s)-\Phi^-(s)$[2]. In order that $\Phi^\pm(s)$ and then $\Phi(s)$ are continuous at $s=0$, it is necessary that $\Phi(0)=0$, i.e., $\Phi^+(0)=\Phi^-(0)$.

(3.2) is the Riemann boundary value problem with discontinuous coefficients:

$$\Phi^+(s)=D(s)\Phi^-(s)+F(s),\quad -\infty<s<+\infty, \tag{3.3}$$

in which we have put

$$D(s)=\frac{a-b\operatorname{sgn}s+K_2(s)}{a-b\operatorname{sgn}s+K_1(s)},\quad F(s)=\frac{G(s)}{a-b\operatorname{sgn}s+K_1(s)}, \tag{3.4}$$

and restricted ourselves to the normal type case

$$K_j(s)\neq\begin{cases}-(a-b), & 0\leqslant s\leqslant+\infty,\\ -(a+b), & -\infty\leqslant s\leqslant 0,\end{cases}\quad j=1,2. \tag{3.5}$$

Noting that $K_j(\infty)=0$ which implies $D(\infty)=1$ and $F(\infty)=0$ since $G(\infty)=0$, we know that $s=\infty$ is not a nodal point of the problem. Its unique nodal point is $s=0$. We require that the solutions of (3.3) should be at least continuous along the whole real axis and $\Phi(\infty)=0$.

According to the method used in [1], take a continuous branch of $\log D(s)$ such that it is continuous at $s=\infty$, e. g., $\log D(\infty)=0$, and denote

$$\gamma_0=\alpha_0+\mathrm{i}\beta_0=\frac{1}{2\pi\mathrm{i}}\{\log D(-0)-\log D(+0)\}. \tag{3.6}$$

Then choose an integer κ, the index of the problem, such that $0\leqslant\alpha=\alpha_0-\kappa<1$. Denote $\gamma=\gamma_0-\kappa=\alpha+\mathrm{i}\beta_0$. Since we require $\Phi(\infty)=0$, so we get: when $\kappa\geqslant 0$, the general solution of (3.3) (without considering the behavior of $\Phi^{\pm}(s)$ at $s=0$ for the time being) is

$$\Phi(z)=X(z)\left\{\Psi(z)+\frac{Q_\kappa(z)}{(z+\mathrm{i})^\kappa}\right\},\quad \Psi(z)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{F(t)\mathrm{d}t}{X^+(t)(t-z)}, \tag{3.7}$$

where

$$Q_{\kappa-1}(z)=C_0+C_1z+\cdots+C_{\kappa-1}z^{\kappa-1} \tag{3.8}$$

is an arbitrary polynomial of degree $\kappa-1$ (in the sequel we understand $Q_\kappa\equiv 0$ when $\kappa<0$); when $\kappa\leqslant -1$, the problem is solvable if and only if the conditions

$$\int_{-\infty}^{+\infty}\frac{F(t)\mathrm{d}t}{X^+(t)(t+\mathrm{i})^j}=0,\quad j=1,\ldots,-\kappa, \tag{3.9}$$

are satisfied, and then the problem has the unique solution (3.7). Here

$$X(z)=\begin{cases}\mathrm{e}^{\Gamma(z)}, & \mathrm{Re}z>0,\\ \left(\dfrac{z+\mathrm{i}}{z-\mathrm{i}}\right)^\kappa\mathrm{e}^{\Gamma(z)}, & \mathrm{Re}z<0,\end{cases} \tag{3.10}$$

where

$$\Gamma(z)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\log D_0(t)}{t-z}\mathrm{d}t,\quad D_0(t)=\left(\frac{t+\mathrm{i}}{t-\mathrm{i}}\right)^\kappa D(z), \tag{3.11}$$

in which we have taken the definite branch of

$$\log D_0(t)=\kappa\log\frac{t+\mathrm{i}}{t-\mathrm{i}}+\log D(t), \tag{3.12}$$

provided we have chosen $\log\dfrac{t+\mathrm{i}}{t-\mathrm{i}}\Big|_{t=\infty}=0$, or, what is the same, $\log\dfrac{t+\mathrm{i}}{t-\mathrm{i}}\Big|_{t=\pm 0}=\pm\mathrm{i}\pi$. It is easy to prove

$$X^+(t)=\sqrt{D_0(t)}\,\mathrm{e}^{\Gamma(t)},\quad X^-(t)=\mathrm{e}^{\Gamma(t)}/\sqrt{D_0(t)}, \tag{3.13}$$

where $\sqrt{D_0(t)}=\exp\left\{\frac{1}{2}\log D_0(t)\right\}$ has definite value. By (3.7), we get

$$\left.\begin{aligned}\Phi^+(s)&=\frac{1}{2}F(s)+X^+(s)\left\{\Psi(s)+\frac{Q_{\kappa-1}(s)}{(s+\mathrm{i})^\kappa}\right\},\\ \Phi^-(s)&=-\frac{1}{2}\frac{F(s)}{D(s)}+X^-(s)\left\{\Psi(s)+\frac{Q_{\kappa-1}(s)}{(s+\mathrm{i})^\kappa}\right\},\end{aligned}\right\} \tag{3.14}$$

and thereby

$$\Phi(s)=\frac{F(s)}{2}\left[1+\frac{1}{D(s)}\right]+[X^+(s)-X^-(s)]\left\{\Psi(s)+\frac{Q_{\kappa-1}(s)}{(s+i)^\kappa}\right\}. \tag{3.15}$$

Since $X^+(s)$ is bounded and $\neq 0$, it is easy to verify $\Phi^\pm(s), \Phi(s)\in L_2(-\infty,+\infty)$ and $\in H$ on any closed interval exterior to $s=0$.

The only thing required to be considered is their behavior near $s=0$. In order that they belong to $\{\{0\}\}$, they ought to be continuous at $s=0$. We prove that it is then necessary for $G(0)=0$.

First, let $s=0$ be an ordinary node, i.e., $0<\alpha<1$. Then $\gamma\neq 0$. It is known that, in the neighborhood of $s=0$,

$$X^+(s)=\sqrt{D_0(s)}\,s^\gamma e^{\Gamma_0(s)},\quad \Gamma_0(s)\in H,$$

where, by (3.12)

$$\sqrt{D_0(\pm 0)}=\exp\frac{1}{2}\{\pm\kappa\pi i+\log D(\pm 0)\},\quad \frac{\sqrt{D_0(+0)}}{\sqrt{D_0(-0)}}=e^{-\gamma\pi i}. \tag{3.16}$$

On the other hand, from [1], § 26, 4°, when $s>0$,

$$\Psi(s)=\frac{e^{-\Gamma_0(s)}}{s^\gamma}\left\{\frac{\cot\gamma\pi}{2i}\frac{F(+0)}{\sqrt{D_0(+0)}}-\frac{e^{-\gamma\pi i}}{2i\sin\gamma\pi}\frac{F(-0)}{\sqrt{D_0(-0)}}\right\}+\Psi^*(s), \tag{3.17}$$

where $\Psi^*(s)=\Psi^{**}(s)/|s|^{\alpha'}$, $0<\alpha'<\alpha$ and $\Psi^{**}(s)\in H$. Then, by (3.16), after simplifying (3.14), we may get

$$\Phi^+(+0)=\frac{e^{\gamma\pi i}}{2i\sin\gamma\pi}[F(+0)-e^{-3\gamma\pi i}F(-0)]. \tag{3.18}$$

When $s<0$, instead of (3.17), we have

$$\Psi(s)=\frac{e^{-\Gamma_0(s)}}{s^\gamma}\left\{\frac{e^{\gamma\pi i}}{2i\sin\gamma\pi}\frac{F(+0)}{\sqrt{D_0(+0)}}-\frac{\cot\gamma\pi}{2i}\frac{F(-0)}{\sqrt{D_0(-0)}}\right\}+\Psi^*(s), \tag{3.17$'$}$$

and then

$$\Phi^+(-0)=\frac{e^{2\gamma\pi i}}{2i\sin\gamma\pi}[F(+0)-e^{-3\gamma\pi i}F(-0)]. \tag{3.18$'$}$$

On comparing (3.18) with (3.18)$'$, we know that $\Phi(s)$ is continuous at $s=0$ if and only if (regarding $e^{\gamma\pi i}\neq 1$)

$$F(+0)=e^{-3\gamma\pi i}F(-0). \tag{3.19}$$

And then we have $\Phi^+(0)=0$. Since we require $\Phi(0)=0$, we must also require $\Phi^-(0)=0$. Returning to (3.2), we know that we must have $F(0)=0$ and hence $G(0)=0$. But once $G(0)=0$, we really have $\Phi^\pm(0)=\Phi(0)=0$ and $\Phi^\pm(s)$, $\Phi(s)\in H$ in the neighborhood of $s=0$, and therefore $\Phi^\pm(s)$, $\Phi(s)$ belong to $\{\{0\}\}$.

Now, let $s=0$ be a special node, i.e., $\alpha=0$. Then $\gamma=i\beta_0$. If $\beta_0\neq 0$, then (3.17) and (3.17)$'$ remain valid, with $\Psi^*(s)\in H_0$, i.e., $\Psi^*(\pm 0)$ exist but do not equal to each other possibly. In place of (3.18), we have

$$\Phi^+(+0)=\frac{e^{\gamma\pi i}}{2i\sin\gamma\pi}[F(+0)-e^{-3\gamma\pi i}F(-0)]+\sqrt{D_0(+0)}e^{\Gamma_0(0)}\lim_{s\to+0}s^{i\beta_0}[\Psi^*(s)+A_0],$$

where

$$A_0=\begin{cases}C_0/i^\kappa, & \kappa>0,\\ 0, & \kappa\leqslant 0,\end{cases}$$

and a similar formula for $\Phi^+(-0)$. In order that $\Phi^+(\pm 0)$ exist, we must have $\Psi^*(\pm 0)=-A_0$. And then we are back to (3.18) and (3.18)′ and hence again to (3.19). Thus, we have $G(0)=0$ again.

Once $G(0)=0$ is fulfilled, then $F(0)=0$ and therefore $\Psi(s)\in H$ near $s=0$. Thus, in order that $\Phi^+(s)$ is continuous at $s=0$, the constant term of $Q_{\kappa-1}(z)$ should take the value

$$C_0=\frac{i^{\kappa+1}}{2\pi}\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)t} \tag{3.20}$$

if $\kappa\geqslant 1$; and an additional condition of solvability

$$\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)t}=0 \tag{3.21}$$

should be supplemented if $\kappa\leqslant 0$. When C_0 is taken to be (3.20) or (3.21) is fulfilled, it is readily seen $\Phi^\pm(0)=\Phi(0)=0$.

If $\beta_0=0$, i.e., $\gamma=0$, then $D(s)$ is continuous at $s=0$. Since $b\neq 0$, we know at once $K_1(0)=K_2(0)=0$. So $D(0)=1$ and hence again we must have $F(0)=0$ and then $G(0)=0$ in order that $\Phi(0)=0$. Thus, $s=0$ is not a nodal point at all. There is no problem in this case.

In all of the above cases, $\Phi^\pm(s), \Phi(s)\in\hat{H}$ undoubtedly.

Thus, we have

Theorem 2. *Under supposition, in the normal type case, equation (3.1) is possibly solvable in class {0} only when $G(0)=0$. Assume that this is fulfilled. If $s=0$ is an ordinary node, then, when the index $\kappa\geqslant 0$, it always has the solution $\varphi=V^{-1}\Phi$, where Φ is given by (3.15); when $\kappa\leqslant -1$, it has the (unique) solution as above, provided the conditions of solvability (3.9) are fulfilled. If $s=0$ is a special node and $K_1(0)=K_2(0)$, the above statements remain true; in case $K_1(0)\neq K_2(0)$, then, it has the solution as above with C_0 to be taken as (3.20) if $\kappa\geqslant 1$, and it is solvable as above if and only if the conditions of solvability (3.21) and (3.9) are fulfilled if $\kappa\leqslant 0$ (the latter disappears when $\kappa=0$).*

Remark. In applications, we often have real equation (3.1), in which $a=A$ is real, $b=Bi$ is pure imaginary ($\neq 0$) and k_1, k_2 are real functions. In this case

$$D(\pm 0)=\frac{A+k_2(0)\mp Bi}{A+k_1(0)\mp Bi}$$

are conjugate to each other, and $\alpha_0=\frac{1}{2\pi}\{\arg D(-0)-\arg D(+0)\}$ so that α_0 is not an integer. Thus, $s=0$ is an ordinary node for the real equation (3.1).

§ 4 Singular Integral Equation of Wiener-Hopf Type

In this section we consider the method of solution for the equation

$$a\varphi(t)+\frac{b}{\pi i}\int_0^{+\infty}\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}k(t-\tau)\varphi(\tau)d\tau=g(t),$$

$$b\neq 0,\quad 0<t<+\infty, \tag{4.1}$$

where a, b are constants, $k, g=g_+\in\langle 0\rangle$ and $\varphi=\varphi_+$ is required to be in $\{0\}$.

Let $k, g\in\langle 0\rangle^{\mu}$ $(\frac{1}{2}<\mu<1)$.

On extending (4.1) to $-\infty<t<0$, the right-hand side of (4.1) is augmented with an unknown function $\varphi_-(t)$. Taking the Fourier transforms of both of its sides, by Lemma 1, we get

$$[a-b\ \mathrm{sgn}s+K(s)]\Phi^+(s)=G(s)+\Phi^-(s).$$

Restricted to the normal type case, i.e., $K(s)$ satisfying (2.2), it can be written as (3.3) with

$$D(s)=\frac{1}{a-b\mathrm{sgn}s+K(s)},\quad F(s)=D(s)G(s). \tag{4.2}$$

Denote

$$\gamma_\infty=\alpha_\infty+i\beta_\infty=\frac{1}{2\pi i}\{\log D(+\infty)-\log D(-\infty)\}=\frac{1}{2\pi i}\log\frac{a+b}{a-b},$$

where $\log D(s)$ is taken to be continuous for $s>0$ and $s<0$ respectively such that $0\leqslant\alpha_\infty<1$. Note that $\gamma_\infty\neq 0$ since $b\neq 0$.

Then take $\gamma_0=\alpha_0+i\beta_0$, $0\leqslant\alpha=\alpha_0-\kappa<1$, $\gamma=\alpha+i\beta_0$ as in § 3, κ being the index of the problem. We also have $\gamma\neq 0$ since $b\neq 0$.

Therefore both $s=0$ and $s=\infty$ are nodes. Note that $\Phi^-(\infty)=\Phi(\infty)=0$ since we require $\Phi^+(\infty)=0$.

Let $s=\infty$ be an ordinary node at first. From [1], we know that, the general solution of (3.3) is

$$\Phi(z)=X(z)\left\{\Psi_1(z)+\frac{Q_\kappa(z)}{(z+i)^\kappa}\right\},\quad \Psi_1(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{z+i}{t+i}\frac{F(t)dt}{X^+(t)(t-z)}, \tag{4.3}$$

when $\kappa\geqslant -1$, where $Q_\kappa(z)=C_0+C_1z+\cdots+C_\kappa z^\kappa$ is an arbitrary polynomial of degree κ and $X(z)$ is still given by (3.10) ①. Since then $X^-(s)=\chi^+(s)/s^\gamma$, $\chi^+(s)\in H_0(N_\infty)$, hence we write $Q_\kappa(z)$ in (4.3) instead of $Q_{\kappa-1}(z)$, which is sufficient for $\Phi(\infty)=0$. Moreover, it should be noted that $\Psi_1(z)$ cannot be separated as, in general,

$$\Psi_1(z)=\Psi(z)-\Psi(-i)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t-z)}-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F(t)dt}{X^+(t)(t+i)}, \tag{4.4}$$

because the integrals in the right-hand member may be divergent. When $\kappa\leqslant -2$, the con-

① However, a factor $(s+i)/(t+i)$ should be multiplied in the integrand of (3.11).

ditions of solvability read

$$\int_{-\infty}^{+\infty}\frac{F(t)\mathrm{d}t}{X^+(t)(t+\mathrm{i})^j}=0,\quad j=2,\ldots,-\kappa. \tag{4.5}$$

In this case,

$$\Phi^+(s)=\frac{1}{2}F(s)+X^+(s)\left\{\Psi_1(s)+\frac{Q_\kappa(s)}{(s+\mathrm{i})^\kappa}\right\}. \tag{4.6}$$

Since $F(\infty)=0$, we know that $\Psi_1(s)=\Psi_1^*(s)/|s|)^{\alpha'}$, $\alpha'<\alpha$ and $\Psi_1^*(s)\in H$ in the neighborhood of $s=0$. Hence it is sure that $\Phi^+(\infty)=0$. We consider the following two cases.

1° Let $\alpha_\infty>\frac{1}{2}$. If $\mu>\alpha_\infty$, $\Psi_1(s)$ is bounded and so $X^+(s)\Psi_1(s)=O(1/|s|^{\alpha_\infty})$ near $s=\infty$. And since $F(s)=O(1/|s|^\mu)$, $\Phi^+(s)=O(1/|s|^\mu)$. If $\mu\leqslant\alpha_\infty$, again by [1], we know $X^+(s)\Psi_1(s)=O(1/|s|^{\alpha_\infty-\varepsilon})$ with $\varepsilon>0$ arbitrarily small. Take ε such that $\alpha_\infty-\varepsilon>\frac{1}{2}$. Therefore, in any case, $\Phi^+(s)=O(1/|s|^\nu)$, $\nu=\min\{\mu,\ \alpha_\infty-\varepsilon\}>\frac{1}{2}$.

2° Let $\alpha_\infty\leqslant\frac{1}{2}$. Since $F(t)\in H^\mu(N_\infty)$, so, by [1], § 6, we know that $F(t)/X^+(t)\in H^{\mu-\alpha_\infty}(N_\infty)$. In this case, (4.4) becomes valid, the integrals in which are convergent now, and thereby $X^+(s)\Psi(s)\in H(N_\infty)$, being of $O(1/|s|^\mu)$. In order to guarantee $\Phi^+(s)\in L_2(-\infty,+\infty)$, we are obliged to take

$$C_\kappa=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{F(t)\mathrm{d}t}{X^+(t)(t+\mathrm{i})}$$

in (4.6) when $\kappa\geqslant0$, that is to say, (4.6) may be written as (3.14) in this case. When $\kappa<0$, there is an additional condition of solvability

$$\int_{-\infty}^{+\infty}\frac{F(t)\mathrm{d}t}{X^+(t)(t+\mathrm{i})}=0$$

which should be supplemented to (4.5) if $\kappa<-1$. Thus, the conditions of solvability return back to (3.9).

Next, let $s=\infty$ be a special node: $\alpha_\infty=0$, $\gamma_\infty=\mathrm{i}\beta_\infty\neq0$. In this case, (4.4) remains valid and the discussion is the same as 2° above.

Thus, $\Phi^+(s)$ satisfies our requirement near $s=\infty$: $\Phi^+(s)\in H$ and $\in L_2(-\infty,+\infty)$ (actually being of $O(1/|s|^\nu)$, $\nu>\frac{1}{2}$).

Now we consider the situation near $s=0$. The discussion is similar to that in § 3. By the requirement $\Phi^+(+0)=\Phi^+(-0)$, we may get (3.19) again. But since we have $F(\pm0)=G(0)D(\pm0)$ now, it may be rewritten as

$$G(0)[D(+0)-e^{3\gamma\pi\mathrm{i}}D(-0)]=0.$$

By noting that $D(-0)/D(+0)=e^{2\gamma_0\pi\mathrm{i}}$ and $\gamma=\gamma_0-\kappa\neq0$, so

$$D(+0)-e^{3\gamma\pi\mathrm{i}}D(-0)=D(+0)(1-e^{-\gamma\pi\mathrm{i}})\neq0$$

and then again we get $G(0)=0$.

The remaining discussions are the same as in § 3. But we should note that, in case α_∞

$>\frac{1}{2}$, if $\kappa\geqslant 0$, since $\Phi^+(s)$ is then given by (4.6), the constant term in Q_κ must be taken as

$$C_0=\frac{\mathrm{i}^{\kappa-1}}{2\pi}\int_{-\infty}^{+\infty}\frac{F(t)\mathrm{d}t}{X^+(t)(t+\mathrm{i})t} \tag{4.7}$$

instead of (3.20), and if $\kappa\leqslant -1$, there is an additional condition of solvability

$$\int_{-\infty}^{+\infty}\frac{F(t)\mathrm{d}t}{x^+(t)(t+\mathrm{i})t}=0. \tag{4.8}$$

Finally we get $\Phi^+(0)=0$, $\Phi^+(s)\in\hat{H}$.

Futhermore, it is seen that actually $\Phi^+(s)\in((0))$.

Thus, we obtain

Theorem 3. *Under supposition, in case of normal type, the necessary condition for the equation (4.1) to be solvable in $\{0\}$ (actually in (0)) is $G(0)=0$. Assume that this is fulfilled. In case $s=0$ is an ordinary node, if $\alpha_\infty>\frac{1}{2}$, then it has the general solution $\varphi=V^{-1}\Phi^+$ when $\kappa\geqslant -1$, where $\Phi^+(s)$ is given by (4.6), and (4.5) is the condition of solvability when $\kappa\leqslant -2$; if $\alpha_\infty\leqslant\frac{1}{2}$, it is solvable as above when $\kappa\geqslant 0$ with Φ^+ given by (3.14), and the condition of solvability is (3.9) when $\kappa\leqslant -1$. In case $s=0$ is a special node, if $\alpha_\infty>\frac{1}{2}$, Φ^+ is again given by (4.6) when $\kappa\geqslant 0$ with the constant term of Q_κ taken as (4.8), and when $\kappa\leqslant -1$, besides (4.5), the condition of solvability (4.8) should be supplemented; if $\alpha_\infty\leqslant\frac{1}{2}$, then the constant term of $Q_{\kappa-1}$ in (3.14) should be taken as (3.20) when $\kappa\geqslant 1$, and when $\kappa\leqslant 0$, besides (3.9), the condition of solvability (3.21) should be supplemented.*

Remark 1. It is seen from the above discussions, when $\alpha_\infty>\frac{1}{2}$, in fact, the obtained $\Phi^+\in\langle\langle 0\rangle\rangle$ and hence $\varphi\in\langle 0\rangle$.

Remark 2. When (4.1) is a real equation, as shown at the end of §3, $s=0$ must be an ordinary node. It is also easily seen that the characteristic feature for $\alpha_\infty>\frac{1}{2}$ or $\leqslant\frac{1}{2}$. Denote $a=A$ and $b=B\mathrm{i}$ as before. By definition of α_∞, it is obvious that

$$0\leqslant\alpha_\infty=\frac{1}{2\pi}\arg\frac{A+B\mathrm{i}}{A-B\mathrm{i}}<1.$$

Hence $\alpha_\infty>\frac{1}{2}$ means $\pi<\arg\frac{A+B\mathrm{i}}{A-B\mathrm{i}}<2\pi$, i.e., $A+B\mathrm{i}$ lies in the quadrant Ⅱ or Ⅳ; $\alpha_\infty\leqslant\frac{1}{2}$ means $A+B\mathrm{i}$ lies in the quadrant Ⅰ or Ⅲ (including the case $A=0$).

§ 5 Dual Singular Integral Equations

The above method is applicable to solving the dual singular integral equations

$$\begin{cases} a_1\omega(t)+\dfrac{b_1}{\pi\,\mathrm{i}}\displaystyle\int_{-\infty}^{+\infty}\frac{\omega(\tau)}{\tau-t}\mathrm{d}\tau+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_1(t-\tau)\omega(\tau)\mathrm{d}\tau=g(t), & 0<t<+\infty, \\ a_2\omega(t)+\dfrac{b_2}{\pi\,\mathrm{i}}\displaystyle\int_{-\infty}^{+\infty}\frac{\omega(\tau)}{\tau-t}\mathrm{d}\tau+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_2(t-\tau)\omega(\tau)\mathrm{d}\tau=g(t), & -\infty<t<0, \end{cases} \tag{5.1}$$

where a_j, b_j are constants, $k_j, g\in\langle 0\rangle$ $(j=1,2)$. Find its solution $\varphi\in\{0\}$. Assume b_1, b_2 are not equal to zero simultaneously.

Rewrite (5.1) as

$$\begin{cases} a_1\omega+b_1T\omega+k_1*\omega=g-\varphi_-, \\ a_2\omega+b_2T\omega+k_2*\omega=g+\varphi_+, \end{cases}$$

where $\varphi_\pm$ are unknown functions, to be required belonging to $\{0\}$ too. Taking Fourier transforms, we get

$$\begin{cases} (a_1-b_1\mathrm{sgn}s+K_1)\Omega=G+\Phi^-, \\ (a_2-b_2\mathrm{sgn}s+K_2)\Omega=G+\Phi^+. \end{cases}$$

Since we require Ω is continuous at $s=0$, we must have $\Omega(0)=0$. Restricted to the normal type case:

$$K_j(s)=\begin{cases} -(a_j-b_j), & 0\leqslant s\leqslant+\infty, \\ -(a_j+b_j), & -\infty\leqslant s\leqslant 0, \end{cases} \quad j=1,2, \tag{5.2}$$

we then have

$$\Omega(s)=\frac{G(s)+\Phi^-(s)}{a_1-b_1\mathrm{sgn}s+K_1(s)}=\frac{G(s)+\Phi^+(s)}{a_2-b_2\mathrm{sgn}s+K_2(s)}. \tag{5.3}$$

Therefore, we should solve the Riemann boundary value problem (3.3) again, in which

$$D(s)=\frac{a_2-b_2\mathrm{sgn}s+K_2(s)}{a_1-b_1\mathrm{sgn}s+K_1(s)}, \quad F(s)=[D(s)-1]G(s). \tag{5.4}$$

In order that $\Omega(s)$ is continuous at $s=0$, it is necessary that $\Phi^\pm(s)$ are continuous at $s=0$ and $\Phi^\pm(0)=-G(0)$. Discussions may be made fully analogous to those in § 4. Since we also require $\Phi^+(+0)=\Phi^+(-0)$, we get $G(0)=0$ again. Hence all the results as stated in Theorem 3 remain true and $\omega=V^{-1}\Omega$ in which Ω is given by (5.3). The only difference lies in that γ_∞ or (and) γ may be zero, for instance, $\gamma_\infty=0$ if $a_1/a_2=b_1/b_2$, in which case the analysis will be even simpler. It is also obvious that the solution ω in $\{0\}$ belongs also to $\langle 0\rangle$.

Finally we remark that the methods of this paper may be used to solve the equations mentioned above in the exceptional cases.

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带复平移的奇异积分方程组*

摘　要

本文讨论了在实轴上带复平移的奇异积分方程组，包括含单个平移和两个平移的情况，给出了可解的充分条件和解的级数形式，并将其应用于带未知函数共轭和复平移的奇异积分方程.

我们在[1]中曾讨论过带复平移的奇异积分方程，给出了唯一可解的充分条件以及解的级数表示. 本文将把这些结果推广到方程组的情形，其中可以含有一个复平移，也可允许其中含有两个平移(可以都是向上平移，也可以一个向上一个向下). 最后还将它们应用于求解既带复平移又带未知函数的共轭的奇异积分方程.

(一) 算子 T^{α} 及其性质

我们将考虑实轴上的函数类 $\hat{H}$ (记号见[2])；同时还考虑函数类 $\hat{H}_0$，$f(x)\in\hat{H}_0$ 表示 $f(x)\in\hat{H}$ 且 $f(\pm\infty)=0$. 一个 N 维函数向量 $f\in\hat{H}$ 或 $\hat{H}_0$ 意指其每一分量都有此性质. 设 $f\in\hat{H}$ 或 $\hat{H}_0$ 是一个 N 维函数向量，除和通常那样把奇异积分算子

$$Tf(x)=\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(t)}{t-x}dt,\quad x\in\mathbf{R} \tag{1.1}$$

外，还引进算子

$$T^{\alpha}f(x)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(t)}{t-x-\alpha}dt,\quad x\in\mathbf{R},\ \mathrm{Im}\,\alpha\neq 0. \tag{1.2}$$

当然积分(1.1)在 $t=x$ 处以及无穷远处都要理解为 Cauchy 主值，而积分(1.2)一般说来在无穷远处也要理解为主值，只是当 $f(t)\in\hat{H}_0$ 时它便成为通常的收敛积分.

算子 T^{α} 有下列性质(以后 A,B,C 等表 N 阶方阵，I 为 N 阶单位方阵)：

1°
$$TT^{\alpha}=\begin{cases}T^{\alpha}, & 当\ \mathrm{Im}\alpha>0\ 时；\\ -T^{\alpha}, & 当\ \mathrm{Im}\alpha<0\ 时.\end{cases}$$

2°
$$T^{\alpha}T^{\beta}=\begin{cases}T^{\alpha+\beta}, & 当\ \mathrm{Im}\alpha>0,\ \mathrm{Im}\beta>0\ 时；\\ -T^{\alpha+\beta}, & 当\ \mathrm{Im}\alpha<0,\ \mathrm{Im}\beta<0\ 时；\\ 0, & 当\ \mathrm{Im}\alpha,\ \mathrm{Im}\beta\ 异号时.\end{cases}$$

作为 2°的推论，有

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3° $$T^{\alpha}T^{\beta}=T^{\beta}T^{\alpha},\quad T^{\alpha}T^{-\alpha}=0,\quad T^{\alpha}T^{\bar{\alpha}}=0,$$

$$(T^{\alpha})^{n}=\begin{cases}T^{n\alpha}, & \text{当 } \operatorname{Im}\alpha>0 \text{ 时；}\\ (-1)^{n-1}T^{n\alpha}, & \text{当 } \operatorname{Im}\alpha<0 \text{ 时.}\end{cases}$$

4° $$(CT^{\alpha})f=T^{\alpha}(Cf),\ (CT)f=T(Cf).$$

5° 如 k 为一 N 维常向量，则 $Tk=0$，

$$T^{\alpha}k=\begin{cases}\dfrac{1}{2}k, & \text{当 } \operatorname{Im}\alpha>0 \text{ 时，}\\ -\dfrac{1}{2}k, & \text{当 } \operatorname{Im}\alpha<0 \text{ 时，}\end{cases}$$

6° $$(T^{\alpha}f)(\pm\infty)=\begin{cases}\dfrac{1}{2}f(\pm\infty), & \text{当 } \operatorname{Im}\alpha>0 \text{ 时，}\\ -\dfrac{1}{2}f(\pm\infty), & \text{当 } \operatorname{Im}\alpha<0 \text{ 时.}\end{cases}$$

特别地，

7° 如果 $f\in\hat{H}_0$，则 $(T^{\alpha}f)(\pm\infty)=0$.

8° 如果 $f\in\hat{H}_0$，则 $(T^{n\alpha}f)(x)$ 当 $n\to\infty$ 时对 x 一致地趋于零.

所有这些性质，都很容易证明. 例如 6°，只要作变换 $\zeta=1/z$，然后应用 Plemelj 公式即可证得.

（二） 单个平移的情形

本节讨论奇异积分方程(组)：

$$(AI+BT+CT^{\alpha})\varphi=f,\quad \operatorname{Im}\alpha\neq 0,\tag{2.1}$$

其中 A,B,C 均为 N 阶已知常数方阵，而已知的 f 和未知的 φ 均为 N 维向量，且 $\in\hat{H}$.

如果 $f_{\infty}=f(\pm\infty)\neq 0$，且(2.1)有解，并记 $\varphi_{\infty}=\varphi(\pm\infty)$，则必

$$\left(A-\frac{1}{2}C\right)\varphi_{\infty}=f_{\infty}.\tag{2.2}$$

如果 $A-\dfrac{1}{2}C$ 满秩，则有

$$\varphi_{\infty}=\left(A-\frac{1}{2}C\right)^{-1}f_{\infty}.\tag{2.3}$$

因此，如记 $\varphi^{*}=\varphi-\varphi_{\infty}$，$f^{*}=f-f_{\infty}$，则方程立即可化归 $f_{\infty}=\varphi_{\infty}=0$ 的情形，亦即 $f,\varphi\in\hat{H}_0$ 的情形. 如上述方阵奇异，立即可找出(2.2)可解的充要条件并求出其一般解来，从而方程(2.1)仍可化归 $f,\varphi\in\hat{H}_0$ 的情形；而若这些条件不满足，则(2.2)无解，因而(2.1)也无解. 总之，问题总可化归 $f,\varphi\in\hat{H}_0$ 的情形. 我们以后将永远作此假定.

记

$$\mathscr{S}=A+B,\quad \mathscr{D}=A-B.\tag{2.4}$$

本文将只讨论正则型情况，即恒假定 $\det\mathscr{S}\neq 0$，$\det\mathscr{D}\neq 0$. 这时，问题的总指标和各偏指标都是零[2].

又记

$$A^{*}=\frac{1}{2}(\mathscr{S}^{-1}+\mathscr{D}^{-1}),\quad B^{*}=\frac{1}{2}(\mathscr{S}^{-1}-\mathscr{D}^{-1}).\tag{2.5}$$

熟知，方程$(AI+BT)\varphi=g$ $(g\in\hat{H}_0)$的唯一解为

$$\varphi=(A^{*}I+B^{*}T)g.$$

将(2.1)改写为$(AI+BT)\varphi=f-CT^{\alpha}\varphi$，立即可知它等价于

$$\varphi=(A^{*}I+B^{*}T)(f-CT^{\alpha}\varphi).$$

我们先设$\operatorname{Im}\alpha>0$. 这时$T^{\alpha}$为一向上平移.

如再记

$$f^{*}=(A^{*}I+B^{*}T)f,\tag{2.6}$$

$$C^{*}=(A^{*}+B^{*})C=\mathscr{S}^{-1}C,\tag{2.7}$$

则注意到性质1°，从上方程可写为

$$\varphi=f^{*}-C^{*}T^{\alpha}\varphi,\tag{2.8}$$

且由性质7°知，$f^{*}\in\hat{H}_0$.

对(2.8)作迭代，并注意到性质3°和4°，形式地可得

$$\varphi=f^{*}+\sum_{n=1}^{\infty}(-1)^{n}C^{*n}T^{n\alpha}f^{*}.\tag{2.9}$$

如对任意N阶方阵$c=(c_{jk})$，取按行范数[8]

$$\|c\|=\max_{j}\sum_{k=1}^{n}|c_{jk}|,\tag{2.10}$$

则由性质8°知，当

$$\|C^{*}\|=\|\mathscr{S}^{-1}C\|<1\tag{2.11}$$

时(2.9)一致收敛. (2.11)还可改为

$$\|C\|<1/\|\mathscr{S}^{-1}\|<1\ \text{或者}\ \|C\|<\|\mathscr{S}\|.\tag{2.12}$$

于是，当(2.11)或(2.12)成立时，(2.9)的确是原方程(组)的解.

如果$\operatorname{Im}\alpha<0$，即方程(2.1)中带有一个向下平移时，可类似上述讨论而得出(2.1)的解

$$\varphi=f^{*}-\sum_{n=1}^{\infty}D^{*n}T^{n\alpha}f^{*},\tag{2.13}$$

这里已记

$$D^{*}=(A^{*}-B^{*})C=\mathscr{D}^{-1}C;\tag{2.14}$$

而这里收敛条件为

$$\|D^{*}\|=\|\mathscr{D}^{-1}C\|<1,\tag{2.15}$$

或改为

$$\|C\|<1/\|\mathscr{D}^{-1}\|<1\ \text{或者}\ \|C\|<\|\mathscr{D}\|.\tag{2.16}$$

（三） 两个平移的情形

本节考虑奇异积分方程(组)中含有两个复平移的情形：

$$(AI+BT+CT^{\alpha}+DT^{\beta})\varphi=f,\tag{3.1}$$

其中 D 也是 N 阶常数方阵，$\mathrm{Im}\alpha$，$\mathrm{Im}\beta$ 均不为零. $\mathscr{S},\mathscr{D},A^*,B^*$ 记号仍同前，并仍限于正则型情况.

先设 $\mathrm{Im}\alpha$ 和 $\mathrm{Im}\beta$ 异号. 不妨设 $\mathrm{Im}\alpha>0$，$\mathrm{Im}\beta<0$，仍按(二)中方法，易知(3.1)等价于

$$\varphi=f^*-(C^*T^\alpha+D^*T^\beta)\varphi,$$

其中 f^*,C^* 仍如前，而

$$D^*=(A^*-B^*)D=\mathscr{D}^{-1}D. \tag{3.2}$$

经过迭代，并注意性质 2°，形式地可得

$$\varphi=f^*+\sum_{n=1}^{\infty}(-1)^n C^{*n}T^{n\alpha}f^*-\sum_{n=1}^{\infty}D^{*n}T^{n\beta}f^*. \tag{3.3}$$

易见，当(2.11)，(2.15)或者(2.12)，(2.16)同时成立时，(3.3)确为(3.1)的解.

次设 $\mathrm{Im}\alpha$，$\mathrm{Im}\beta$ 同号，例如不妨设它们都大于零. 这时，仍如前迭代，则有

$$\begin{aligned}\varphi&=f^*+\sum_{n=1}^{\infty}(-1)^n(C^*T^\alpha+D^*T^\beta)^n f^*\\&=f^*+\sum_{n=1}^{\infty}(-1)^n\sum_{r+s=0}^{n}\binom{n}{r}C^{*r}D^{*s}T^{r\alpha+s\beta}f^*\end{aligned} \tag{3.4}$$

(已记 $T^0=I$)，其中 C^* 仍由(2.7)给出；而

$$D^*=(A^*+B^*)D=\mathscr{S}^{-1}D; \tag{3.5}$$

且当

$$\|C^*\|+\|D^*\|=\|\mathscr{S}^{-1}C\|+\|\mathscr{S}^{-1}D\|<1 \tag{3.6}$$

或改为

$$\|C\|+\|D\|<1/\|\mathscr{S}^{-1}\| \text{ 或者 } \|C\|+\|D\|<\|\mathscr{S}\| \tag{3.7}$$

时，(3.4)一致收敛，从而确实是(3.1)的解.

当方程(组)中含有更多个复平移时，本节方法原则上完全适用，当然解的形式和收敛条件会相应复杂些.

(四)　应用于带共轭的方程

本节将把上面结果，应用于奇异积分方程中既带复平移又带未知函数共轭的方程.

首先，我们来求解方程

$$\begin{aligned}&a\psi(x)+b\overline{\psi(x)}+\frac{c}{\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\psi(t)}{t-x}\mathrm{d}t+\frac{d}{\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\overline{\psi(t)}}{t-x}\mathrm{d}t+\frac{e}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\psi(t)}{t-x-\alpha}\mathrm{d}t\\&\quad+\frac{f}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\overline{\psi(t)}}{t-x-\alpha}\mathrm{d}t=h(x),\quad x\in\mathbf{R},\ \mathrm{Im}\alpha>0,\end{aligned} \tag{4.1}$$

其中 $a,b,\cdots,f$ 均为常数，而 ψ,h 均为 $\in\hat{H}$ 的函数. 仍记 $\psi(\pm\infty)=\psi_\infty$ 等等. 易见，如(4.1)有解，则必

$$\psi_\infty=\left[(\bar{a}+\frac{1}{2}\bar{e})h_\infty-(b+\frac{1}{2}f)\bar{h}_\infty\right]\Big/\left[|a+\frac{1}{2}c|^2|b+\frac{1}{2}f|^2\right]. \tag{4.2}$$

因此，仍不妨设 $\psi_\infty=h_\infty=0$，亦即 $\psi,h\in\hat{H}_0$.

将(4.1)两端取共扼后，与(4.1)一起可写成方程组形式

$$(AI+BT+CT^{\alpha}+DT^{\bar{\alpha}})\varphi=g, \tag{4.3}$$

其中已令

$$\varphi=\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\quad g=\begin{pmatrix}h\\ \bar{h}\end{pmatrix}=\begin{pmatrix}g_1\\ g_2\end{pmatrix}, \tag{4.4}$$

$$A=\begin{pmatrix}a & b\\ \bar{b} & \bar{a}\end{pmatrix},\ B=\begin{pmatrix}c & d\\ -\bar{d} & -\bar{c}\end{pmatrix},\ C=\begin{pmatrix}e & f\\ 0 & 0\end{pmatrix},\ D=\begin{pmatrix}0 & 0\\ -\bar{f} & -\bar{e}\end{pmatrix} \tag{4.5}$$

这是(三)中讨论过的带一个向上平移 α 和一个向下平移 $\bar{\alpha}$ 的方程组($N=2$). 只要求出其解 φ 满足

$$\bar{\varphi}_2=\varphi_1 \tag{4.6}$$

者，则 $\psi=\varphi_1$ 便是(4.1)的解；反之，如 ψ 为(4.1)的解，则 φ 必是满足(4.6)的方程(4.1)的解.

按照(三)中记号，现在

$$\mathscr{S}=\begin{bmatrix}a+c & b+d\\ \bar{b}-\bar{d} & \bar{a}-\bar{c}\end{bmatrix},\quad \mathscr{D}=\begin{bmatrix}a-c & b-d\\ \bar{b}+\bar{d} & \bar{a}+\bar{c}\end{bmatrix}, \tag{4.7}$$

因此，

$$\begin{aligned}&\Delta=\det\mathscr{S}=|a|^2-|b|^2-|c|^2+|d|^2+2\mathrm{i}\,\mathrm{Im}(\bar{a}c-\bar{b}d),\\ &\bar{\Delta}=\det\mathscr{D}.\end{aligned} \tag{4.8}$$

我们仍只考虑正则型情况，即设 $\Delta\neq0$.

现在

$$\mathscr{S}^{-1}=\begin{pmatrix}\bar{a}-\bar{c} & -(b+d)\\ -(\bar{b}-\bar{d}) & a+c\end{pmatrix}\Big/\Delta,\quad \mathscr{D}^{-1}=\begin{bmatrix}\bar{a}+\bar{c} & b-d\\ \bar{b}+\bar{d} & a-c\end{bmatrix}\Big/\bar{\Delta}; \tag{4.9}$$

$$C^{*}=\begin{bmatrix}e(\bar{a}-\bar{c}) & f(\bar{a}-\bar{c})\\ -e(\bar{b}-\bar{d}) & -f(\bar{b}-\bar{d})\end{bmatrix}\Big/\Delta,\quad D^{*}=\begin{bmatrix}\bar{f}(b-d) & \bar{e}(b-d)\\ -\bar{f}(a-c) & -\bar{e}(a-c)\end{bmatrix}\Big/\bar{\Delta}. \tag{4.10}$$

于是由(3.3)知，(4.4)的形式解为

$$\varphi=g^{*}+\sum_{n=1}^{\infty}(-1)^nC^{*n}T^{n\alpha}g^{*}-\sum_{n=1}^{\infty}D^{*n}T^{n\bar{\alpha}}g^{*}, \tag{4.11}$$

其中

$$g^{*}=(A^{*}I+B^{*}T)g. \tag{4.12}$$

收敛条件(2.12),(2.16)现在可化为同一条件

$$(|e|+|f|)\max\{|a-c|,|b-d|\}<|\Delta|. \tag{4.13}$$

现在我们将证实(4.6)确乎成立. 为此，我们引进倒置矩阵(包括倒置向量)的概念. 设 M 为一矩阵，将其各行次序颠倒，同时又将每一行中元素的次序也颠倒，所得矩阵称为 M 的倒置矩阵，记为 M^{Λ}. 例如，若 M 为一方阵，而[4]

$$V=\begin{bmatrix}0 & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0\end{bmatrix},$$

则

$$M^{\Lambda}=VMV.$$

因此，若φ为一列向量，则φ^{Λ}仍为一列向量，但其分量次序与φ的完全颠倒. 容易验证，如M,N为任意矩阵，则

$$(M^{\Lambda})^{\Lambda}=M,\ \overline{M}^{\Lambda}=\overline{M^{\Lambda}},\ (MN)^{\Lambda}=M^{\Lambda}N^{\Lambda},\ (M^{n})^{\Lambda}=(M^{\Lambda})^{n},\ (M^{-1})^{\Lambda}=(M^{\Lambda})^{-1},$$

只要它们有意义. 又若X为方程$MX=Y$的解，则X^{Λ}为$M^{\Lambda}X^{\Lambda}=Y^{\Lambda}$的解. 这些都极易验证的.

由(4.7),(4.9)易见，

$$\overline{\mathscr{D}}^{\Lambda}=\mathscr{S},\quad \overline{D^{*}}^{\Lambda}=-C^{*}. \tag{4.14}$$

注意到由于$\overline{g}^{\Lambda}=g$，$\overline{A}^{\Lambda}=A$，$\overline{B}^{\Lambda}=-B$，故由(4.12)知$\overline{g}^{*\Lambda}=g^{*}$. 再注意到$\overline{T^{\alpha}g^{*}}^{\Lambda}=-T^{\bar{\alpha}}g^{*}$，所以由(4.11)立即可推得$\overline{\varphi}^{\Lambda}=\varphi$，亦即(4.6)的确成立.

因此，当条件(4.13)满足时，写出(4.11)的第一个分量$\varphi_1=\psi$即为所求解：

$$\psi=g_1^{*}+\sum_{n=1}^{\infty}(-1)^{n}(C_{11}^{*n}T^{n\alpha}g_1^{*}+C_{12}^{*n}T^{n\alpha}g_2^{*})-\sum_{n=1}^{\infty}D_{11}^{*n}T^{n\bar{\alpha}}g_1^{*}+D_{12}^{*n}T^{n\bar{\alpha}}g_2^{*}). \tag{4.15}$$

由(4.9)，现在

$$A^{*}=\frac{1}{2}\begin{pmatrix}\dfrac{\bar{a}-\bar{c}}{\Delta}+\dfrac{\bar{a}+\bar{c}}{\overline{\Delta}} & -\dfrac{b+d}{\Delta}-\dfrac{b-d}{\overline{\Delta}}\\ -\dfrac{\bar{b}-\bar{d}}{\Delta}-\dfrac{\bar{b}+\bar{d}}{\overline{\Delta}} & \dfrac{a+c}{\Delta}+\dfrac{a-c}{\overline{\Delta}}\end{pmatrix},$$

$$B^{*}=\frac{1}{2}\begin{pmatrix}\dfrac{\bar{a}-\bar{c}}{\Delta}-\dfrac{\bar{a}+\bar{c}}{\overline{\Delta}} & -\dfrac{b+d}{\Delta}+\dfrac{b-d}{\overline{\Delta}}\\ -\dfrac{\bar{b}-\bar{d}}{\Delta}+\dfrac{\bar{b}+\bar{d}}{\overline{\Delta}} & \dfrac{a+c}{\Delta}-\dfrac{a-c}{\overline{\Delta}}\end{pmatrix}.$$

这样，如果记

$$h^{*}=[h(\bar{a}-\bar{c})-\overline{h}(b+d)]/\Delta, \tag{4.16}$$

则由(4.12)知，$g_1^{*}=h^{*}, g_2^{*}=\overline{h}^{*}$. 又由于$(\overline{D}^{*\Lambda})^{n}=(-1)^{n}C^{*n}$，故$(-1)^{n}C_{12}^{*n}=\overline{D}_{12}^{*n}$，$(-1)^{n}C_{11}^{*n}=\overline{D}_{22}^{*n}$. 因此方程(4.1)的解(4.15)最后可写为

$$\psi=h^{*}+\sum_{n=1}^{\infty}(\overline{D}_{22}^{*n}T^{n\alpha}h^{*}+\overline{D}_{21}^{*n}T^{n\alpha}\overline{h}^{*}-D_{11}^{*n}T^{n\bar{\alpha}}h^{*}-D_{12}^{*n}T^{n\bar{\alpha}}\overline{h}^{*}), \tag{4.17}$$

其中D_{jk}^{*n}为$D^{*n}=(D_{jk}^{*n})$的元，而D^{*}由(4.10)给出，h^{*}由(4.16)给出.

我们还可考虑另一种方程

$$a\psi(x)+b\,\overline{\psi(x)}+\frac{c}{\pi i}\int_{-\infty}^{+\infty}\frac{\psi(t)}{t-x}dt+\frac{d}{\pi i}\int_{-\infty}^{+\infty}\frac{\overline{\psi(t)}}{t-x}dt+\frac{e}{2\pi i}\int_{-\infty}^{+\infty}\frac{\psi(t)}{t-x-\alpha}dt$$
$$+\frac{f}{2\pi i}\int_{-\infty}^{+\infty}\frac{\overline{\psi(t)}}{t-x-\bar{\alpha}}dt=h(x),\quad x\in\mathbf{R},\ \mathrm{Im}\alpha>0. \tag{4.18}$$

仍用前面记号. 如$h_{\infty}\neq0$，则代替(4.2)有

$$\psi_{\infty}=\left[(\bar{a}+\frac{1}{2}\bar{e})h_{\infty}-(b-\frac{1}{2}f)\overline{h}_{\infty}\right]\Big/\left[|a+\frac{e}{2}|^{2}|b-\frac{f}{2}|^{2}\right]. \tag{4.19}$$

于是仍可不妨设$\psi,h\in\hat{H}_0$. 仍如前讨论，可得：(4.18)等价于方程(4.4)且要求(4.6)成立. 这时，A,B因而$\mathscr{S},\mathscr{D}$均同前，但

$$C=\begin{pmatrix}e & 0\\ -\bar{f} & 0\end{pmatrix},\quad D=\begin{pmatrix}0 & f\\ 0 & -\bar{e}\end{pmatrix}. \tag{4.20}$$

因此$\overline{C}^{\Lambda}=-D$仍成立，而

$$C^*=\begin{pmatrix} C_1^* & 0 \\ C_2^* & 0 \end{pmatrix}\Big/\Delta,\quad D^*=\begin{pmatrix} 0 & D_1^* \\ 0 & D_2^* \end{pmatrix}\Big/\overline{\Delta}, \tag{4.21}$$

其中

$$D_1^*=f(\overline{a}+\overline{c})+\overline{e}(b-d),\quad D_2^*=-f(\overline{b}+\overline{d})-\overline{e}(a-c), \tag{4.22}$$
$$C_1^*=-\overline{D}_2^*,\quad C_2^*=-\overline{D}_1^*;$$

且易见

$$C_1^{*n}=\begin{pmatrix} C^{*n} & 0 \\ C_1^{*n-1}C_2 & 0 \end{pmatrix}\Big/\Delta^n,\quad D^{*n}=\begin{pmatrix} 0 & D_1^*D_2^{*n-1} \\ 0 & D_2^{*n} \end{pmatrix}\Big/\overline{\Delta}^n, \tag{4.23}$$

所以，解(4.17)现在可写为

$$\psi=h^*+\sum_{n=1}^{\infty}\{\overline{D}_2^{*n}T^{n\alpha}\overline{h}^*/\Delta^n-D_1^*D_2^{*n-1}T^{n\overline{\alpha}}h^*/\overline{\Delta}^n\}, \tag{4.24}$$

而其收敛条件 $\|C^*\|<1$，$\|D^*\|<1$ 现在是

$$\max\{|e(\overline{a}-\overline{c})+\overline{f}(b+d)|,\ |e(\overline{b}-\overline{d})+\overline{f}(a+c)|\}<|\Delta|. \tag{4.25}$$

以上方法不难应用于方程中既含 $T^\alpha\psi$, $T^{\overline{\alpha}}\psi$, 又含 $T^{\overline{\alpha}}\psi$, $T^{\overline{\alpha}}\psi$ 的项；由于仍有 $\overline{C}^\Delta=-D$，所以论证不会有新的困难，只是结果较复杂.

如方程中含有不止一个平移 α，也照样可以讨论.

又，带共轭的方程组照样也可讨论，这些在原则上也无特殊困难.

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ON SYSTEMS OF SINGULAR INTEGRAL EQUATIONS WITH COMPLEX TRANSLATIONS

Abstract

Systems of singular integral equations along the real axis with a single translation or two translations are discussed. Sufficient conditions of solvability are obtained and the solutions are expressed in terms of series. The results are applied to solve singular integral equations with a translation when the conjugate of the unknown function also appears.

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CONVOLUTION EQUATIONS WITH REFLECTION AND TRANSLATION SHIFT*

Abstract. In this paper, four kinds of integral equations of convolution type are solved, in which the reflection occurs, that is, besides the unknown $f(t)$, $f(-t)$ is also appeared. Moreover, it is mentioned that the methods of solution for two of them are still effective when translation shifts, i.e., $f(t+\lambda_j)$ or/and $f(-t-\mu_j)$, occur in addition.

Key words. convolution equation, Riemann boundary value problem, singular integral equation, reflection, translation shift.

1. Introduction

It is well-known that convolution equations are closely related to Riemann boundary value problems or singular integral equations.[1] When there occurs reflection in convolution equations, in the related boundary value problems or integral equations reflection will also occur.

The method of solution for some kinds of such equations will be discussed in this paper. The method here used is also effective for certain kinds of such equations with a system of translation shifts and reversed ones.

The variable of functions appeared in this paper is taken on the real axis while their values are complex.

The Fourier transform of a function $f(t)\in L_2(-\infty,+\infty)$ will be denoted by $F(x)$:

$$F(x)=V\{f(t)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)e^{ixt}dt \tag{1.1}$$

(that of $g(t)$ by $G(x)$, etc.). Then we have

$$V\{f(-t)\}=V^{-1}\{f(t)\}=F(-x). \tag{1.2}$$

The Fourier transform of the convolution of $k(t)$ and $f(t)$

$$k*f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k(t-s)f(s)ds \tag{1.3}$$

is

$$V\{k*f\}=Vk\cdot Vf=KF. \tag{1.4}$$

A typical but simple equation of the above mentioned type is

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$$Af(t)+Bf(-t)+\frac{C}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k(t-s)f(s)\mathrm{d}s+\frac{D}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h(t-s)f(-s)\mathrm{d}s$$
$$=g(t),\quad -\infty<t<+\infty, \tag{1.5}$$

where A,B,C,D are constants and all the functions appeared in it belong to class $\{0\}$ which means that their Fourier transforms belong to class $\{\{0\}\}\in L_2(-\infty,+\infty)\cap H$, H being the class of Hölder continuous functions (for notation, cf. [1]).

By taking Fourier transform, (1.5) is readily reduced to

$$[A+CK(x)]F(x)+[B+DH(x)]F(-x)=G(x). \tag{1.6}$$

We assume

$$\Delta(x)=\begin{vmatrix} A+CK(x) & B+DH(x) \\ B+DH(-x) & A+CK(-x) \end{vmatrix}\neq 0 \tag{1.7}$$

(including $\Delta(\pm\infty)=0$, i. e., $A^2-B^2\neq 0$), which is called the normal case of (1.5). Then, we obtain, by (1.6),

$$F(x)=\frac{1}{\Delta(x)}\begin{vmatrix} G(x) & B+DH(x) \\ G(-x) & A+CK(-x) \end{vmatrix}. \tag{1.8}$$

Since $1/\Delta(x)$ is bounded on the whole real axis, so $F(x)\in H$ and hence $F(x)\in\{\{0\}\}$. Thus, (1.5) has the unique solution $f(t)=V^{-1}\{F(x)\}$ in $\{0\}$.

2. Equations with Two Pairs of Kernels

Consider the equation

$$Af(t)+Bf(-t)+\frac{C_1}{\sqrt{2\pi}}\int_0^{+\infty}k_1(t-s)f(s)\mathrm{d}s+\frac{C_2}{\sqrt{2\pi}}\int_{-\infty}^0 k_2(t-s)f(s)\mathrm{d}s$$
$$+\frac{D_1}{\sqrt{2\pi}}\int_0^{+\infty}h_1(t-s)f(-s)\mathrm{d}s+\frac{D_2}{\sqrt{2\pi}}\int_{-\infty}^0 h_2(t-s)f(-s)\mathrm{d}s$$
$$=g(t),\quad -\infty<t<+\infty, \tag{2.1}$$

where A,B,C_j,D_j $(j=1,2)$ are constants and all the functions appeared belong to $\{0\}$.

Taking Fourier transform, we reduce (2.1) to

$$AF(x)+BF(-x)+C_1K_1(x)F^+(x)-C_2K_2(x)F^-(x)$$
$$+D_1H_1(x)F^+(-x)-D_2H_2(x)F^-(-x)=G(x), \tag{2.2}$$

where $F^\pm(x)$are boundary values of the Cauchy type integral

$$F(z)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{F(x)}{x-z}\mathrm{d}x,\quad \mathrm{Im}z\neq 0, \tag{2.3}$$

and it is well-known that $F^\pm(x)$ are the one-sided Fourier transforms of $f(t)$:

$$F^+(x)=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}f(t)\mathrm{e}^{\mathrm{i}xt}\mathrm{d}t,\quad F^-(x)=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 f(t)\mathrm{e}^{\mathrm{i}xt}\mathrm{d}t. \tag{2.4}$$

It is evident

$$F(x)=F^+(x)-F^-(x) \tag{2.5}$$

so that (2.2) may be written as

$$[A+C_1K_1(x)]F^+(x)-[A+C_2K_2(x)]F^-(x)+[B+D_1H_1(x)]F^+(-x)$$
$$-[B+D_2H_2(x)]F^-(-x)=G(x). \tag{2.6}$$

This is a Riemann boundary value problem with reflection.

Note that $F(\pm\infty)=0$ are required since $F\in\{\{0\}\}$, or, by using

$$\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{F(x)}{x-t}dx=F^+(t)+F^-(t), \tag{2.7}$$

(2.2) may be also reduced to a singular integral equation with reflection:

$$a(x)F(x)+b(x)F(-x)+\frac{c(x)}{\pi i}\int_{-\infty}^{+\infty}\frac{F(t)}{t-x}dt-\frac{d(x)}{\pi i}\int_{-\infty}^{+\infty}\frac{F(-t)}{t-x}dx=2G(x), \tag{2.8}$$

where

$$\begin{aligned} a(x)&=2A+C_1K_1(x)+C_2K_2(x),\\ b(x)&=2B+D_1H_1(x)+D_2H_2(x),\\ c(x)&=C_1K_1(x)-C_2K_2(x),\\ d(x)&=D_1H_1(x)-D_2H_2(x). \end{aligned} \tag{2.9}$$

To solve (2.3), we replace x by $-x$ in it and let $\Omega(x)=(F(x),F(-x))'$. Then, a system of singular integral equations of dimension 2 in class $\{\{0\}\}$ is obtained:

$$\begin{pmatrix} a(x) & b(x)\\ b(-x) & a(-x)\end{pmatrix}\Omega(x)+\begin{pmatrix} c(x) & -d(x)\\ d(-x) & -c(-x)\end{pmatrix}\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\Omega(t)}{t-x}dt=2\begin{pmatrix} G(x)\\ G(-x)\end{pmatrix}. \tag{2.10}$$

Similar to the discussion in § 6.2 of [2], we know that (2.8) is solvable iff (2.10) is solvable and its solution may be obtained after the latter is solved. In fact, assume $\Omega(x)=(\Omega_1(x),\Omega_2(x))'$ is a solution of (2.10). Then, it is easily verified that $\Omega_*(x)=(\Omega_2(-x),\Omega_1(-x))'$ is also a solution of it and so does $[\Omega(x)+\Omega_*(x)]/2$. Hence

$$F(x)=\frac{1}{2}[\Omega_1(x)+\Omega_2(-x)] \tag{2.11}$$

is a solution of (2.6). Finally, $f=V^{-1}F$ is a solution of (2.1).

(2.10) is a characteristic singular integral equation and its method of solution as well as the condition of its solvability is well-known as shown in [3] in case of normal type: assuming

$$\begin{vmatrix} a(x)\pm c(x) & b(x)\mp d(x)\\ b(-x)\pm d(-x) & a(-x)\mp c(-x)\end{vmatrix}\neq 0$$

all over the real axis (including $x=\pm\infty$, i.e., $A^2-B^2\neq 0$). The details will be omitted.

3. Dual Equation

Let us now consider the following dual equation with reflection:

$$A_1\varphi(t)+B_1\varphi(-t)+\frac{C_1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_1(t-s)\varphi(s)ds+\frac{D_1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h_1(t-s)\varphi(-s)ds$$
$$=g(t),\quad 0<t<+\infty,$$

$$A_2\varphi(t)+B_2\varphi(-t)+\frac{C_2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_2(t-s)\varphi(s)\mathrm{d}s+\frac{D_2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h_2(t-s)\varphi(-s)\mathrm{d}s$$
$$=g(t),\quad -\infty<t<0. \tag{3.1}$$

Extending t in the first equation of (3.1) to $-\infty<t<0$, we get an equation

$$A_1\varphi(t)+B_1\varphi(-t)+\frac{C_1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_1(t-s)\varphi(s)\mathrm{d}s+\frac{D_1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h_1(t-s)\varphi(-s)\mathrm{d}s$$
$$=g(t)+f_-(t),\quad -\infty<t<+\infty,$$

where $f_-(t)$ is an unknown function in $\{0\}$ with $f_-(t)=0$ when $0<t<+\infty$. Similarly, extending t in the second one of (2.1) to $0<t<+\infty$, we have

$$A_2\varphi(t)+B_2\varphi(-t)+\frac{C_2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_2(t-s)\varphi(s)\mathrm{d}s+\frac{D_2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h_2(t-s)\varphi(-s)\mathrm{d}s$$
$$=g(t)+f_+(t),\quad -\infty<t<+\infty,$$

where $f_+(t)$ is an unknown function in $\{0\}$ with $f_+(t)=0$ when $-\infty<t<0$.

Taking Fourier transform in both obtained equations and denoting $F^+=Vf_+, F^-=-Vf_-$, we obtain

$$\begin{aligned}&[A_1+C_1K_1(x)]\Phi(x)+[B_1+D_1H_1(x)]\Phi(-x)=G(x)+F^-(x),\\&[A_2+C_2K_2(x)]\Phi(x)+[B_2+D_2H_2(x)]\Phi(-x)=G(x)+F^+(x).\end{aligned} \tag{3.2}$$

Note that $F^{\pm}(x)$ are boundary values of the Cauchy type integral (2.3) where $F(x)$ is given by (2.5).

By eliminating $\Phi(x)$ in (3.2), it gives rise to

$$\begin{aligned}&\Delta(-x)[A_1+C_1K_1(x)]F^+(x)-\Delta(-x)[A_2+C_2K_2(x)]F^-(x)\\&\quad+\Delta(x)[B_1+D_1H_1(-x)]F^+(-x)-\Delta(x)[B_2+D_2H_2(-x)]F^-(-x)\\&=\Delta(-x)[A_1-A_2+C_1K_1(x)-C_2K_2(x)]G(x)\\&\quad-\Delta(x)[B_1-B_2+D_1H_1(-x)-D_2H_2(-x)]G(-x),\end{aligned} \tag{3.3}$$

where $\Delta(x)$ is the coefficient determinant of (3.2). This is a problem similar to (2.6), which may be discussed by using the same method as shown in § 2.

4. Wiener-Hopf Equation

Method mentioned above may be also used to solve the following Wiener-Hopf equation with reflection:

$$Af(x)+\frac{C}{\sqrt{2\pi}}\int_0^{+\infty}k(t-s)f(s)\mathrm{d}s+\frac{D}{\sqrt{2\pi}}\int_{-\infty}^{0}h(t-s)f(-s)\mathrm{d}s$$
$$=g(t),\quad 0<t<+\infty, \tag{4.1}$$

where $f(t), g(t), h(t)\in L^2(0,+\infty)$ and their one-sided Fourier transforms belong to Hölder continuous class on $[0,+\infty]$.

By extending t to $-\infty<t<0$ and denoting $f(t)=f_+(t)$, $g(t)=g_+(t)$, $h_+(t)=h(t)$, (4.1) may be written as

$$A f_+(t)+\frac{C}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k(t-s)f_+(s)\mathrm{d}s+\frac{D}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h_+(t-s)f_+(-s)\mathrm{d}s$$
$$=g_+(t)+f_-(t),\quad -\infty<t<+\infty, \tag{4.2}$$

where $f_-(t)$ is unknown with $f(t)=0$ when $-\infty<t<0$.

Taking Fourier transform in both sides of (4.2), we obtain

$$[A+CK(x)]F^+(x)+D H^+(x)F^+(-x)=G^+(x)-F^-(x), \tag{4.3}$$

which is again a problem similar to (2.6). Further discussions will be omitted also.

5. Comments

The method of solution used above is also in effect for equations similar to (1.6) and (3.1) with a finite set of translation shifts and reversed ones, namely,

$$\sum_{j=1}^{n}\left\{A_j f(t+\lambda_j)+\frac{B_j}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_j(t-s)f(s+\lambda_j)\mathrm{d}s+C_j f(-t-\mu_j)\right.$$
$$\left.+\frac{D_j}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h_j(t-s)f(-s-\mu_j)\mathrm{d}s\right\}=g(t),\quad -\infty<t<+\infty, \tag{5.1}$$

and

$$\begin{aligned}&\sum_{j=1}^{n}\left\{A_j^1 f(t+\lambda_j)+\frac{B_j^1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_j^1(t-s)f(s+\lambda_j)\mathrm{d}s+C_j^1 f(-t-\mu_j)\right.\\&\qquad\left.+\frac{D_j^1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h_j^1(t-s)f(-s-\mu_j)\mathrm{d}s\right\}=g(t),\quad 0<t<+\infty,\\&\sum_{j=1}^{n}\left\{A_j^2 f(t+\lambda_j)+\frac{B_j^2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_j^2(t-s)f(s+\lambda_j)\mathrm{d}s+C_j^2 f(-t-\mu_j)\right.\\&\qquad\left.+\frac{D_j^2}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h_j^2(t-s)f(-s-\mu_j)\mathrm{d}s\right\}=g(t),\quad -\infty<t<0,\end{aligned} \tag{5.2}$$

where λ_j, μ_j and all the coefficients appeared are constants.

But for equations similar to (2.1) or (4.1) of the same type, the above described method is not effective.

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Peano Derivatives and Singular Integrals of Arbitrary Order

Abstract. The Peano derivatives are introduced for functions along an arc in the complex plane. Singular integrals of arbitrary order with singularities at its end-points are defined so that a unified theory for such integrals and Cauchy principal value integrals is established.

Key words. Peano derivatives, singular integrals of arbitrary order, Cauchy principal value integrals.

0. Introduction

The finite part of a real integral was introduced by Hadamard in [1], which was extended to complex ones by Fox in [2]. From then on, there were many discussions about it, for instance in [3,4]. In recent years, it was extended further to the case where there exist singularities of higher order at the ends of path of integration[5,6]. All these investigations were discussed under the assumption that the higher derivatives involved satisfy the Hölder continuity.

The concept of Peano derivatives will be introduced below, which makes the class of functions appeared to be much wider and the condition subjected to the path of integration may be weakened. In the main time, the notion of the finite part of a limit is also proposed, which makes the discussions much clearer. Then a unified theory for Cauchy principal value integrals and singular integrals of any order in general is established. The results in [5,6] are special ones of this paper.

1. Peano Derivatives

Let $L=\widehat{ab}$ be an oriented open rectifiable continuous arc in the complex plane and $f(t)$ be a function defined on L. Define recurrently

$$\begin{aligned} &f^{[0]}(a)=f(a), \\ &f^{[k]}(a)=k!\lim_{t\to a}\Big[f(t)-\sum_{m=0}^{k-1}\frac{f^{[m]}(a)}{m!}(t-a)^m/(t-a)^k\Big] \quad (k=1,2,\dots) \end{aligned} \tag{1.1}$$

provided that the limit on the right side exists, and $f^{[k]}(a)$ is called the Peano derivative of k-th order of $f(t)$ at $t=a$.

Evidently, $f^{[1]}(a)=f'(a)$. Nevertheless, the existence of $f^{[k]}(a)$ $(k\geqslant 2)$ does not

imply the existence of the ordinary derivative $f^{(k)}(a)$ or those of lower order. For example, for

$$f(t)=(t-a)^2\varphi(t),\quad t\in L,$$

where $\varphi(t)$ is continuous but not differentiable at $t=a$, it is obvious that $f^{[2]}(a)=0$ but $f''(a)$ does not exist. However, if $f^{(k)}(a)$ exists, then it is easily seen that $f^{[k]}(a)=f^{(k)}(a)$.

If $f(t)$ has the n-th Peano derivative at $t=a$ and consequently has all the Peano derivatives of lower order there, then we say that $f(t)$ belongs to the Peano class of order n at a, denoted by $f(t)\in \mathrm{P}^n(a)$. Thus, in this case, we may write, by (1.1),

$$f(t)=\sum_{k=0}^{n}\frac{f^{[k]}(a)}{k!}(t-a)^k+r^n(f,a;t)(t-a)^n \tag{1.2}$$

where $r^n(f,a;t)\to 0$ as $t\to a$. In fact, (1.2) may be regarded as the definition of $f^{[k]}(a)$ too. (1.2) is called the Peano expansion of $f(t)$ at $t=a$.

For the relation between the Peano derivatives and the ordinary ones, we have

Lemma 1. *If $f'(t)\in \mathrm{P}^{n-1}(a)$, $n\geqslant 1$, then*

$$f(t)\in \mathrm{P}^n(a) \text{ and } f'^{[k]}(a)=f^{[k+1]}(a)\quad (k=0,1,\dots,n-1).$$

Proof. By assumption,

$$f'(t)=f'(a)+\sum_{k=1}^{n-1}\frac{f'^{[k]}(a)}{k!}(t-a)^k+r^{n-1}(f',a;t)(t-a)^{n-1}$$

with $r^{n-1}(f',a;t)\to 0$ as $t\to a$. Then, by integrating both sides from a to t along L, we obtain

$$f(t)=f(a)+\sum_{k=1}^{n}\frac{f'^{[k-1]}(a)}{k!}(t-a)^k+r^n(f,a;t)(t-a)^n \tag{$*$}$$

where

$$r^n(f,a;t)=\int_a^t r^{n-1}(f',a;\tau)(\tau-a)^{n-1}\mathrm{d}\tau\Big/(t-a)^n.$$

Therefore

$$\lim_{t\to a} r^n(f,a;t)=\lim_{t\to a}\frac{r^{n-1}(f',a;t)}{n}=0$$

i.e., $f(t)\in \mathrm{P}^n(a)$. $f^{[k]}(a)=f'^{[k-1]}(a)$ follows from ($*$).

If

$$\int_a^b \frac{r^n(f,a;t)}{t-a}\mathrm{d}t \tag{1.3}$$

converges, where $\int_a^b$ means the integral taken along the arc $L=\widehat{ab}$ from a to b, then $f(t)$ is said to belong to Dini class of order n, denoted by $f(t)\in \mathrm{D}^n(a)$. If

$$|r^n(f,a;t)|\leqslant A|t-a|^\mu,\quad 0<\mu\leqslant 1,\ t\in L, \tag{1.4}$$

where A and μ are constants, then denote $f(t)\in \mathrm{H}^n(a)$ (Hölder class of order n at $t=a$). Evidently, for smooth arc,

$$\mathrm{P}^n(a)\supset \mathrm{D}^n(a)\supset \mathrm{H}^n(a). \tag{1.5}$$

We may define analogously the Peano derivatives, Peano expansion and $r^n(f,b;t)$ of $f(t)$

at $t=b$ as well as classes $P^n(b)$, $D^n(b)$ and $H^n(b)$.

If c is an inner point of L, we may define the backward and the forward (one-sided) Peano derivatives of $f(t)$ of order k at $t=c$, denoted by $f_-^{[k]}(c)$ and $f_+^{[k]}(c)$ respectively. Thus, actually $f^{[k]}(a)=f_+^{[k]}(a)$ and $f_{[k]}(b)=f_-^{[k]}(b)$. If $f_+^{[k]}(c)=f_-^{[k]}(c)$, then their common value, denoted by $f^{[k]}(c)$, is called the (two-sided) k-th Peano derivative of $f(t)$ at $t=c$. The meanings of the symbols $P_\pm^n(c)$, $P^n(c)$, $r_\pm^n(f,c;t)$, etc. are obvious.

Lemma 1 remains true for $t=b$ as well as for $t=c$.

2. Singular Integrals of Arbitrary Order With Constant Kernel Density

Before discussing singular integrals of any order in the general case, we consider first those with constant kernel density. When $n>1$, define

$$\int_a^b \frac{\mathrm{d}t}{(t-a)^n}=-\frac{1}{(n-1)(b-a)^{n-1}} \quad (n>1). \tag{2.1}$$

It may be understood as follows. After the primitive of $1/(t-a)^n$ has been found, the term containing $\lim\limits_{t\to a} 1/(t-a)^{n-1}=\infty$ which makes the integral divergent is omitted during double substitution.

Similarly, we define

$$\int_a^b \frac{\mathrm{d}t}{(t-b)^n}=-\int_b^a \frac{\mathrm{d}t}{(t-b)^n}=\frac{1}{(n-1)(a-b)^{n-1}} \quad (n>1). \tag{2.2}$$

If c is an inner point of L, we define

$$\int_a^b \frac{\mathrm{d}t}{(t-c)^n}=\int_a^c+\int_c^b=\frac{1}{n-1}\left(\frac{1}{(a-c)^{n-1}}-\frac{1}{(b-c)^{n-1}}\right) \quad (n>1). \tag{2.3}$$

The situation is different when $n=1$. In this case, we assume L has a directed tangent T_a at $t=a$, issuing from a. The divergence of the integral $\int_a^b \frac{\mathrm{d}t}{t-a}$ is due to

$$\operatorname{Re}\log(t-a)=\ln|t-a|\to-\infty$$

as $t\to a$, while the increment of $\operatorname{Im}\log(t-a)$ is finite as t describes L from a to b and equal to the angle from T_a to $\overrightarrow{ab}$ (Fig. 1), denoted by

$$\theta_{ab}=[\arg(t-a)]_L. \tag{2.4}$$

Then we define

$$\int_a^b \frac{\mathrm{d}t}{t-a}=\ln|b-a|+\mathrm{i}\theta_{ab}. \tag{2.5}$$

Similarly, assuming L has a tangent T_b issuing from b, we define

$$\int_a^b \frac{\mathrm{d}t}{t-b}=-\int_b^a \frac{\mathrm{d}t}{t-b}=-\ln|b-a|-\mathrm{i}\theta_{ba}. \tag{2.6}$$

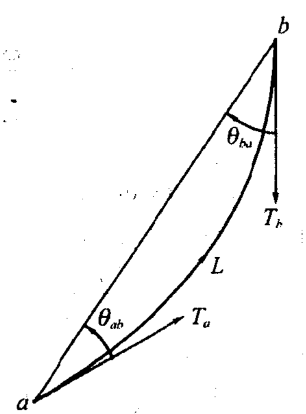

Fig. 1 Singular point at the ends

If c is an inner point of L with two one-sided tangents

T_{c+} and T_{c-} both issuing from c in its positive and negative sides respectively, then we define:

$$\int_a^b \frac{dt}{t-c} = \int_a^c + \int_c^b = \ln\left|\frac{b-c}{c-a}\right| + i(\theta_{cb} - \theta_{ca}). \tag{2.7}$$

Let θ_c be the angle from T_{c+} to T_{c-} and θ_{bca} be the angle from $\overrightarrow{cb}$ to $\overrightarrow{ca}$ such that both of them lie in $[0, 2\pi]$ (Fig. 2). When the two angles in consideration are coincident, it will be equal to 0 or 2π by using a limiting process. It may be shown that

$$\theta_c = \theta_{bca} + \theta_{cb} - \theta_{ca}. \tag{2.8}$$

Thus, (2.7) may be rewritten as

$$\int_a^b \frac{dt}{t-c} = \ln\left|\frac{b-c}{c-a}\right| + i(\theta_c - \theta_{bca}). \tag{2.7}'$$

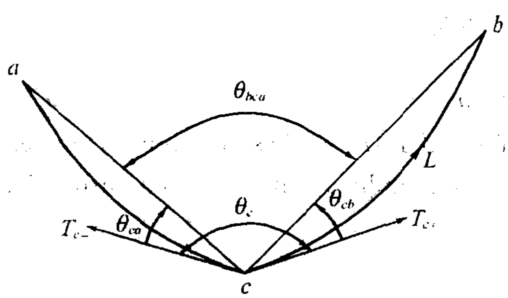

Fig. 2. Singular point at an inner point

In particular, if L is a closed Jordan contour and oriented counter-clockwisely, then, for $c \in L$, we have, by taking any $a = b\ (\neq c)$ on L,

$$\int_L \frac{dt}{(t-c)^n} = 0, \quad n > 1, \tag{2.9}$$

and

$$\int_L \frac{dt}{t-c} = i\theta_c \tag{2.10}$$

since $\theta_{bca} = 0$ in this case, which is identical to the Cauchy principal value integral in ordinary sense.

The advantage of the definition for singular integrals of any order given in this and the next sections lies in that the condition subjected to L is weakened. The smoothness (or arc-wise smoothness) for the path of integration is not assumed and hence the so-called standard radius as well as the boundedness of the ratio for the arc-length to the corresponding length of the chord[7] is not involved.

3. Singular Integrals of Arbitrary Order in the General Case

Now we may define singular integrals of any order in general as follows.

Let $f(t) \in D^n(a)\ (n \geqslant 0)$. Define

$$\int_a^b \frac{f(t)dt}{(t-a)^{n+1}} = \sum_{k=0}^{n} \frac{f^{[k]}(a)}{k!} \int_a^b \frac{dt}{(t-a)^{n-k+1}} + \int_a^b \frac{r^n(f,a;t)}{t-a} dt, \tag{3.1}$$

or, what is the same,

$$\int_a^b \frac{f(t)dt}{(t-a)^{n+1}} = \sum_{k=0}^{n} \frac{f^{[k]}(a)}{k!} \frac{1}{(n-k)(b-a)^{n-k}} + \frac{f^{[n]}(a)}{n!}[\ln|b-a| + i\theta_{ab}] + \int_a^b \frac{r^n(f,a;t)}{t-a} dt \tag{3.1}'$$

as the singular integral of order $n+1$ for $f(t)$ at $t=a$, provided that T_a exists.

Similarly, when $f(t)\in \mathrm{D}^n(b)(n\geqslant 0)$ and T_b exists, then define

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-b)^{n+1}}=-\int_b^a \frac{f(t)\mathrm{d}t}{(t-b)^{n+1}}$$
$$=\sum_{k=0}^{n-1}\frac{f^{[k]}(b)}{k!}\frac{1}{(n-k)(a-b)^{n-k}}-\frac{f^{[n]}(b)}{n!}[\ln|b-a|+\mathrm{i}\theta_{ba}]$$
$$+\int_a^b \frac{r^n(f,b;t)}{t-b}\mathrm{d}t. \tag{3.2}$$

If $t=c$ is an inner point of L and $f(t)\in \mathrm{D}^n_\pm(c)$, then define

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-c)^{n+1}}=\int_a^c+\int_c^b \tag{3.3}$$

as usual, provided that both $T_{c\pm}$ exist. In particular, if $f(t)\in \mathrm{D}^n(c)$, then

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-c)^{n+1}}=\sum_{k=0}^{n-1}\frac{f^{[k]}(c)}{k!\,(n-k)}\left(\frac{1}{(a-c)^{n-k}}-\frac{1}{(b-c)^{n-k}}\right)$$
$$+\frac{f^{[n]}(c)}{n!}\left[\ln\left|\frac{b-c}{a-c}\right|+\mathrm{i}(\theta_c-\theta_{bca})\right]+\int_a^b \frac{r^n(f,c;t)}{t-c}\mathrm{d}t. \tag{3.4}$$

For $f(t)\in \mathrm{D}(c)$ $(=\mathrm{D}^0(c))$, we have

$$\int_a^b \frac{f(t)}{t-c}\mathrm{d}t=f(c)\left[\ln\left|\frac{b-c}{c-a}\right|+\mathrm{i}(\theta_c-\theta_{bca})\right]+\int_a^b \frac{f(t)-f(c)}{t-c}\mathrm{d}t \tag{3.5}$$

since $r^0(f,c;t)=f(t)-f(c)$. For the special case where L is a closed Jordan contour oriented counter-clockwisely, (3.4) and (3.5) become respectively

$$\int_L \frac{f(t)\mathrm{d}t}{(t-c)^{n+1}}=\mathrm{i}\theta_c\frac{f^{[n]}(c)}{n!}+\int_a^b \frac{r^n(f,c;t)}{t-c}\mathrm{d}t \tag{3.4$'$}$$

and

$$\int_L \frac{f(t)}{t-c}\mathrm{d}t=\mathrm{i}\theta_c f(c)+\int_a^b \frac{f(t)-f(c)}{t-c}\mathrm{d}t. \tag{3.5$'$}$$

The latter is identical to the well-known formula for Cauchy principal value integrals.

In the sequel, we always assume that the tangents at the points of singularity considered exist.

To compare with ordinary derivatives, we have

Lemma 2. *If $f'(t)\in \mathrm{D}^{n-1}(a)$, $n\geqslant 1$, then $f(t)\in \mathrm{D}^n(a)$ and*

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-a)^{n+1}}=-\frac{f(b)}{n(b-a)^n}+\frac{f^{[n]}(a)}{n!n}+\frac{1}{n}\int_a^b \frac{f'(t)}{(t-a)^n}\mathrm{d}t. \tag{3.6}$$

Proof. By Lemma 1, $f(t)\in \mathrm{P}^n(a)$. Then, by integration by parts,

$$\int_a^b \frac{r^n(f,a;t)}{t-a}\mathrm{d}t=\int_a^b \frac{f(t)-\sum_{k=0}^{n}\frac{f^{[k]}(a)}{k!}(t-a)^k}{(t-a)^{n+1}}\mathrm{d}t$$
$$=-\frac{r^n(f,a;t)}{n}\Big|_a^b+\frac{1}{n}\int_a^b \frac{r^{n-1}(f',a;t)}{t-a}\mathrm{d}t$$

which is convergent since $f'(t)\in \mathrm{D}^{n-1}(a)$ and $r^n(f,a;t)\to 0$ as $t\to a$. Therefore $f(t)\in \mathrm{D}^n(a)$.

Moreover, since

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-a)^{n+1}}=\int_a^b \frac{f(t)-\sum\limits_{k=0}^{n}\frac{f^{[k]}(a)}{k\,!}(t-a)^k}{(t-a)^{n+1}}\mathrm{d}t-\sum_{k=0}^{n-1}\frac{f^{[k]}(a)}{k\,!(n-k)(b-a)^{n-k}}+\frac{f^{[n]}(a)}{n\,!}\int_a^b\frac{\mathrm{d}t}{t-a},$$

so, by taking again integration by parts for the first integral in the right-hand member, we get immediately

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-a)^{n+1}}=-\frac{f(b)}{n(b-a)^n}+\frac{1}{n}\sum_{k=0}^{n}\frac{f^{[k]}(a)}{k\,!(b-a)^{n-k}}+\frac{1}{n}\int_a^b\frac{f'(t)-\sum\limits_{k=1}^{n}\frac{f^{[k]}(a)(t-a)^{k-1}}{(k-1)!}}{(t-a)^n}\mathrm{d}t+\sum_{k=0}^{n-1}\frac{f^{[k]}(a)}{k\,!(n-k)(b-a)^{n-k}}+\frac{f^{[n]}(a)}{n\,!}\int_a^b\frac{\mathrm{d}t}{t-a}.$$

By using (2.1) and simplifying, we obtain (3.6).

Furthermore, we have

Lemma 3. *If $f^{(n)}(t)\in \mathrm{D}(a)$, then*

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-a)^{n+1}}=-\sum_{k=0}^{n-1}\frac{1}{n(n-1)\cdots(n-k)}\frac{f^{(k)}(b)}{(b-a)^{n-k}}+\frac{f^{(n)}(a)}{n\,!}\sum_{k=0}^{n-1}\frac{1}{k}+\frac{1}{n\,!}\int_a^b\frac{f^{(n)}(t)}{t-a}\mathrm{d}t. \tag{3.7}$$

This may be easily obtained by using (3.6) repeatedly.

We have similar formulas for $t=b$. For example, if $f^{(n)}(t)\in\mathrm{D}(b)$, then

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-b)^{n+1}}=\sum_{k=0}^{n-1}\frac{1}{n(n-1)\cdots(n-k)}\frac{f^{(k)}(a)}{(a-b)^{n-k}}-\frac{f^{(n)}(b)}{n\,!}\sum_{k=0}^{n-1}\frac{1}{k}+\frac{1}{n\,!}\int_a^b\frac{f^{(n)}(t)}{t-b}\mathrm{d}t. \tag{3.8}$$

In [5], (3.7) and (3.8) are regarded as definition when $f^{(n)}(t)\in\mathrm{H}(L)$.

For an inner point c on L, if $f^{(n)}(t)\in\mathrm{D}_\pm(c)$, then, by (3.7) and (3.8), we have

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-c)^{n+1}}=\sum_{k=0}^{n-1}\frac{1}{n(n-1)\cdots(n-k)}\left[\frac{f^{(k)}(a)}{(a-c)^{n-k}}-\frac{f^{(k)}(b)}{(b-c)^{n-k}}\right]+\frac{f_+^{(n)}(c)-f_-^{(n)}(c)}{n\,!}\sum_{k=1}^{n-1}\frac{1}{k}+\frac{1}{n\,!}\int_a^b\frac{f^{(n)}(t)}{t-c}\mathrm{d}t. \tag{3.9}$$

In particular, if $f^{(n)}(t)\in\mathrm{D}(c)$, then the second term in the right-hand member disappears and the resulting formula becomes the classical one[3].

We may interpret (3.7) and (3.8) by introducing the notion of the finite part of a limit. Define

$$\lim_{t\to a}\frac{f(t)}{(t-a)^n}=\lim_{t\to a}\left\{\frac{f(t)}{(t-a)^n}-\sum_{k=0}^{n-1}\frac{\alpha_k}{(t-a)^{n-k}}\right\}, \tag{3.10}$$

provided that the limit on the right side in the usual sense exists for suitably (and uniquely) chosen constants $\alpha_0,\ldots,\alpha_{n-1}$. This limit, say α_n, is called the finite part of the limit on the left side. Obviously, $\alpha_k=f^{[k]}(a)/k\,!$ $(k\leqslant n)$ if $f(t)\in\mathrm{P}^n(a)$.

Thus, if $f'(t)\in\mathrm{D}^{n-1}(a)$, $n\geqslant 1$, then, by formal integration by parts, we have

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-a)^{n+1}} = -\frac{1}{n}\left.\frac{f(t)}{(t-a)^n}\right|_a^b + \frac{1}{n}\int_a^b \frac{f'(t)\mathrm{d}t}{(t-a)^n} \tag{3.11}$$

and get (3.6) at once by (3.10).

4. Further Generalizations

The Peano expansion of $f(t)$ at $t=a$ is in fact the decomposition of $f(t)-f(a)$ into infinitesimals of different orders. Generally speaking, they are not necessarily of integral order. In general, $f(t)$ is said to belong to $\mathrm{P}^{\lambda_n}(a)$ if

$$f(t) = \sum_{k=0}^{n} \alpha_k (t-a)^{\lambda_k} + r^{\lambda_n}(f,a;t)(t-a)^{\lambda_n}, \quad 0=\lambda_0<\lambda_1<\cdots<\lambda_n, \tag{4.1}$$

where $\alpha_0 = f(a)$, $\alpha_1, \ldots, \alpha_n$ are certain definite constants (the term with $\alpha_k=0$ may be omitted) with $r^{\lambda_n}(f,a;t) \to 0$ as $t\to a$ along L. Here, the branch of

$$(t-a)^{\lambda_k} = \exp\{\lambda_k \ln|t-a| + \mathrm{i}\lambda_k \arg(t-a)\},$$

or, that of $\arg(t-a)|_{t=a} = \arg T_a$, may be taken arbitrarily, which only influences the value of α_k $(k=1,\ldots,n)$. Then the definition for $\mathrm{D}^{\lambda_n}(a)$ is evident.

Thus, after

$$\int_a^b \frac{\mathrm{d}t}{(t-a)^{\lambda+1}} = -\frac{1}{\lambda(b-a)^{\lambda+1}} \quad (\lambda>0) \tag{4.2}$$

has been defined, we may define, for $f(t)\in \mathrm{D}^{\lambda_n}(a)$,

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-a)^{\lambda_n+1}} = -\sum_{k=0}^{n-1} \frac{\alpha_k}{(\lambda_n-\lambda_k)(b-a)^{\lambda_n-\lambda_k}} + \alpha_n(\ln|b-a| + \mathrm{i}\theta_{ab}) + \int_a^b \frac{r^{\lambda_n}(f,a;t)}{t-a}\mathrm{d}t. \tag{4.3}$$

Similarly, the notion of the finite part of a limit may be extended also. Define, for $f(t)$ given on L,

$$\lim_{t\to a} \frac{f(t)}{(t-a)^{\lambda}} = \lim_{t\to a}\left\{\frac{f(t)}{(t-a)^{\lambda}} - \sum_{k=0}^{n} \frac{\alpha_k}{(t-a)^{\lambda_k}}\right\}, \quad 0=\lambda_0<\lambda_1<\cdots<\lambda_n<\lambda,\ \alpha_0=f(a), \tag{4.4}$$

provided that the limit, say α, on its right side exists for certain set of $\{\lambda_k\}$ and $\{\alpha_k\}$.

Thus, if $f'(t)\in \mathrm{D}^{\lambda}(a)$, $\lambda>0$, then we have the extension of (3.6):

$$\int_a^b \frac{f(t)\mathrm{d}t}{(t-a)^{\lambda+1}} = \frac{f(b)}{\lambda(b-a)^{\lambda}} + \frac{\alpha}{\lambda} + \frac{1}{\lambda}\int_a^b \frac{f'(t)\mathrm{d}t}{(t-a)^{\lambda}}. \tag{4.5}$$

Formula analogous to (3.7) may be derived by recurrence. The situation may be also extended to the case of singularity at $t=b$ or $t=c$. These extensions contain the results in [6] as special case.

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(三)在弹性和断裂力学中的应用

关于不同弹性材料的平面焊接问题

摘　要

本文讨论不同弹性材料的平面焊接问题,这里设焊接后的弹性域是多连通的,且材料间可以相互套着焊接.通过柯西型积分,本文把第一、第二基本焊接问题化为某种(一个未知函数的)奇异积分方程.本文证明了它们解的存在和唯一.证明过程中用到了上述两个基本焊接问题的唯一性定理,后者在本文中也给出了严格的数学证明.文末并举出一些实际中重要的例子:

关于各向同性的不同弹性材料平面焊接问题,曾经有人研究过[1,2,3]. Д. И. Шерман 于 1943 年提出了在已知外力情况下(第一基本问题),把上述问题化为弗雷德霍姆积分方程的一般方法[4]. 他所用的方法比较复杂,即使在他所考虑的有一个洞的弹性材料中焊接上一个材料(成为一个单连通域)时就已经是这样;而且在焊接边界上假定了两个未知函数,于是所得方程本质上是方程组.后来他对这种情况又改用了别的方法以图简化[5,6],并变成一个方程[7]. 他对一个多洞材料焊接一些别的材料使成为一个单连通域的情况只作了一般的提示[4](其复杂程度可以想见).而对第二基本问题则并没有讨论.

本文的目的是要把上述第一和第二基本问题,在极为一般的情况下(即焊接后的域仍可以是多连通的,且各种材料间可一层套一层地焊接)化为一种相对比较简单的奇异积分方程,并利用Г. Ф. Манджавидзе[8]的研究,证明其解的存在和唯一.这里所用方法事实上就是从Д. И. Шерман 在解决一个均匀材料的基本问题时所用过的方法[9,10](或参看[11])启发而得来的.在证明过程中,我们还用到了上述两个基本问题本身的唯一性定理.就第一基本问题的唯一性定理而言,在[4]中已应用过;但在那里是从力学观点看来自然地得出的,并未加以严格论证.在本文§3中,给出了这两个定理的数学证明.最后(§4)还给出了一些特例.

§1　第一基本问题

设有一有界多连通平面弹性区域 S,其边界 $L=L_0+L_1+\cdots+L_m$ 由 $m+1$ 个互不相交

的封闭光滑曲线 L_k $(k=0,1,\cdots,m)$ 构成，其中 L_0 是最外层边界①. 这个弹性区域是这样构成的：它是由 $p+1$ 种不同（或有某些相同）材料相互焊接在一起而形成的，这些相互焊接着的边界是 S 中 p 个互不相交的封闭光滑曲线 $\gamma_1,\cdots,\gamma_p$. 所有 L_k $(k=0,1,\cdots,m)$, γ_j $(j=1,\cdots,p)$ 都假定有满足赫尔窦条件 (H) 的曲率. 为确定起见，我们取 L_0 的反时针方向为正向，$L_1,\cdots,L_m$ 的顺时针方向为正向，因此 S 保持在 L 的左侧；所有 γ_j 例如都取反时针方向为正向，并记 $\gamma=\gamma_1+\cdots+\gamma_p$.

所有 γ_j 把 S 分割成 $p+1$ 个各具同样材料的子域. 我们以后讲到 S 的子域时都是指的这种子域. 例如图 1 是表示 $m=2$, $p=3$ 的一种情况，同一子域用同种斜线表示. 在各子域中，弹性常数 κ 与 μ 一般各不相同. 以 L_k 为边界的子域的相应常数我们就记作 κ_k,μ_k；在 γ_j 左（右）侧的子域的相应常数记作 κ_j^+, $\mu_j^+(\kappa_j^-,\mu_j^-)$. 于是，其中某些记号虽不同，但实际上表示同一值. 例如，就图 1 而言，$\kappa_0=\kappa_2=\kappa_1^-=\kappa_3^-$, $\kappa_1=\kappa_1^+$, $\kappa_2^-=\kappa_3^+$，对 μ 也有类似的等式. 此外，对 S 中任意一点 z（不在 γ 上）处相应的弹性常数就记作 κ_z,μ_z；因此，它们是 S 中的分区常数，即在 S 的每一子域中是一常数②.

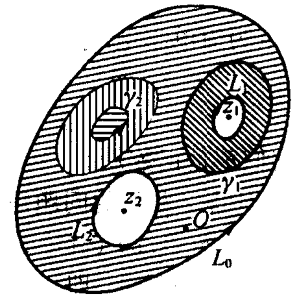

图 1

我们永远假定，坐标原点 O 取在区域 S 内，且在所有 γ_j 之外. 这样，例如在图 1 中，κ_0,μ_0 的记号既可理解为 O 点处的弹性常数，也可理解为 L_0 上各点的，而不会混淆.

我们还假设，在未焊接前，在各 γ_j 的正负侧上还有已知的位移差：

$$g(t)=g_j(t)=[u_j^+(t)+iv_j^+(t)]-[u_j^-(t)+iv_j^-(t)],\quad t\in\gamma_j,\ j=1,\cdots,p,$$

并且假定 $g'(t)\in H$.

以上的说明在本文中将一直采用.

现在如果还知道在 L 上各点处的外应力 $X_n(t)+iY_n(t)$，求弹性平衡. 这便是不同材料平面焊接的第一基本问题的一般提法.

如果在 L_k 上的外应力主矢量是 X_k+iY_k $(k=1,\cdots,m)$，则 Колосов 函数的形式显然是

$$\left.\begin{aligned}\varphi(z)&=-\frac{1}{2\pi}\sum_{k=1}^{m}\frac{X_k+iY_k}{1+\kappa_z}\log(z-z_k)+\varphi_0(z),\\ \psi(z)&=\frac{1}{2\pi}\sum_{k=1}^{m}\frac{\kappa_z(X_k-iY_k)}{1+\kappa_z}\log(z-z_k)+\psi_0(z),\end{aligned}\right\}\tag{1.1}$$

其中 $\varphi_0(z),\psi_0(z)$ 在 S 中分区全纯，而 z_k 是 L_k 所围洞内的任意一点. 这时，

$$f(t)=i\int_0^s(X_n+iY_n)ds,\quad t=t(s),\ t\in L_k\ (k=0,1,\cdots,m)\tag{1.2}$$

也是 L_k 上的多值函数. 在 L_k 上我们有边值条件：

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f(t)+C_k,\quad t\in L_k,\tag{1.3}$$

其中 C_k 是待定常数，但不妨假设 $C_0=0$. 我们可以把(1.1)中含对数的项并入 $f(t)$ 中，而得 $\varphi_0(z),\psi_0(z)$ 的类似于(1.3)的条件，这时其右边就是单值函数了. 以下，为简单起见，我们

① 如果 L_0 不存在，则只要略加修改，本文所论仍成立.

② 以后我们永远约定：用下标 z 的文字代表区域 S 中分区常数，例如 c_z,ε_z 等等.

就假定所有的 $X_k+\mathrm{i}Y_k\ (k=0,1,\cdots,m)$ 都等于零，从而(1.3)中的 $f(t)$ 是单值连续函数. 我们永远假定 $f'(t)\in H$.

在各 γ_j 上，由于焊接的条件，我们应有边值条件：

$$\varphi^+(t)+t\overline{\varphi^{+\prime}(t)}+\overline{\psi^+(t)}=\varphi^-(t)+t\overline{\varphi^{-\prime}(t)}+\overline{\psi^-(t)},\quad t\in\gamma,\tag{1.4}$$

$$\alpha_j^+\varphi^+(t)-\beta_j^+[t\overline{\varphi^{+\prime}(t)}+\psi^+(t)]=\alpha_j^-\varphi^-(t)-\beta_j^-[t\overline{\varphi^{-\prime}(t)}+\overline{\psi^-(t)}]+2g_j(t),$$
$$t\in\gamma_j,\quad j=1,\cdots,p,\tag{1.5}$$

这里为简单起见，已令

$$\alpha_j^\pm=\frac{\kappa_j^\pm}{\mu_j^\pm},\quad \beta_j^\pm=\frac{1}{\mu_j^\pm}\tag{1.6}$$

(它们都是正数).

于是，我们的问题是：要寻找在 S 中的一对分区全纯函数 $\varphi(z),\psi(z)$，使满足边值条件(1.3)～(1.5)（其中 C_k 待定，但 $C_0=0$). 不要忘掉，为使问题有解，在 L 上的外应力主力矩必须等于零，即

$$\mathrm{Re}\int_L f(t)\mathrm{d}\,\bar{t}=0.\tag{1.7}$$

当 $z\in S$ 但 $\overline{\in}\gamma$ 时，令

$$\varphi(z)=\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\omega(t)}{t-z}\mathrm{d}t+\sum_{k=1}^m\frac{b_k}{z-z_k},\tag{1.8}$$

$$\psi(z)=\frac{1}{2\pi\mathrm{i}}\int_L\frac{\overline{\omega(t)}-\bar{t}\omega'(t)}{t-z}\mathrm{d}t-\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{\overline{\omega(t)}+\bar{t}\omega'(t)}{t-z}\mathrm{d}t+\sum_{k=1}^m\frac{b_k}{z-z_k},\tag{1.9}$$

其中 $\omega(t)$ 是 $L+\gamma$ 上的未知函数，设其导数 $\omega'(t)\in H$，而 b_k 是一些待定实常数，它们与 $\omega(t)$ 间有下列关系：

$$b_k=\mathrm{i}\int_{L_k}\omega(t)\mathrm{d}\,\bar{t}-\overline{\omega(t)}\mathrm{d}t.\tag{1.10}$$

以 $\varphi(z),\psi(z)$ 的表达式(1.8),(1.9)代入(1.3)中，化简后得

$$\omega(t_0)+\frac{1}{2\pi\mathrm{i}}\int_L\omega(t)\mathrm{d}\log\frac{t-t_0}{\bar{t}-\bar{t}_0}+\frac{1}{\pi\mathrm{i}}\int_\gamma\omega(t)\mathrm{d}\log|t-t_0|-\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\overline{\omega(t)}\mathrm{d}\frac{t-t_0}{\bar{t}-\bar{t}_0}$$
$$+\sum_{k=0}^m\left\{\frac{b_k}{t_0-z_k}+\frac{\overline{b_k}}{\bar{t}_0-\bar{z}_k}\left(1-\frac{t_0}{\bar{t}_0-\bar{z}_k}\right)\right\}-C(t_0)=f(t_0),\quad t_0\in L,\tag{1.11}$$

这里我们已添加了有关 b_0 的项(认为 $z_0=0$)，b_0 是一纯虚常数，以

$$b_0=\frac{1}{2\pi\mathrm{i}}\int_{L_0}\left\{\frac{\omega(t)}{t^2}\mathrm{d}t+\frac{\overline{\omega(t)}}{\bar{t}^2}\mathrm{d}\,\bar{t}\right\}\tag{1.12}$$

定义. 此外，我们还认为

$$C_k=-\int_{L_k}\omega(t)\mathrm{d}s\quad(k=1,\cdots,m).\tag{1.13}$$

由 $\varphi(z),\psi(z)$ 的表达式(1.8),(1.9)立刻可见，(1.4)已告满足，而把它们代入(1.5)中经化简后，成为

$$(\alpha_j^++\alpha_j^-+\beta_j^++\beta_j^-)\omega(t_0)+\frac{\alpha_j^+-\alpha_j^--\beta_j^++\beta_j^-}{\pi\mathrm{i}}\int_{L+\gamma}\frac{\omega(t)}{t-t_0}\mathrm{d}t$$
$$+\frac{\beta_j^+-\beta_j^-}{\pi\mathrm{i}}\left\{\int_\gamma\omega(t)\mathrm{d}\log\frac{t-t_0}{\bar{t}-\bar{t}_0}+2\int_k\omega(t)\mathrm{d}\log|t-t_0|+\int_{L+\gamma}\overline{\omega(t)}\mathrm{d}\frac{t-t_0}{\bar{t}-\bar{t}_0}\right\}$$

$$+2(\alpha_j^+-\alpha_j^-)\sum_{k=1}^{m}\frac{b_k}{z-z_k}-2(\beta_j^+-\beta_j^-)\sum_{k=1}^{m}\frac{\overline{b_k}}{\overline{t_0}-\overline{z_k}}\left(1-\frac{t_0}{\overline{t_0}-\overline{z_k}}\right)$$
$$=4g_j(t_0),\quad t_0\in\gamma_j. \tag{1.14}$$

把(1.11)与(1.14)合在一起，就成为未知函数$\omega(t)$在$L+\gamma$上的一个奇异积分方程，其特征部分为

$$A(t_0)\omega(t_0)+\frac{B(t_0)}{\pi\,\mathrm{i}}\int_{L+\gamma}\frac{\omega(t)}{t-t_0}\mathrm{d}t=0,$$

其中

$$A(t_0)=\begin{cases}1, & t_0\in L,\\ \alpha_j^++\alpha_j^-+\beta_j^++\beta_j^-, & t_0\in\gamma_j;\end{cases}$$
$$B(t_0)=\begin{cases}0, & t_0\in L,\\ \alpha_j^+-\alpha_j^--\beta_j^++\beta_j^-, & t_0\in\gamma_j.\end{cases}$$

易见，$A(t_0)\pm B(t_0)$在$L+\gamma$上处处不为零. 所以我们的奇异方程是正则型的，且其指标为0.

由于已设$f'(t),g'(t)$分别在L,γ上$\in H$，故由文献[8]可知，$\omega(t),\omega'(t)\in H$.

现在证明，如方程(1.11),(1.14)有解，则必$b_0=0$. 事实上，由(1.8),(1.9)可知，(1.11)可改写为

$$\varphi(t)+t\,\overline{\varphi'(t)}+\overline{\psi(t)}+b_0\left(\frac{1}{t}-\frac{1}{\bar t}+\frac{t}{\bar t^2}\right)-C_k=f_k(t),\quad t\in L_k. \tag{1.15}$$

注意

$$\psi(z)=\psi_1(z)-\frac{1}{2\pi\,\mathrm{i}}\int_{\gamma}\frac{\overline{\omega(t)}+\bar t\omega'(t)}{t-z}\mathrm{d}t,$$

其中$\psi_1(z)$是S中的全纯函数，连续到L上者，所以，

$$\int_L\psi(t_0)\mathrm{d}t_0=-\frac{1}{2\pi\,\mathrm{i}}\int_L\mathrm{d}t_0\int_{\gamma}\frac{\overline{\omega(t)}+\bar t\omega'(t)}{t-t_0}\mathrm{d}t=\int_{\gamma}\overline{\omega(t)}\mathrm{d}t-\omega(t)\mathrm{d}\,\bar t.$$

在(1.15)中乘上$\mathrm{d}\,\bar t$，并沿L积分，得

$$\int_L[\varphi(t)\mathrm{d}\,\bar t-\overline{\varphi(t)}\mathrm{d}t]+\int_{\gamma}[\omega(t)\mathrm{d}\,\bar t-\overline{\omega(t)}\mathrm{d}t]+b_0\int_L\left(\frac{\mathrm{d}\,\bar t}{t}+\frac{\mathrm{d}t}{\bar t}\right)-2\pi\,\mathrm{i}b_0=\int_L f(t)\mathrm{d}\,\bar t.$$

注意到(1.7)，上式中除$-2\pi\,\mathrm{i}b_0$一项为实数外，其余各项都是虚数，故$b_0=0$.

这样，在(1.7)成立的条件下，方程(1.11),(1.14)的解必然给出所提出的基本问题的解，而C_k必须由(1.13)给出.

现在证明方程(1.11),(1.14)恒可解，且解为唯一. 由于其指标是0，由文献[8]可知，只需证明其相应齐次方程，即当$f(t)=0$，$g(t)=0$时，只有零解. 为此，设$\omega_0(t)$是齐次方程的一解. 并设由(1.10),(1.13),(1.8),(1.9)算得的相应结果是$b_k^0,C_k^0(k=1,\cdots,m)$，$\varphi_0(z),\psi_0(z)$；当然，由(1.12)所得的相应$b_0^0=0$. $\varphi_0(z),\psi_0(z)$是在L上外应力恒为0、γ上无位移差的条件下所解决的第一基本问题(且$C_0^0=0$)，故由唯一性定理(参看§3定理1附注)，有

$$\varphi_0(z)=\mathrm{i}\varepsilon_z z+c_z,\quad \psi_0(z)=-\overline{c_z}, \tag{1.16}$$

其中ε_z是S中分区实常数，c_z是分区常数，且$C_k^0=0$，又

$$(\alpha_j^++\beta_j^+)\varepsilon_j^+=(\alpha_j^-+\beta_j^-)\varepsilon_j^-,\ (\alpha_j^++\beta_j^+)c_j^+=(\alpha_j^-+\beta_j^-)c_j^-,\quad j=1,\cdots,p. \tag{1.17}$$

所以，这时(1.8),(1.9)成为

$$\mathrm{i}\varepsilon_z z + c_z = \frac{1}{2\pi \mathrm{i}}\int_{L+\gamma}\frac{\omega_0(t)}{t-z}\mathrm{d}t + \sum_{k=1}^{m}\frac{b_k^0}{z-z_k}, \tag{1.18}$$

$$-\overline{c_z} = \frac{1}{2\pi \mathrm{i}}\int_L\frac{\overline{\omega_0(t)} - \bar{t}\omega_0'(t)}{t-z}\mathrm{d}t - \frac{1}{2\pi \mathrm{i}}\int_\gamma\frac{\overline{\omega_0(t)} + \bar{t}\omega_0'(t)}{t-z}\mathrm{d}t + \sum_{k=1}^{m}\frac{b_k^0}{z-z_k}. \tag{1.19}$$

在(1.18)中对 γ 上的 t_0 点用 Plemelj 公式，立得

$$\omega_0(t_0) = \mathrm{i}t_0(\varepsilon_j^+ - \varepsilon_j^-) + (c_j^+ - c_j^-), \quad t_0 \in \gamma_j. \tag{1.20}$$

把它代回(1.18),(1.19)中，得

$$\mathrm{i}\varepsilon_z z + c_z = \frac{1}{2\pi \mathrm{i}}\int_L\frac{\omega_0(t)}{t-z}\mathrm{d}t + \sum_{j=1}^{p}\frac{\mathrm{i}z(\varepsilon_j^+ - \varepsilon_j^-) + (c_j^+ - c_j^-)}{2\pi \mathrm{i}}\int_{\gamma_j}\frac{\mathrm{d}t}{t-z} + \sum_{k=1}^{m}\frac{b_k^0}{z-z_k}, \tag{1.18$'$}$$

$$-\overline{c_z} = \frac{1}{2\pi \mathrm{i}}\int_L\frac{\overline{\omega_0(t)} - \bar{t}\omega_0'(t)}{t-z}\mathrm{d}t - \sum_{j=1}^{p}\frac{c_j^+ - c_j^-}{2\pi \mathrm{i}}\int_{\gamma_j}\frac{\mathrm{d}t}{t-z} + \sum_{k=1}^{m}\frac{b_k^0}{z-z_k}. \tag{1.19$'$}$$

如果我们令

$$\chi_1(z) = \mathrm{i}\varepsilon_z z + c_z - \sum_{j=1}^{p}\frac{\mathrm{i}z(\varepsilon_j^+ - \varepsilon_j^-) + (c_j^+ - c_j^-)}{2\pi \mathrm{i}}\int_{\gamma_j}\frac{\mathrm{d}t}{t-z}, \tag{1.21}$$

$$\chi_2(z) = -\overline{c_z} + \sum_{j=1}^{p}\frac{c_j^+ - c_j^-}{2\pi \mathrm{i}}\int_{\gamma_j}\frac{\mathrm{d}t}{t-z}, \tag{1.22}$$

则由(1.18)′,(1.19)′可知，$\chi_1(z)$,$\chi_2(z)$都是 S 内的全纯函数，且连续到 L 上.

现在引进记号：

$$\mathrm{i}\varphi^*(t) = \omega_0(t) + \sum_{k=1}^{m}\frac{b_k^0}{t-z_k} - \chi_1(t), \tag{1.23}$$

$$\mathrm{i}\psi^*(t) = \overline{\omega_0(t)} - \bar{t}\omega_0'(t) + \sum_{k=1}^{m}\frac{b_k^0}{t-z_k} - \chi_2(t), \tag{1.24}$$

则容易证明

$$\frac{1}{2\pi \mathrm{i}}\int_L\frac{\varphi^*(t)}{t-z}\mathrm{d}t = 0, \quad \frac{1}{2\pi \mathrm{i}}\int_L\frac{\psi^*(t)}{t-z}\mathrm{d}t = 0, \quad z \in S.$$

所以 $\varphi^*(t)$,$\psi^*(t)$是 $S_0, S_1, \cdots, S_m$(S_k 是 L_k 所围右侧的区域)中的全纯函数 $\varphi^*(z)$,$\psi^*(z)$ 的边值,且 $\varphi^*(\infty) = \psi^*(\infty) = 0$. 因此,

$$0 = \frac{1}{2\pi \mathrm{i}}\int_{L_0}\frac{\varphi^*(t)}{t^2}\mathrm{d}t = \frac{1}{2\pi \mathrm{i}}\int_{L_0}\frac{\omega_0(t)}{t^2}\mathrm{d}t + \sum_{k=1}^{m}\frac{b_k^0}{2\pi \mathrm{i}}\int_L\frac{\mathrm{d}t}{t^2(t-z_k)} - \frac{1}{2\pi \mathrm{i}}\int_{L_0}\frac{\chi_1(t)}{t^2}\mathrm{d}t,$$

化简得

$$\frac{1}{2\pi \mathrm{i}}\int_{L_0}\frac{\omega_0(t)}{t^2}\mathrm{d}t = \chi_1'(0) = \mathrm{i}\varepsilon_0.$$

由此又有

$$0 = b_0^0 = \frac{1}{2\pi \mathrm{i}}\int_{L_0}\left\{\frac{\omega_0(t)}{t^2}\mathrm{d}t + \frac{\overline{\omega_0(t)}}{\bar{t}^2}\mathrm{d}\,\bar{t}\right\} = 2\mathrm{i}\varepsilon_0.$$

由(1.17)可知，$\varepsilon_z = 0$.

这样一来，(1.21),(1.22)成为

$$\chi_1(z) = c_z - e_z, \quad \chi_2(z) = -\overline{c_z} + e_z, \tag{1.25}$$

其中已令

$$e_z=\sum_{j=1}^{p}\frac{c_j^+-c_j^-}{2\pi\mathrm{i}}\int_{\gamma_j}\frac{\mathrm{d}t}{t-z},\tag{1.26}$$

显然它是 S 中的分区常数. 既然 $\chi_1(z),\chi_2(z)$ 在 S 中全纯，又是分区常数，从而恒等于常数；令 $z=0$ 代入(1.26)，立刻可见 $e_0=0$，于是由(1.25)，

$$\chi_1(z)=c_0,\quad \chi_2(z)=-\overline{c_0}.\tag{1.25$'$}$$

在(1.23),(1.24)中消去 $\omega_0(t)$，并以(1.25)$'$代入，得

$$\overline{\varphi^*(t)}+\bar{t}\varphi^{*\prime}(t)+\psi^*(t)=\mathrm{i}\sum_{k=1}^{m}b_k^0\Big[\frac{1}{\bar{t}-\bar{z}_k}-\frac{1}{t-z_k}+\frac{\bar{t}}{(t-z_k)^2}\Big]-2\mathrm{i}\,\overline{c_0}.$$

把它两端乘以 $\mathrm{d}t$ 并沿 L_k 积分，并注意 b_k^0 都是实常数，立刻可知 $b_k^0=0$. 这样，

$$\overline{\varphi^*(t)}+\bar{t}\varphi^{*\prime}(t)+\psi^*(t)=-2\mathrm{i}\,\overline{c_0}.$$

根据 S_0 中第一基本问题的唯一性定理，可知 $c_0=0$，从而 $c_z=0$. 这样，由(1.20)知，

$$\omega_0(t)=0,\quad t\in\gamma.$$

再像文献[11]第 318 页下面那样处理，便可知

$$\omega_0(t)=0,\quad t\in L.$$

这就证明了我们方程解的存在和唯一.

§2 第二基本问题

所谓第二基本问题，边值条件(1.4),(1.5)仍同§1，不过(1.3)现在要改作

$$\kappa_k\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}=f(t),\quad t\in L_k,\ k=0,1,\cdots,m,\tag{2.1}$$

其中

$$f(t)=2\mu_k[g_1(t)+\mathrm{i}g_2(t)],\quad t\in L_k,\tag{2.2}$$

而 $g_1(t)+\mathrm{i}g_2(t)$ 是 L_k 上的已知位移矢量. 我们仍假定 $f'(t)\in H$.

这次我们令($z\in S$，但$\overline{\in}\gamma$)

$$\varphi(z)=\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\omega(t)}{t-z}\mathrm{d}t+\frac{1}{\kappa_z+1}\sum_{k=1}^{m}A_k\log(z-z_k),\tag{2.3}$$

$$\begin{aligned}\psi(z)=&-\sum_{k=0}^{m}\frac{\kappa_k}{2\pi\mathrm{i}}\int_{L_k}\frac{\overline{\omega(t)}}{t-z}\mathrm{d}t-\frac{1}{2\pi\mathrm{i}}\int_{\gamma}\frac{\overline{\omega(t)}}{t-z}\mathrm{d}t-\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\bar{t}\omega'(t)}{t-z}\mathrm{d}t\\&-\frac{\kappa_z}{\kappa_z+1}\sum_{k=1}^{m}\overline{A_k}\log(z-z_k)+\frac{1}{2\pi\mathrm{i}}\int_{\gamma}\frac{\overline{H(t)}}{t-z}\mathrm{d}t,\end{aligned}\tag{2.4}$$

其中已令 $A_k=-\dfrac{X_k+\mathrm{i}Y_k}{2\pi}$，是未知(待定)常数，而 $H(t)$ ($t\in L$)为一待定函数，将在下面确定. 这里诸对数仍任意取定分支. 我们还令

$$A_k=\int_{L_k}\omega(t)\mathrm{d}s,\quad k=1,\cdots,m.\tag{2.5}$$

将(2.3),(2.4)代入(2.1)中，则得

$$\begin{aligned}&\kappa_k\omega(t_0)+\frac{\kappa_k}{2\pi\mathrm{i}}\int_{L_k}\omega(t)\mathrm{d}\log\frac{t-t_0}{\bar{t}-\bar{t}_0}\\&\quad+\sum_{j\neq k}\frac{1}{2\pi\mathrm{i}}\Big[\kappa_k\int_{L_j}\frac{\omega(t)}{t-t_0}\mathrm{d}t-\kappa_j\int_{L_j}\frac{\omega(t)}{\bar{t}-\bar{t}_0}\mathrm{d}\,\bar{t}\Big]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2\pi\mathrm{i}}\left[\kappa_k\int_\gamma \frac{\omega(t)}{t-t_0}\mathrm{d}t-\int_\gamma \frac{\omega(t)}{\bar{t}-\bar{t}_0}\mathrm{d}\,\bar{t}+\int_\gamma \overline{\omega(t)}\mathrm{d}\,\frac{t-t_0}{\bar{t}-\bar{t}_0}\right]\\
&+\frac{2\kappa_k}{\kappa_k+1}\sum_{r=1}^{m}A_r\ln|t_0-z_r|-\frac{t_0}{\kappa_k+1}\sum_{r=1}^{m}\frac{\overline{A_r}}{\bar{t}_0-\bar{z}_r}\\
&=f(t_0)-\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{H(t)}{\bar{t}-\bar{t}_0}\mathrm{d}\,\bar{t},\quad t_0\in L_k,\ k=0,\cdots,m.
\end{aligned}\tag{2.6}
$$

将它们代入(1.4)，则所有含 $\omega(t)$ 的项均消失，而成为

$$
\begin{aligned}
&\frac{1}{\kappa_j^++1}\sum_{k=1}^{m}A_k\log(t_0-z_k)+\frac{t_0}{\kappa_j^++1}\sum_{k=1}^{m}\frac{\overline{A_k}}{\bar{t}_0-\bar{z}_k}\\
&\quad-\frac{\kappa_j^+}{\kappa_j^++1}\sum_{k=1}^{m}A_k\,\overline{\log(t_0-z_k)}+\frac{1}{2}H(t_0)\\
&\quad=\frac{1}{\kappa_j^-+1}\sum_{k=1}^{m}A_k\log(t_0-z_k)+\frac{\bar{t}_0}{\kappa_j^-+1}\sum_{k=1}^{m}\frac{\overline{A_k}}{\bar{t}_0-\bar{z}_k}\\
&\quad-\frac{\kappa_j^-}{\kappa_j^-+1}\sum_{k=1}^{m}A_k\,\overline{\log(t_0-z_k)}-\frac{1}{2}H(t_0),\quad t_0\in\gamma.
\end{aligned}
$$

易见此式两端的多值性质是相同的，亦即，它又可写为

$$
\begin{aligned}
H(t_0)&=\left(\frac{\kappa_j^+-1}{\kappa_j^++1}-\frac{\kappa_j^--1}{\kappa_j^-+1}\right)\sum_{k=1}^{m}A_k\ln|t_0-z_k|\\
&\quad-\left(\frac{1}{\kappa_j^++1}-\frac{1}{\kappa_j^-+1}\right)t_0\sum_{k=1}^{m}\frac{\overline{A_k}}{\bar{t}_0-\bar{z}_k}\\
&=-\left(\frac{1}{\kappa_j^++1}-\frac{1}{\kappa_j^-+1}\right)\left[2\sum_{k=1}^{m}A_k\ln|t_0-z_k|+t_0\sum_{k=1}^{m}\frac{\overline{A_k}}{\bar{t}_0-\bar{z}_k}\right],\quad t_0\in\gamma_j.
\end{aligned}
$$

因此，如果我们即以此式定义(2.4)中的函数 $H(t)$，则(2.4)就会自动成立.①以下已设如此做了.

再将(2.3),(2.4)代入(2.5)，则得

$$
\begin{aligned}
&\frac{\alpha_j^++\alpha_j^-+\beta_j^++\beta_j^-}{2}\omega(t_0)+\frac{\alpha_j^+-\alpha_j^-}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\omega(t)}{t-t_0}\mathrm{d}t\\
&\quad-\frac{\beta_j^+-\beta_j^-}{2\pi\mathrm{i}}\int_\gamma\frac{\omega(t)}{\bar{t}-\bar{t}_0}\mathrm{d}\,\bar{t}+\frac{\beta_j^+-\beta_j^-}{2\pi\mathrm{i}}\int_\gamma\overline{\omega(t)}\mathrm{d}\,\frac{t-t_0}{\bar{t}-\bar{t}_0}\\
&\quad-\frac{\beta_j^+-\beta_j^-}{2\pi\mathrm{i}}\sum_{k=1}^{m}\kappa_k\int_{L_k}\frac{\omega(t)}{\bar{t}-\bar{t}_0}d\,\bar{t}\\
&\quad-\left(\frac{\beta_j^+}{\alpha_j^++1}-\frac{\beta_j^-}{\alpha_j^-+1}\right)\sum_{k=1}^{m}\frac{\overline{A_k}}{\bar{t}_0-\bar{z}_k}\\
&\quad+\left(\frac{\alpha_j^+}{\kappa_j^++1}-\frac{\alpha_j^-}{\kappa_j^-+1}\right)\sum_{k=1}^{m}A_k\log(t_0-z_k)\\
&\quad+\left(\frac{\kappa_j^+\beta_j^+}{\kappa_j^++1}-\frac{\kappa_j^-\beta_j^-}{\kappa_j^-+1}\right)\sum_{k=1}^{m}A_k\,\overline{\log(t_0-z_k)}\\
&=\frac{\beta_j^++\beta_j^-}{2}H(t_0)-\frac{\beta_j^+-\beta_j^-}{2\pi\mathrm{i}}\int_\gamma\frac{H(t)}{\bar{t}-\bar{t}_0}\mathrm{d}\,\bar{t}+2h(t_0),\quad t_0\in\gamma_j.
\end{aligned}
$$

---

① 第一基本问题中，如 $X_j+\mathrm{i}Y_j$ 不为零，也可作类似处理.

但注意到 $\kappa_j^{\pm}\beta_j^{\pm}=\alpha_j^{\pm}$，故上式中的多值性也消失了，所以它又可改写成

$$
\begin{aligned}
&(\alpha_j^+ + \alpha_j^- + \beta_j^+ + \beta_j^-)\omega(t_0) + \frac{\alpha_j^+ - \alpha_j^- - \beta_j^+ + \beta_j^-}{\pi \mathrm{i}}\int_\gamma \frac{\omega(t)}{t-t_0}\mathrm{d}t \\
&\quad + \frac{\beta_j^+ - \beta_j^-}{\pi \mathrm{i}}\int_\gamma \omega(t)\mathrm{d}\log\frac{t-t_0}{\bar{t}-\bar{t}_0} + \frac{\beta_j^+ - \beta_j^-}{\pi \mathrm{i}}\int_\gamma \overline{\omega(t)}\mathrm{d}\frac{t-t_0}{\bar{t}-\bar{t}_0} \\
&\quad - \frac{\beta_j^+ - \beta_j^-}{\pi \mathrm{i}}\sum_{k=1}^m \kappa_k \int_{L_k}\frac{\overline{\omega(t)}}{\bar{t}-\bar{t}_0}\mathrm{d}\bar{t} \\
&\quad - 4\left(\frac{\beta_j^+}{\kappa_j^+ + 1} - \frac{\beta_j^-}{\kappa_j^- + 1}\right)\sum_{k=1}^m A_k \ln|t_0 - z_k| \\
&\quad - 2\left(\frac{\beta_j^+}{\kappa_j^+ + 1} - \frac{\beta_j^-}{\kappa_j^- + 1}\right)\sum_{k=1}^m \frac{\overline{A_k}}{\bar{t}_0 - \bar{z}_k} \\
&= 4h(t_0) + (\beta_j^+ + \beta_j^-)H(t_0) - \frac{\beta_j^+ - \beta_j^-}{\pi \mathrm{i}}\int_\gamma \frac{H(t)}{\bar{t}-\bar{t}_0}\mathrm{d}\bar{t}, \quad t_0 \in \gamma_j. \qquad (2.7)
\end{aligned}
$$

和上节一样，容易看出(2.6)，(2.7)一起构成 $\omega(t)$ 在 $L+\gamma$ 上的一个正则型奇异积分方程，指标为 0.

为了求证此方程解的存在和唯一，只要求证 $f\equiv 0$，$h\equiv 0$ 时其解 $\omega_0(t)$ 恒等于零.

由 $\omega_0(t)$ 算出的函数和常数记为 $\varphi_0(z)$，$\psi_0(z)$，A_k^0. 立即可以知道，$\varphi_0(t)$，$\psi_0(t)$ 满足(2.1)，(2.4)，(2.5)的相应零边界条件. 根据唯一性定理(参看§3 定理 2)，便可知道

$$\varphi_0(z)=c_z, \quad \psi_0(z)=\kappa_z \bar{c}_z, \qquad (2.8)$$

且有

$$(\kappa_j^+ + 1)c_j^+ = (\kappa_j^- + 1)c_j^-, \quad j=1,\cdots,p. \qquad (2.9)$$

由于 $\varphi_0(z)$ 的单值性，由(2.3)可知 $A_k^0=0$，即

$$\int_{L_k}\omega_0(t)\mathrm{d}s = 0, \quad k=1,\cdots,m. \qquad (2.10)$$

现在我们有

$$c_z = \frac{1}{2\pi \mathrm{i}}\int_{L+\gamma}\frac{\omega_0(t)}{t-z}\mathrm{d}t, \qquad (2.11)$$

$$\kappa_z \bar{c}_z = -\sum_{k=1}^m \frac{\kappa_k}{2\pi \mathrm{i}}\int_{L_k}\frac{\overline{\omega_0(t)}}{t-z}\mathrm{d}t - \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{\overline{\omega_0(t)}}{t-z}\mathrm{d}t - \frac{1}{2\pi \mathrm{i}}\int_{L+\gamma}\frac{\bar{t}\omega_0'(t)}{t-z}\mathrm{d}t. \qquad (2.12)$$

由(2.11)，立即可知

$$\omega_0(t) = c_j^+ - c_j^-, \quad t \in \gamma_j. \qquad (2.13)$$

如果限定 z 在 L_0 左侧的子域 S_0 中，于是它就位于所有 γ_j 的外域内. 由(2.11)和(2.13)，立即可知

$$c_0 = \frac{1}{2\pi \mathrm{i}}\int_L \frac{\omega_0(t)}{t-z}\mathrm{d}t. \qquad (2.14)$$

但右端实际上在整个 S 中全纯，可见此式对一切 $z\in S$ 恒成立. 对(2.12)作同样考虑，则又可得

$$\kappa_0 \bar{c}_0 = -\sum_{k=0}^m \frac{\kappa_k}{2\pi \mathrm{i}}\int_{L_k}\frac{\overline{\omega_0(t)}}{t-z}\mathrm{d}t - \frac{1}{2\pi \mathrm{i}}\int_L \frac{\bar{t}\omega_0'(t)}{t-z}\mathrm{d}t \qquad (2.15)$$

也对一切 $z\in S$ 成立.

令

$$
\left.\begin{aligned}
&\mathrm{i}\varphi_*(t)=\omega_0(t)-c_0, \quad t\in L;\\
&-\mathrm{i}\psi_*(t)=\kappa_k\overline{\omega_0(t)}+\bar{t}\omega_0'(t)+\kappa_0\overline{c_0}, \quad t\in L_k,
\end{aligned}\right\} \tag{2.16}
$$

则由(2.14),(2.15)，立即可得

$$
\frac{1}{2\pi\mathrm{i}}\int_L\frac{\varphi_*(t)}{t-z}\mathrm{d}t=0,\ \frac{1}{2\pi\mathrm{i}}\int_L\frac{\psi_*(t)}{t-z}\mathrm{d}t=0,\quad z\in S,
$$

因此 $\varphi_*(t),\psi_*(t)$ 是 S 的各个余域中全纯函数 $\varphi_*(z),\psi_*(z)$ 的边值，且 $\varphi_*(\infty)=\psi_*(\infty)=0$. 在(2.16)中消去 $\omega_0(t)$，则得

$$
\kappa_k\varphi_*(t)-t\overline{\varphi_*'(t)}-\overline{\psi_*(t)}=2\mathrm{i}\kappa_0c_0, \quad t\in L_k. \tag{2.17}
$$

先令 $k=0$. 此式是 L_0 外域 S_0^- 中假想弹性材料(κ_0 为弹性常数)的零边界条件的第二基本问题，因此由唯一性定理，并注意到 $\varphi_*(\infty)=\psi_*(\infty)=0$，故 $\varphi_*(z)=\psi_*(z)=0$ 于 S_0^- 中，从而 $c_0=0$. 且由(2.16)知，$\omega_0(t)=0$ 于 L_0 上. 再取 $k>0$. 这时(2.17)是 L_k 所围内域 S_k^- 中零边界条件的第二基本问题(且已知 $c_0=0$)，故再由唯一性定理知，

$$
\varphi_k(z)=d_k,\ \psi_*(z)=\kappa_k\overline{d_k}, \quad z\in S_k^-,\ k>0.
$$

因此 $\omega_0(t)=\mathrm{i}\varphi_*(t)=\mathrm{i}d_k$. 代入(2.10)，立即可知 $d_k=0$. 这样，$\omega_0(t)=0$ 于 L_k 上($k>0$).

今任取为 S_0 边界一部分的 γ_j. 则 c_j^- 实际上就是 c_0，故 $c_j^-=0$，由(2.9)又可知 $c_j^+=0$. 因此由(2.13)知，$\omega_0(t)=0$ 于 γ_j 上. 再取靠里一层的 γ_k(即 γ_k 与前述的 γ_j 为同一子域的边界)，则 $c_k^-=c_j^+=0$. 再由(2.9)，有 $c_k^+=0$. 于是 $\omega_0(t)=0$ 于 γ_k 上. 如此逐层进行下去，就可证得 $\omega_0(t)=0$ 于整个 γ 上.

$\omega_0(t)=0$ 于 $L+\gamma$ 上已经证得，问题完全解决.

§3　唯一性定理

我们前面证明中，用了第一和第二基本焊接问题的唯一性定理，在这一节中予以证明. 先证下面一引理(记号同前两节).

引理　*如果 $\psi(z)$ 在 S 中分区全纯，在 L_k 上取常数值 $h_k(k=0,1,\cdots,m)$，而在 γ_j 上 $\psi(z)$ 有常数跳跃：*

$$
\psi^+(t)-\psi^-(t)=g_j, \quad t\in\gamma_j,\ j=1,\cdots,p,
$$

则 $\psi(z)$ 在 S 中是分区常数：$\psi(z)=h_z$.

证　令

$$
\psi_1(z)=\sum_{j=1}^{p}\frac{g_j}{2\pi\mathrm{i}}\int_{\gamma_j}\frac{\mathrm{d}t}{t-z},
$$

则 $\psi_2(z)=\psi(z)-\psi_1(z)$ 显然在 S 中全纯，并在 L_k 上取某些常数值，从而它本身是一常数(参看[12]第 247 页). 另一方面，$\psi_1(z)$ 又是 S 中的分区常数. 引理于是得证.

现在来证明第一基本焊接问题的唯一性定理.

定理 1　*如果在 L 上无外应力，在 γ 上无位移差，则 Колосов 函数有如下形式：*

$$
\varphi(z)=\mathrm{i}\varepsilon_z+c_z, \quad \psi(z)=-\overline{c_z}+d, \tag{3.1}
$$

其中 ε_z 是分区实常数，c_z 是分区(复)常数，d 是常数，但它们间满足下列关系式：

$$
(\alpha_j^++\beta_j^+)\varepsilon_j^+=(\alpha_j^-+\beta_j^-)\varepsilon_j^-, \tag{3.2}
$$

$$(\alpha_j^+ + \beta_j^+)c_j^+ - \beta_j^+ d = (\alpha_j^- + \beta_j^-)c_j^- - \beta_j^- d. \tag{3.3}$$

我们说这是唯一性定理，意即：这时在 S 中到处没有应力，而除整个弹性区域的刚性平移和旋转外，也无相对位移存在.

证 在定理假设条件下，$\varphi(z)$，$\psi(z)$要满足条件(1.3)，(1.4)，(1.5)——但令其中 $f(t)=0$，$g(t)=0$，其中 C_k 为某些待定常数. 与文献[11]，§42 中证明相类似，得知

$$\mathrm{Re}\,\varphi'(z)=0, \quad z\in S.$$

不要忘记，现在 $\varphi(z)$是分区全纯的，由此立刻得到(3.1)的前一式.

把它代入(1.3)中，立刻可以看出，$\psi(t)$在 L_k 上取常数值. 但由(1.4)又可看出，$\psi(z)$在 γ_j 上有常数跳跃：

$$\psi^+(t)-\psi^-(t)=\overline{c_j^-}-\overline{c_j^+} \quad (j=1,\cdots,p).$$

因此由引理知道，$\psi(z)$是一分区常数：

$$\psi(z)=c_z'.$$

再把此式代入(1.5)中，可见

$$c_j^+ + \overline{c_j'}^+ = c_j^- + \overline{c_j'}^- \quad (j=1,\cdots,p),$$

也就是，

$$c_z + \overline{c_z'} = d$$

是一常数，于是得到(3.1)中第二式.

最后把(3.1)代入(1.6)，便可得到(3.2)，(3.3)：定理证毕.

附注 由定理的证明可以看出，(1.4)中的所有C_k $(k=0,1,\cdots,m)$都应相等，且等于 d. 所以，如果不计整个弹性区域的一个刚性平移，则不妨例如可假设$C_0=0$，从而一切 $C_k=0$，且 $d=0$. 这时代替(3.1)，(3.3)，就有较简形式.

定理 2 *如果 L 上无位移，在 γ 上无位移差，则有*

$$\varphi(z)=c_z, \quad \psi(z)=\kappa_z\overline{c_z}, \tag{3.4}$$

且

$$(\kappa_j^+ + 1)c_j^+ = (\kappa_j^- + 1)c_j^-. \tag{3.5}$$

证 同文献[11]第121页，容易证明 $\varphi(z)=c_z$. 由于现在有(2.1) ($f(t)=0$)，立刻可以看出，

$$\psi(t)=\kappa_k\overline{c_k}, \quad t\in L_k,\ k=0,1,\cdots,m. \tag{3.6}$$

又由(1.5)，易见 $\psi(z)$在 γ 上有常数跳跃，故由引理，$\psi(z)$是分区常数. 由(3.6)就可看出，

$$\psi(z)=\kappa_z\overline{c_z}, \quad z\in S.$$

把(3.4)代入(1.4)中，便得(3.5)(且这时(1.6)已告满足——其中 $g(t)=0$). 定理证毕.

§4 全平面中的焊接情况，一些特例

如果各种焊接材料充满着全平面，即没有 L 出现，则问题特别简单. 设在 $z=\infty$处与应力和旋转相关的常数为 Γ 与 Γ' (参看[11]，§36). 这时可设

$$\varphi(z)=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{\omega(t)}{t-z}\mathrm{d}t+\Gamma z, \quad \psi(z)=-\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{\overline{\omega(t)}+\bar{t}\omega'(t)}{t-z}\mathrm{d}t+\Gamma' z. \tag{4.1}$$

于是，方程(1.11)不存在，只剩下方程(1.14)，而现在有如下形式：

$$
\begin{aligned}
&(\alpha_j^- + \alpha_j^- + \beta_j^+ + \beta_j^-)\omega(t_0) + \frac{\alpha_j^+ - \alpha_j^- - \beta_j^+ + \beta_j^-}{\pi \mathrm{i}} \int_\gamma \frac{\omega(t)}{t - t_0} \mathrm{d}t \\
&\quad + \frac{\beta_j^+ - \beta_j^-}{\pi \mathrm{i}} \int_\gamma \omega(t) \mathrm{d} \log \frac{t - t_0}{\overline{t} - \overline{t_0}} + \frac{\beta_j^+ - \beta_j^-}{\pi \mathrm{i}} \int_\gamma \overline{\omega(t)} \mathrm{d} \frac{t - t_0}{\overline{t} - \overline{t_0}} \\
&= 4g_j(t_0) - 2[(\alpha_j^+ - \alpha_j^-)\Gamma - (\beta_j^+ - \beta_j^-)\overline{\Gamma}]t_0 \\
&\quad + 2(\beta_j^+ - \beta_j^-)\overline{\Gamma'}\overline{t_0}, \quad t_0 \in \gamma_j.
\end{aligned} \tag{4.2}
$$

这是一个在 γ 上的指标为 0 的正则型奇异方程.

作为(4.2)的应用，我们来考虑两个特殊情况.

1. 两个半平面的焊接

设有两种不同弹性材料分别占据着上半平面 S^+ 和下半平面 S^-（图 2），它们的弹性常数分别为 κ^+, μ^+ 和 κ^-, μ^-. 设在 x 轴的区间 $|t| \leqslant a$ 上，两种材料表面不平滑，在 $x = t$ 处，上、下岸纵坐标之差为 $h(t)$（$\geqslant 0$ 或 $\leqslant 0$），而当 $|t| > a$ 时，两材料表面都是与 x 轴相重合的. 我们假定 $h'(t) \in H$，且自然要求

$$
h(a) = h(-a) = 0. \tag{4.3}
$$

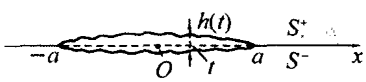

图 2

现在把它们沿整个 x 轴焊接起来，且使同一横坐标上的两点焊接在一起. 为简单起见，并设在 $z = \infty$ 处无应力也无旋转. 所以现在

$$
g(t) = \begin{cases} -h(t)\mathrm{i}, & \text{当 } |t| \leqslant a, \\ 0, & \text{当 } |t| > a; \end{cases} \quad \text{且 } \Gamma = \Gamma' = 0. \tag{4.4}
$$

这时，方程(4.2)成为①

$$
(\alpha^+ + \alpha^- + \beta^+ + \beta^-)\omega(t_0) + \frac{\alpha^+ - \alpha^- - \beta^+ + \beta^-}{\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\omega(t)\mathrm{d}t}{t - t_0} = 4g(t_0),
$$
$$
-\infty < t_0 < +\infty. \tag{4.5}
$$

按照文献[12]的 § 47，容易求出方程(4.5)的唯一解为

$$
\begin{aligned}
\omega(t_0) &= \left(\frac{1}{\alpha^+ + \beta^-} + \frac{1}{\alpha^- + \beta^+} \right) g(t_0) \\
&\quad + \left(\frac{1}{\alpha^+ + \beta^-} - \frac{1}{\alpha^- + \beta^+} \right) \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{g(t)\mathrm{d}t}{t - t_0}.
\end{aligned} \tag{4.6}
$$

把(4.6)代入(4.1)中，并注意(4.4)，化简后，易得

$$
\varphi(z) = \begin{cases} -\dfrac{1}{(\alpha^+ + \beta^-)\pi} \displaystyle\int_{-a}^{a} \frac{h(t)}{t - z} \mathrm{d}t, & \text{当 } z \in S^+, \\ -\dfrac{1}{(\alpha^- + \beta^+)\pi} \displaystyle\int_{-a}^{a} \frac{h(t)}{t - z} \mathrm{d}t, & \text{当 } z \in S^-; \end{cases} \tag{4.7}
$$

$$
\psi(z) = \begin{cases} \dfrac{\alpha^+ + \beta^-}{\alpha^- + \beta^-} \varphi(z) - z\varphi'(z), & \text{当 } z \in S^+, \\ \dfrac{\alpha^- + \beta^+}{\alpha^+ + \beta^-} \varphi(z) - z\varphi'(z), & \text{当 } z \in S^-. \end{cases} \tag{4.8}
$$

或者，如果只着眼于应力分布，令 $\Phi(z) = \varphi'(z)$，$\Psi(z) = \psi'(z)$，则有

---

① 虽然这里 γ 延伸到无穷远，不是封闭曲线，但不难证明 § 2 中的理论在这里完全适合. 也不难从最后所得结果的合理性而证实. 又由(4.6)，当 $t_0 \to \pm \infty$ 时，$\omega(t_0) = o(1)$，故(4.5)中的反常积分是收敛的.

$$\Phi(z)=\begin{cases}-\dfrac{1}{(\alpha^{+}+\beta^{-})\pi}\displaystyle\int_{-a}^{a}\frac{h'(t)}{t-z}\mathrm{d}t, & \text{当 } z\in S^{+},\\ -\dfrac{1}{(\alpha^{-}+\beta^{+})\pi}\displaystyle\int_{-a}^{a}\frac{h'(t)}{t-z}\mathrm{d}t, & \text{当 } z\in S^{-};\end{cases} \tag{4.7$'$}$$

$$\Psi(z)=\begin{cases}\left(\dfrac{\alpha^{+}+\beta^{-}}{\alpha^{-}+\beta^{+}}-1\right)\Phi(z)-z\Phi'(z), & \text{当 } z\in S^{+},\\ \left(\dfrac{\alpha^{-}+\beta^{+}}{\alpha^{+}+\beta^{-}}-1\right)\Phi(z)-z\Phi'(z), & \text{当 } z\in S^{-}.\end{cases} \tag{4.8$'$}$$

例 1 设

$$h(t)=\frac{\varepsilon}{a^{2}}(a^{2}-t^{2}),\quad h'(t)=-\frac{2\varepsilon t}{a^{2}}. \tag{4.9}$$

即，在 $|t|\leqslant a$ 时有一上下侧都是抛物线形的空隙，中间最宽处为 ε，而将此空隙连同平滑部分上下垂直地焊接起来(图 3).

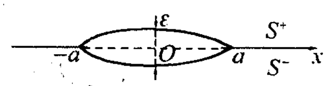

图 3

容易算出，

$$\Phi(z)=\begin{cases}\dfrac{2\varepsilon}{(\alpha^{+}+\beta^{-})a^{2}\pi}\left(2a+z\log\dfrac{z-a}{z+a}\right), & z\in S^{+},\\ \dfrac{2\varepsilon}{(\alpha^{-}+\beta^{+})a^{2}\pi}\left(2a+z\log\dfrac{z-a}{z+a}\right), & z\in S^{-},\end{cases} \tag{4.10}$$

其中对数这样来理解：沿 $|t|\leqslant a$ 剖开全平面，取其一支，使

$$\lim_{z\to\infty}\log\frac{z-a}{z+a}=0.$$

我们只来算出 x 轴上即焊接处的上、下岸上的应力分布. 先设 $z=t$ 在上半平面的岸上，用 $\sigma_x^{+}(t),\sigma_y^{+}(t),\tau_{xy}^{+}(t)$ 表示其应力，则只要注意到当 $|t|<a$ 时，$\arg\left(\dfrac{t-a}{t+a}\right)=\pi$，当 $|t|>a$ 时，它为 0，根据熟知的公式：

$$\sigma_x^{+}(t)+\sigma_y^{+}(t)=4\operatorname{Re}\Phi^{+}(t),$$

$$\sigma_y^{+}(t)-\sigma_x^{+}(t)+2i\tau_{xy}^{+}(t)=2\{t\Phi^{+\prime}(t)+\Psi^{+}(t)\},$$

容易得到

$$\sigma_y^{+}(t)=\frac{4\varepsilon}{a^{2}\pi}\left(\frac{1}{\alpha^{+}+\beta^{-}}+\frac{1}{\alpha^{-}+\beta^{+}}\right)\left(2a+t\ln\left|\frac{t-a}{t+a}\right|\right), \tag{4.11}$$

$$\sigma_x^{+}(t)=\frac{4\varepsilon}{a^{2}\pi}\left(\frac{3}{\alpha^{+}+\beta^{-}}-\frac{1}{\alpha^{-}+\beta^{+}}\right)\left(2a+t\ln\left|\frac{t-a}{t+a}\right|\right), \tag{4.12}$$

$$\tau_{xy}^{+}(t)=\begin{cases}\dfrac{2\varepsilon}{a^{2}}\left(\dfrac{1}{\alpha^{-}+\beta^{+}}-\dfrac{1}{\alpha^{+}+\beta^{-}}\right)t, & \text{当 } |t|<a,\\ 0, & \text{当 } |t|>a.\end{cases} \tag{4.13}$$

当 $z=t$ 在下半平面的岸上时，本来只要在以上诸式中把 α^{+} 与 α^{-} 以及 β^{+} 与 β^{-} 对调即得，但要注意这时 $|t|<a$ 时 $\arg\left(\dfrac{t-a}{t+a}\right)=-\pi$，故得

$$\sigma_y^{+}(t)=\sigma_y^{-}(t),\quad \tau_{xy}^{+}(t)=\tau_{xy}^{-}(t),$$

这自然在意料之中；而由(4.12)知，

$$\sigma_x^{-}(t)=\frac{4\varepsilon}{a^{2}\pi}\left(\frac{3}{\alpha^{-}+\beta^{+}}-\frac{1}{\alpha^{+}+\beta^{-}}\right)\left(2a+t\ln\left|\frac{t-a}{t+a}\right|\right). \tag{4.12$'$}$$

由此可见，当且仅当$\alpha^++\beta^-=\alpha^-+\beta^+$时，即$\frac{\kappa^+-1}{\mu^+}=\frac{\kappa^--1}{\mu^-}$时，才有$\sigma_x^+(t)=\sigma_x^-(t)$①.

2. 带圆孔的平面与垫圈的焊接

考虑一无限平面，带有一单位圆孔，其弹性常数为κ^-,μ^-. 今在其内焊接一实心垫圈，其弹性常数为κ^+,μ^+；且设在$t=e^{i\theta}$处，焊接点的位移差为$g(t)$，而$g'(t)\in H$. 又设在无穷远处$\Gamma=\Gamma'=0$. 这时方程(4.2)成为

$$(\alpha^++\alpha^-+\beta^++\beta^-)\omega(t_0)+\frac{\alpha^+-\alpha^--\beta^++\beta^-}{\pi i}\int_\gamma\frac{\omega(t)}{t-t_0}dt$$
$$+\frac{\beta^+-\beta^-}{\pi i}\int_\gamma\frac{\omega(t)}{t}dt-\frac{(\beta^+-\beta^-)t_0}{\pi i}\int_\gamma\overline{\omega(t)}dt=4g(t_0). \tag{4.14}$$

如果令

$$A=\frac{\beta^+-\beta^-}{4\pi i}\int_\gamma\frac{\omega(t)}{t}dt,\quad B=\frac{\beta^+-\beta^-}{4\pi i}\int_\gamma\overline{\omega(t)}dt, \tag{4.15}$$

暂把它们看作已知常数，则(4.14)可写成

$$(\alpha^++\alpha^-+\beta^++\beta^-)\omega(t_0)+\frac{\alpha^+-\alpha^--\beta^++\beta^-}{\pi i}\int_\gamma\frac{\omega(t)}{t-t_0}dt=4[g(t_0)-A+Bt_0]. \tag{4.14$'$}$$

这方程的唯一解是

$$\omega(t_0)=\left(\frac{1}{\alpha^++\beta^-}+\frac{1}{\alpha^-+\beta^+}\right)g(t_0)+\left(\frac{1}{\alpha^++\beta^-}-\frac{1}{\alpha^-+\beta^+}\right)\frac{1}{\pi i}\int_\gamma\frac{g(t)}{t-t_0}dt$$
$$-\frac{2}{\alpha^++\beta^-}(A-Bt_0). \tag{4.16}$$

把(4.16)代入(4.15)中，容易算出

$$\left.\begin{aligned}A&=\frac{\beta^+-\beta^-}{\alpha^++\beta^+}\frac{1}{2\pi i}\int_\gamma\frac{g(t)}{t}dt,\\ \mathrm{Re}B&=\frac{\beta^+-\beta^-}{\alpha^+-\beta^++2\beta^-}\mathrm{Re}\frac{1}{2\pi i}\int_\gamma\overline{g(t)}dt,\\ \mathrm{Im}B&=\frac{\beta^+-\beta^-}{\alpha^++\beta^+}\mathrm{Im}\frac{1}{2\pi i}\int_\gamma\overline{g(t)}dt.\end{aligned}\right\} \tag{4.17}$$

再由(4.1)和(4.16)，不难算出：

$$\varphi(z)=\begin{cases}\dfrac{1}{\alpha^++\beta^-}\dfrac{1}{\pi i}\displaystyle\int_\gamma\frac{g(t)}{t-z}dt-\dfrac{2}{\alpha^++\beta^-}(A-Bz), & \text{当}|z|<1,\\ \dfrac{1}{\alpha^-+\beta^+}\dfrac{1}{\pi i}\displaystyle\int_\gamma\frac{g(t)}{t-z}dt, & \text{当}|z|>1;\end{cases} \tag{4.18}$$

$$\psi(z)=\begin{cases}-\dfrac{1}{\alpha^-+\beta^+}\dfrac{1}{\pi i}\displaystyle\int_\gamma\frac{\overline{g(t)}}{t-z}dt-\left(\dfrac{1}{\alpha^++\beta^-}-\dfrac{1}{\alpha^-+\beta^+}\right)\dfrac{1}{\pi i}\displaystyle\int_\gamma\frac{\overline{g(t)}}{t}dt\\ \quad-\dfrac{\varphi'(z)-\varphi'(0)}{z}, & \text{当}|z|<1,\\ -\dfrac{1}{\alpha^++\beta^-}\dfrac{1}{\pi i}\displaystyle\int_\gamma\frac{\overline{g(t)}}{t-z}dt+\dfrac{2\overline{B}}{\alpha^++\beta^-}\dfrac{1}{z}-\dfrac{\varphi'(z)-\varphi'(0)}{z}, & \text{当}|z|>1.\end{cases} \tag{4.19}$$

---

① 当$\kappa^+=\kappa^-$，$\mu^+=\mu^-$时，文献[13]，§47.4中曾讨论过，不过问题提法稍有不同.

如果只着眼于应力分布，则有

$$\Phi(z)=\begin{cases}\dfrac{1}{\alpha^{+}+\beta^{-}}\dfrac{1}{\pi\mathrm{i}}\displaystyle\int_{\gamma}\dfrac{\mathrm{d}g(t)}{t-z}+\dfrac{2B}{\alpha^{+}+\beta^{-}}, & \text{当}|z|<1,\\ \dfrac{1}{\alpha^{-}+\beta^{+}}\dfrac{1}{\pi\mathrm{i}}\displaystyle\int_{\gamma}\dfrac{\mathrm{d}g(t)}{t-z}, & \text{当}|z|>1;\end{cases}\tag{4.18$'$}$$

$$\Psi(z)=\begin{cases}-\dfrac{1}{\alpha^{-}+\beta^{+}}\dfrac{1}{\pi\mathrm{i}}\displaystyle\int_{\gamma}\dfrac{\mathrm{d}\,\overline{g(t)}}{t-z}+\dfrac{\Phi(z)-\Phi(0)}{z^{2}}-\dfrac{\Phi'(z)}{z}, & \text{当}|z|<1,\\ -\dfrac{1}{\alpha^{+}+\beta^{-}}\dfrac{1}{\pi\mathrm{i}}\displaystyle\int_{\gamma}\dfrac{\mathrm{d}\,\overline{g(t)}}{t-z}-\dfrac{2\overline{B}}{\alpha^{+}+\beta^{-}}\dfrac{1}{z^{2}} & \\ \qquad+\dfrac{\Phi(z)-\Phi(0)}{z^{2}}-\dfrac{\Phi'(z)}{z}, & \text{当}|z|>1.\end{cases}\tag{4.19$'$}$$

例 2(同心焊接) 如果垫圈也是一圆，其半径为 $1+\varepsilon$，而 $g(t)=-\varepsilon t$. 由此容易算出

$$\Phi(z)=\begin{cases}-\dfrac{2\varepsilon}{\alpha^{+}-\beta^{+}+2\beta^{-}}, & \text{当}|z|<1,\\ 0, & \text{当}|z|>1;\end{cases}$$

$$\Psi(z)=\begin{cases}0, & \text{当}|z|<1,\\ \dfrac{4\varepsilon}{\alpha^{+}-\beta^{+}+2\beta^{-}}\dfrac{1}{z^{2}}, & \text{当}|z|>1.\end{cases}$$

由此容易算出在 $|t|=1$ 上，

$$\widehat{rr}^{\pm}(t)=-\frac{4\varepsilon}{\alpha^{+}-\beta^{+}+2\beta^{-}}=-\frac{4\varepsilon\mu^{+}\mu^{-}}{(\kappa^{+}-1)\mu^{-}+2\mu^{+}}.\tag{4.20}$$

这和熟知的经典结果是一样的(参看文献[11]，§ 58 第 1 段).

例 3(偏心切触焊接) 带有单位圆孔的平面与垫圈同例 2，但垫圈的位置放置在与孔在 $t=-1$ 处相切，然后沿(孔的)同一半径上两点相焊接(图 4).

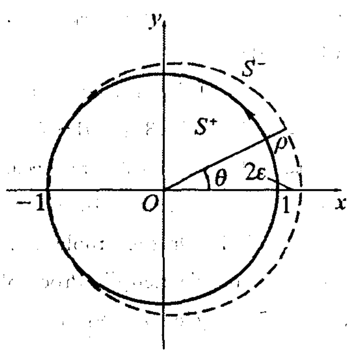

图 4

如果不计 ε^2 的项，则易证在 $t=\mathrm{e}^{\mathrm{i}\theta}$ 处焊接点间的距离是

$$\rho=\varepsilon(1+\cos\theta),$$

从而位移差

$$g(t)=-\rho\mathrm{e}^{\mathrm{i}\theta}=-\frac{\varepsilon}{2}(1+t)^{2}.\tag{4.21}$$

这时，把(4.21)代入(4.18)$'$，(4.19)$'$，可以算出

$$\Phi(z)=\begin{cases}-\dfrac{2\varepsilon z}{\alpha^{+}+\beta^{-}}-\dfrac{2\varepsilon}{\alpha^{+}-\beta^{+}+2\beta^{-}}, & \text{当}|z|<1,\\ 0, & \text{当}|z|>1;\end{cases}\tag{4.22}$$

$$\Psi(z)=\begin{cases}0, & \text{当}|z|<1,\\ \dfrac{2\varepsilon}{(\alpha^{+}+\beta^{-})z^{3}}+\dfrac{4\varepsilon}{\alpha^{+}-\beta^{+}+2\beta^{-}}\cdot\dfrac{1}{z^{2}}, & \text{当}|z|>1.\end{cases}\tag{4.23}$$

利用[11]，§ 39 中的记号，当 $z=\mathrm{e}^{\mathrm{i}\theta}$ 在垫圈周边上及孔的周边上时，不难算出

$$
\begin{aligned}
\widehat{rr}^{+} = \widehat{rr}^{-} &= -\frac{2\varepsilon\cos\theta}{\alpha^{+}+\beta^{-}} - \frac{4\varepsilon}{\alpha^{+}-\beta^{+}+2\beta^{-}},\\
\widehat{r\theta}^{+} = \widehat{r\theta}^{-} &= -\frac{2\varepsilon\sin\theta}{\alpha^{+}+\beta^{-}},\\
\widehat{\theta\theta}^{+} &= -\frac{6\varepsilon\cos\theta}{\alpha^{+}+\beta^{-}} - \frac{4\varepsilon}{\alpha^{+}-\beta^{+}+2\beta^{-}},\\
\widehat{\theta\theta}^{-} &= -\widehat{rr}^{-}.
\end{aligned} \tag{4.24}
$$

我们发现，在$\theta=\pm 90°$时，$\widehat{rr}^{\pm}$与例2中(4.20)表示的相同；而在$\theta=0°$时，它取得最大(绝对)值：

$$
\widehat{rr}^{\pm}\big|_{\theta=0} = -\frac{2\varepsilon}{\alpha^{+}+\beta^{-}} - \frac{4\varepsilon}{\alpha^{+}-\beta^{+}+2\beta^{-}}, \tag{4.25}
$$

其中右端第一项可认为是由于偏心的缘故，在$z=1$处所产生的附加压应力. 同样，在$z=-1$处也有同样大小的附加拉应力.

我们还可注意到，与例2一样，应力状态与α^{-}不发生关系，即与带孔平面的弹性常数κ^{-}无关(当然指精确到ε的一次幂而言).

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ON THE PLANE WELDING PROBLEMS OF DIFFERENT MATERIALS

Abstract

The plane welding problems of different materials are discussed. The elastic domain S after welding is supposed to be multiply connected, and the materials may be welded one interior to another (cf. fig. 1). D. I. Sherman had discussed the first fundamental problem when S is simply connected and reduced it to Fredholm equations. On assuming (1.8), (1.9) (supposing the result and vector of the external forces along any closed contour of the domain to be zero), we reduce the first fundamental problem to singular equations (1.11), (1.14) with one unknown function $\omega(t)(t \in L)$. As regards the second fundamental problem, we assume (2.3), (2.4) and reduce it to equations (2.6), (2.7). Under some general restrictions, we have proved the existence and uniqueness of the solutions of the equations in both cases. In the proofs of them, we have used the uniqueness theorems for the fundamental welding problems themselves, which we also give rigorous mathematical proofs here. Some examples important in practice are illustrated at the end of the paper.

[原载于武汉大学学报（自然科学版），1963，(2)：50～66；转载于高等学校自然科学学报（数学、力学、天文学版），1965，1 (2)：149～163. 此处§2改动较大.]

关于循环对称弹性平面中的数学问题

摘　要

本文讨论具循环对称性（或称循环周期）的弹性平面的第一、第二基本问题. 除循环对称性外，对于弹性域的形状及边值条件，我们不加特殊的条件限制. 在这种一般情形下，简化了 Шерман-Lauricella 积分方程，使其中代表待定常数的项达到最小限度，从而使方程的积分曲线可只限于一个周期角域中的边界上.

设有一各向同性的平面弹性域，无论域的边界以及边值条件，都具有循环对称性，即，如果绕坐标原点作 $2\pi/n$ 角的旋转（$n\geqslant 2$），则弹性域的形状以及边值条件都不变. 这种情况就称作**循环对称的弹性平面**，而 $\omega=\omega_n=e^{\frac{2\pi i}{n}}$ 称为它的**循环周期**.

这类问题包括实际中相当广泛的一类重要问题. 对于其中的某些特殊情况（主要是区域的边界都是圆周），过去已有不少作者讨论过，参看文献[1～4]. 我们这里将讨论最一般性的问题：对区域及边值条件除上述假定外，不再附加其他限制. 这种问题原则上可化为 Шерман-Lauricella 积分方程而解决（例如，参看[5]，§102），但应作适当改造，利用循环对称性使方程简化. 这一点在[2,3]中已注意到，并在某些特殊限制下（区域要具备例如对实轴的对称性）成功地做到了. 本文就要在最一般情况下，来简化上述方程.

§1　Колосов 函数的表达式

我们熟知，令 $z=\rho e^{i\theta}$时，通过极坐标表示，应力与位移可表为（见[5]，§39）：

$$\left.\begin{aligned}&\widehat{rr}+\widehat{\theta\theta}=2[\Phi(z)+\overline{\Phi(z)}],\\&\widehat{\theta\theta}-\widehat{rr}+2i\widehat{r\theta}=2[\bar{z}\Phi'(z)+\Psi(z)]e^{2i\theta},\\&2\mu(v_r+iv_\theta)=[\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}]e^{-i\theta},\end{aligned}\right\}\tag{1.1}$$

其中 $\Phi(z)=\varphi'(z)$，$\Psi(z)=\psi'(z)$是弹性域 S 中的全纯函数，μ,κ 为弹性常数.

由于问题以 ω 为循环周期，故当在(1.1)中把 z 改为 ωz 时，所有应力和位移均不改变.

由(1.1)中第一式知，

$$\operatorname{Re}\Phi(\omega z)=\operatorname{Re}\Phi(z),$$

从而

$$\Phi(\omega z)=\Phi(z)+i\delta,\tag{1.2}$$

其中 δ 是某确定的实常数. 由此立得

$$\Phi'(\omega z)=\bar{\omega}\Phi(z) \quad \left(\bar{\omega}=\frac{1}{\omega}\right).$$

再由(1.1)中第二式，并结合上式考虑，便得

$$\Psi(\omega z)=\bar{\omega}^2\Psi(z). \tag{1.3}$$

我们再来看 $\varphi(z),\psi(z)$的相应变化情况. 由(1.2),(1.3)积分，得

$$\bar{\omega}\varphi(\omega z)=\varphi(z)+\mathrm{i}\delta z+a,$$
$$\omega\psi(\omega z)=\psi(z)+b,$$

其中 a,b 为某二复常数.①

由(1.1)中第三式，容易得到

$$\kappa(\mathrm{i}\delta z+a)+\mathrm{i}\delta z-\bar{b}=0,$$

于是

$$\delta=0, \quad b=\kappa\bar{a}.$$

所以我们有

$$\Phi(\omega z)=\Phi(z), \quad \Psi(\omega z)=\bar{\omega}^2\Psi(z), \tag{1.4}$$

$$\varphi(\omega z)=\omega\varphi(z)+A, \quad \psi(\omega z)=\bar{\omega}\psi(z)+\kappa\bar{A}, \tag{1.5}$$

其中 $A=a\omega$ 是任意复常数. 但由于 $\varphi(z)$改为 $\varphi(z)+C$，$\psi(z)$改为 $\psi(z)+\kappa\bar{C}$(C 为任意复常数)时，不会改变应力和位移，故在(1.5)中不妨取 $A=0$，即

$$\varphi(\omega z)=\omega\varphi(z), \quad \psi(\omega z)=\bar{\omega}\psi(z). \tag{1.5$'$}$$

为了分离出 $\varphi(z),\psi(z)$的多值部分，我们如下进行.

先设弹性域 S 是有界的. 因此它外面必定由某一封闭曲线 L_0 所包围，且 L_0 的形状有循环对称性. 在每一周期基本域

$$S_k: \quad \frac{(2k-1)\pi}{n}<\arg z<\frac{(2k+1)\pi}{n} \quad (k=0,1,\cdots,n-1)$$

中，设有 m 个孔，其边界分别为 $L_{1k},L_{2k},\cdots,L_{mk}$，而 $L_{j0}\equiv l_j,L_{j1},\cdots,L_{j,n-1}$是周期合同的. 此外，在 S 内部，可能还有围绕原点的一孔，其边界为 L_{m+1}，它也有循环对称性. 把 L_0,L_{m+1} 在 S_0 中的部分记作 l_0,l_{m+1}，于是，弹性域 S 在 S_0 中的边界是 $l=\sum\limits_{j=0}^{m+1}l_j$. 我们取 L_0 的反时针方向为正向，其他边界都取顺时针方向为正向. 此外，在所有边界上的边值条件也假定是循环对称的.

图 1 中画的是 $n=2$ 时的一种情况，图 2 中画的是一个基本周期区域 S_0.

由循环对称性知，在 L_0 及 L_{m+1}上外应力主矢量均为 0. 设在 l_j 上的外应力主矢量为 $X_j+\mathrm{i}Y_j$，于是在 L_{jk}上则为$(X_j+\mathrm{i}Y_j)\omega^k$.

在 l_j 所围内域中任取一点 z_j，而在 L_{jk}的内域中则相应地取 $\omega^k z_j$，于是

$$\varphi(z)=-\frac{1}{2\pi(1+\kappa)}\sum_{j,k}(X_j+\mathrm{i}Y_j)\omega^k\log(z-\omega^k z_j)+\varphi_0(z),② \tag{1.6}$$

---

① 由于 $\varphi(z),\psi(z)$ 一般是多值的，这里 $\varphi(\omega z),\psi(\omega z)$ 可这样理解：先取定 $\varphi(z),\psi(z)$ 的值，当 z 沿 S 中某一确定路径连续变到 ωz 时便得 $\varphi(\omega z),\psi(\omega z)$；而当 z 连续变动时，这一路径也连续变形.

② $\sum\limits_{j,k}$指的是 j 从 1 到 m 相加，k 从 0 到 $n-1$ 相加. 下同.

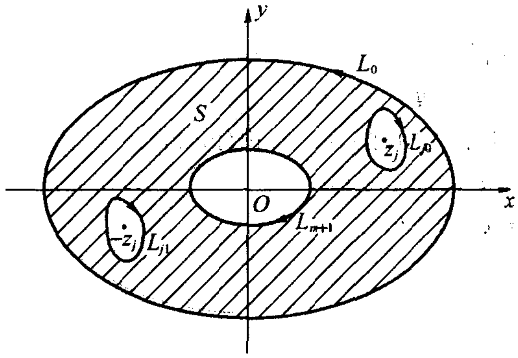

图 1

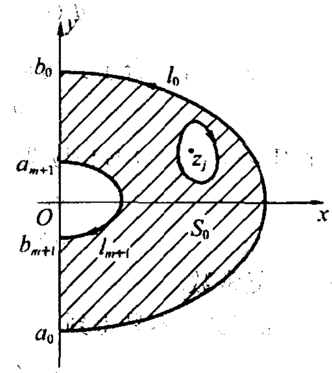

图 2

$$\psi(z)=\frac{\kappa}{2\pi(1+\kappa)}\sum_{j,k}(X_j-\mathrm{i}Y_j)\overline{\omega}^k\log(z-\omega^k z_j)+\psi_0(z), \tag{1.7}$$

其中 $\varphi_0(z),\psi_0(z)$ 是 S 中的全纯函数，而各对数可取任意确定的分支.

当 z 沿前述确定路径变到 ωz 时，如果我们这样认为

$$\log(\omega z-\omega^k z_j)=\log(z-\omega^{k-1}z_j)+\frac{2\pi\mathrm{i}}{n},$$

则由(1.5)′，易得

$$\varphi_0(\omega z)=\omega\varphi_0(z),\quad \psi_0(\omega z)=\overline{\omega}\psi_0(z). \tag{1.8}$$

由此也便得到

$$\Phi_0(\omega z)=\Phi_0(z),\quad \Psi_0(\omega z)=\overline{\omega}^2\Psi_0(z). \tag{1.9}$$

这样，便得 Колосов 函数的一般表达式(1.6)，(1.7)，其中 $\varphi_0(z),\psi_0(z)$ 是满足(1.8)的全纯函数.

如果区域 S 是无限的，即上述 L_0 不存在(L_{m+1} 仍存在或否)，则代替(1.6)，(1.7)便有

$$\varphi(z)=-\frac{1}{2\pi(1+\kappa)}\sum_{j,k}(X_j+\mathrm{i}Y_j)\omega^k\log(z-\omega^k z_j)+\varphi_0(z)+\Gamma z, \tag{1.6$'$}$$

$$\psi(z)=\frac{\kappa}{2\pi(1+\kappa)}\sum_{j,k}(X_j-\mathrm{i}Y_j)\overline{\omega}^k\log(z-\omega^k z_j)+\psi_0(z)+\Gamma' z, \tag{1.7$'$}$$

其中 Γ,Γ' 是表明 $z=\infty$ 处的应力和位移的常数(见[5]，§16)，且 $\varphi_0(\infty),\psi_0(\infty)$ 均有限. 考虑 z 沿着辐角 θ 为常数的半直线趋向 ∞ 时，我们有

$$\widehat{\theta\theta}-\widehat{rr}+2\mathrm{i}\,\widehat{r\theta}\to 2\Gamma' \mathrm{e}^{2\mathrm{i}\theta}.$$

但由假设，左边是循环对称的，故极限值也是如此，即

$$2\Gamma' \mathrm{e}^{2\mathrm{i}\theta}\omega^2=2\Gamma' \mathrm{e}^{2\mathrm{i}\theta}.$$

所以我们知道：当 $n>2$ 时必定 $\Gamma'=0$，而当 $n=2$ 时 Γ' 可以任意. 这在力学上也是明显的：当 $n>2$ 时，在 $z=\infty$ 处只可能是全面拉伸(或压缩)；当 $n=2$ 时，却不必有此限制.

重复有界区域时的讨论，我们仍可得(1.8)，(1.9)式. 特别注意：当 $n=2$ 时，$\omega=\overline{\omega}=-1$，所以(1.7)′中出现 $\Gamma' z$ 的项并不影响推导进行.

我们还指出，当 $z=0$ 属于 S 时即 L_{m+1} 不存在时，由(1.8)我们应有 $\varphi_0(0)=\psi_0(0)=0$. 当 S 为无限域时，$\varphi_0(\infty)=\psi_0(\infty)=0$.

§2 第一基本问题

以后我们将假定区域S的边界L有满足条件H的曲率.

现设已知L上的外应力$X_n(t)+\mathrm{i}Y_n(t)$，求弹性平衡. 由循环对称假设，在ωt处的外应力应为$\omega(X_n+\mathrm{i}Y_n)$. 不失一般性，我们还假定在每一$L_{jk}$上的外应力主矢量为零. 和通常一样，令

$$f(t)=\begin{cases}\mathrm{i}\int_{a_0}^{t}(X_n+\mathrm{i}Y_n)\mathrm{d}s, & t\in L_0,\\ \mathrm{i}\int_{a_{m+1}}^{t}(X_n+\mathrm{i}Y_n)\mathrm{d}s, & t\in L_{m+1},\\ \mathrm{i}\int_{\omega^k t_j}^{t}(X_n+\mathrm{i}Y_n)\mathrm{d}s, & t\in L_{jk},\end{cases} \tag{2.1}$$

其中t_j是l_j上的任一定点，积分路径取在相应的边界上. 我们还把l_0与l_{m+1}和角域S_0的二直线边$\arg z=\pm\frac{2\pi}{n}$的交点分别记为$a_0,b_0;a_{m+1},b_{m+1}$.

先来看$f(\omega t)$与$f(t)$的关系. 当$t\in L_{jk}$时，$\omega t\in L_{j,k+1}$，故

$$f(\omega t)=\mathrm{i}\int_{\omega^{k+1}t_j}^{\omega t}(X_n+\mathrm{i}Y_n)\mathrm{d}s=\omega\mathrm{i}\int_{\omega^k t_j}^{t}(X_n+\mathrm{i}Y_n)\mathrm{d}s=\omega f(t),\quad t\in L_{jk}.$$

设在l_0上外应力主矢量为$X_0'+\mathrm{i}Y_0'$，即

$$X_0'+\mathrm{i}Y_0'=\int_{a_0}^{b_0}(X_n+\mathrm{i}Y_n)\mathrm{d}s. \tag{2.2}$$

所以，当$t\in L_0$时，

$$f(\omega t)=\mathrm{i}\int_{a_0}^{\omega t}(X_n+\mathrm{i}Y_n)\mathrm{d}s=\mathrm{i}\int_{a_0}^{b_0}+\mathrm{i}\int_{b_0}^{\omega t}=\mathrm{i}(X_0'+\mathrm{i}Y_0')+\omega f(t).$$

同理，设l_{m+1}上的外应力主矢量为

$$X_{m+1}'+\mathrm{i}Y_{m+1}'=\int_{a_{m+1}}^{b_{m+1}}(X_n+\mathrm{i}Y_n)\mathrm{d}s, \tag{2.3}$$

则当$t\in L_{m+1}$时(注意L_{m+1}的正向)，

$$f(\bar{\omega}t)=\mathrm{i}\int_{a_{m+1}}^{\bar{\omega}t}(X_n+\mathrm{i}Y_n)\mathrm{d}s=\mathrm{i}(X_{m+1}'+\mathrm{i}Y_{m+1}')+\bar{\omega}f(t);$$

或者，在上式中把t改为ωt，得

$$f(\omega t)=-\omega\mathrm{i}(X_{m+1}'+\mathrm{i}Y_{m+1}')+\omega f(t).$$

由以上所论，如果令

$$f^0(t)=\begin{cases}f(t)+\frac{\mathrm{i}}{\omega-1}(X_0'+\mathrm{i}Y_0'), & \text{当 } t\in L_0,\\ f(t)-\frac{\omega\mathrm{i}}{\omega-1}(X_{m+1}'+\mathrm{i}Y_{m+1}'), & \text{当 } t\in L_{m+1},\\ f(t), & \text{当 } t\in L_{jk},\end{cases} \tag{2.4}$$

则我们便有

$$f^0(\omega t)=\omega f^0(t),\quad t\in L. \tag{2.5}$$

我们将假定 $f^0(t)$有满足条件 H 的导数. 第一基本问题可化为下列边值问题:

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f^0(t)+C(t), \tag{2.6}$$

其中 $C(t)$在每一 L_{jk}以及 L_0, L_{m+1}上都是待定常数. 由(1.5)′以及(2.4)立刻可知,

$$C(\omega t)=\omega C(t).$$

于是, 当 $t\in L_0$ 时, $C(t)=C_0=0$; 当 $t\in L_{m+1}$时, $C(t)=C_{m+1}=0$; 而

$$C(t)=\omega^k C_j, \quad 当\ t\in L_{jk}. \tag{2.7}$$

所以, 待定常数的个数比 S 的连通数少多了.

以下我们暂设 L_0, L_{m+1}都存在而求解(2.6). 这里我们已知 $f^0(t)$满足(2.5), 要求满足(1.5)′的全纯解 $\varphi(t),\psi(t)$; 对于 $C(t)$, 我们暂只假定 $C_0=0$, 而不假定 $C_{m+1}=0$, 也不假定(2.7)成立, 这些都将作为必然的结果推得.

我们将仿照Шерман 的方法, 寻求如下形式的解:

$$\varphi(z)=\frac{1}{2\pi i}\int_L\frac{\rho(t)}{t-z}dt+\sum_{j,k}B_{jk}\xi_{jk}(z)+B_{m+1}z, \tag{2.8}$$

$$\psi(z)=\frac{1}{2\pi i}\int_L\frac{\overline{\rho(t)}}{t-z}dt-\frac{1}{2\pi i}\int_L\frac{\bar t\rho'(t)}{t-z}dt+\sum_{j,k}B_{jk}\eta_{jk}(z)+\frac{B_{m+1}}{z}, \tag{2.9}$$

其中

$$\xi_{jk}(z)=\frac{\omega^{2k}}{nz_j^{n-2}(z-\omega^k z_j)}, \quad \eta_{jk}(z)=\frac{1}{n(z-\omega^k z_j)}, \tag{2.10}$$

而 B_{jk}, B_{m+1}是待定实常数:

$$\left.\begin{aligned}B_{jk}&=ni\int_{L_{jk}}\rho(t)d\,\bar t-\overline{\rho(t)}dt,\\ B_{m+1}&=i\int_{L_{m+1}}\rho(t)d\,\bar t-\overline{\rho(t)}dt,\end{aligned}\right\} \tag{2.11}$$

这里 $\rho(t)$是未知函数.

这里对 $\varphi(z)$的表达式与 Шерман 提出的不完全一样, 为的是希望将来求得的 $\rho(t)$也有下一性质:

$$\rho(\omega t)=\omega\rho(t), \tag{2.12}$$

而且还希望

$$B_{jk}=B_j, \tag{2.13}$$

与 k 无关. 在[2,3]中未注意到这一点, 从而引起了不必要的困难, 并使结果不得不受到一定的局限性.

把(2.8),(2.9)代入(2.6)中, 便得

$$\begin{aligned}K_\rho\equiv{}&\rho(t_0)+\frac{1}{2\pi i}\int_L\rho(t)d\log\frac{t-t_0}{\bar t-\bar t_0}-\frac{1}{2\pi i}\int_L\overline{\rho(t)}d\frac{t-t_0}{\bar t-\bar t_0}\\ &+\sum_{j,k}B_{jk}\{\xi_{jk}(t_0)+t_0\overline{\xi_{jk}'(t_0)}+\overline{\eta_{jk}(t_0)}\}\\ &+B_{m+1}\left\{2t_0+\frac{1}{\bar t_0}\right\}-C(t_0)\\ ={}&f^0(t_0).\end{aligned} \tag{2.14}$$

如果我们再令

$$C(t_0)=\begin{cases}C_{jk}=-n\int_{L_{jk}}\rho(t)|t|^{n-1}\mathrm{d}s, & t_0\in L_{jk},\\ C_{m+1}=-\int_{L_{m+1}}\rho(t)\mathrm{d}s, & t_0\in L_{m+1}\end{cases} \tag{2.15}$$

(C_0 已假定为零)，则(2.14)便是 $\rho(t)$ 的一个 Fredholm 方程.

既然现在 L_0 存在，我们还应有外应力主力矩为零的条件：

$$\mathrm{Re}\int_L \overline{f(t)}\mathrm{d}t=0 \quad 或 \quad \mathrm{Re}\int_L \overline{f^0(t)}\mathrm{d}t=0. \tag{2.16}$$

任取 $z_0\in S$，在(2.14)左边再添加一项

$$\frac{B_0\,\bar{z}^{n-1}}{\bar{z}^n-\bar{z}_0^n},① \tag{2.17}$$

其中已令 B_0 是一虚常数：

$$B_0=\frac{1}{2n\pi\,\mathrm{i}}\int_{L_0}\frac{\rho(t)}{t^2}\mathrm{d}t+\frac{\overline{\rho(t)}}{\bar{t}^2}\mathrm{d}\,\bar{t}. \tag{2.18}$$

这样，我们便得另一 Fredholm 方程：

$$\boldsymbol{K}_0\rho\equiv\boldsymbol{K}\rho+B_0\frac{\bar{t}_0^{\,n-1}}{\bar{t}_0^{\,n}-\bar{z}_0^{\,n}}=f^0(t_0). \tag{2.14'}$$

我们首先证明，如(2.14)′可解，则必 $B_0=0$. 这与[5]，§ 102 中相似，代替那里的(1)′式现在有

$$\varphi(t)+t\,\overline{\varphi'(t)}+\overline{\psi(t)}+\frac{B_0\,\bar{t}^{n-1}}{\bar{t}^n-\bar{t}_0^{\,n}}-C(t)=f^0(t).$$

把此式两边取共轭值，乘以 $\mathrm{d}t$，沿 L 积分，则由(2.16)，得

$$\mathrm{Re}\,\overline{B_0}\int_L\frac{\mathrm{d}t^n}{t^n-z_0^n}=\mathrm{Re}\,\overline{B_0}[\log(t^n-z_0^n)]_L=\mathrm{Re}\,\overline{B_0}\Big[\sum_{k=0}^{n-1}\log(t-\omega^k z_0)\Big]_L$$
$$=\mathrm{Re}\,2n\pi\,\mathrm{i}\,\overline{B_0}=0.$$

由于 B_0 是纯虚数，故必 $B_0=0$. 所以，在(2.16)条件下，(2.14)′的解必是(2.14)的解.

求证(2.14)′永远可解(因而解唯一)时，可令 $f^0\equiv0$ 而求证其解 $\rho_0(t)\equiv0$. 证明时与 Шерман 原证明类似，这里只简略叙述之.

设由 $\rho_0(t)$ 而得的 $\varphi,\psi,B_{jk},\cdots$ 现在是 $\varphi_0,\psi_0,B^0_{jk},\cdots$ ($B^0_0=0$ 已知). 由第一基本问题唯一性定理，

$$\varphi_0(z)=\mathrm{i}\varepsilon z+c,\quad \psi_0(z)=-\bar{c},\quad C^0_{jk}=0,\quad C^0_{m+1}=0.$$

如果令

$$\mathrm{i}\varphi^*(t)=\rho_0(t)+\sum_{j,k}B^0_{jk}\xi_{jk}(t)+B^0_{m+1}t-\mathrm{i}\varepsilon t-c,$$

$$\mathrm{i}\psi^*(t)=\overline{\rho_0(t)}-\bar{t}\rho_0'(t)+\sum_{j,k}B^0_{jk}\eta_{jk}(t)+\frac{B^0_{m+1}}{t}+\bar{c},$$

则易知 $\varphi^*(t),\psi^*(t)$ 是 S 的补域中全纯函数的边值，且在∞处为 0. 于是

① 顺便指出，[5]，§ 102 中(a)式改为$\frac{b_{m+1}}{t_0}$已行了，不必像那里那样麻烦.

$$0=\frac{1}{2\pi}\int_{L_0}\frac{\varphi^*(t)}{t^2}\mathrm{d}t=\frac{1}{2\pi\mathrm{i}}\int_{L_0}\frac{\rho_0(t)\mathrm{d}t}{t^2}+\sum_{j,k}\frac{B_{jk}^0}{2\pi\mathrm{i}}\int_{L_0}\frac{\xi_{jk}(t)\mathrm{d}t}{t^2}+\frac{B_{m+1}^0}{2\pi\mathrm{i}}\int_{L_0}\frac{\mathrm{d}t}{t}-\frac{\varepsilon}{2\pi}\int_{L_0}\frac{\mathrm{d}t}{t},$$

因为 $\int_{L_0}\frac{\xi_{jk}(t)\mathrm{d}t}{t^2}=0$，故

$$\mathrm{i}\varepsilon=\frac{1}{2\pi\mathrm{i}}\int_{L_0}\frac{\rho_0(t)}{t^2}\mathrm{d}t+B_{m+1}^0.$$

注意 B_{m+1}^0是实数，所以

$$2\mathrm{i}\varepsilon=\frac{1}{2\pi\mathrm{i}}\int_{L_0}\frac{\rho_0(t)\mathrm{d}t}{t^2}+\frac{\overline{\rho_0(t)}}{\bar{t}^2}\mathrm{d}\,\bar{t}=B_0^0=0.$$

以下证明 $B_{jk}^0=0$，$B_{m+1}^0=0$，$c=0$ 以及最后 $\rho_0(t)\equiv0$ 均同[5]，§102.

证明了(2.5)′的解存在且唯一后，我们将求证：当 $f^0(t)$满足(2.5)时，其解 $\rho(t)$也满足(2.12). 为此，我们暂令

$$\rho^*(t)=\frac{\rho(\omega t)}{\omega},$$

而由此算出的 $B_{jk},\cdots$则记作 $B_{jk}^*,\cdots$. 由(2.11)，

$$B_{j,k+1}=n\mathrm{i}\int_{L_{j,k+1}}\rho(t)\mathrm{d}\,\bar{t}-\overline{\rho(t)}\mathrm{d}t=n\mathrm{i}\int_{L_{jk}}\rho(\omega t)\bar{\omega}\mathrm{d}\,\bar{t}-\overline{\rho(\omega t)}\omega\mathrm{d}t.$$

$$=n\mathrm{i}\int_{L_{jk}}\overline{\rho^*(t)}\mathrm{d}\,\bar{t}-\rho^*(t)\mathrm{d}t=B_{jk}^*,$$

同理，

$$B_{m+1}=B_{m+1}^*;$$

此外，又有

$$C_{j,k+1}=-\int_{L_{j,k+1}}\rho(t)\mathrm{d}s=-\int_{L_{jk}}\rho(\omega t)\mathrm{d}s=\omega C_{jk}^*,$$

同理，

$$C_{m+1}=\omega C_{m+1}^*.$$

由(2.18)也易见，

$$B_0=B_0^*.$$

再注意到

$$\xi_{jk}(\omega z)=\omega\xi_{j,k-1}(z),\quad \eta_{jk}(\omega z)=\bar{\omega}\eta_{j,k-1}(z),$$

则容易看出，$\rho^*(t)$也是(2.14)′的解；而由解的唯一性，即知(2.12)成立.

这样一来，由以上所论，即知(2.13)也成立，且 $C_{m+1}=0$，而(2.7)也成立. 这时方程(2.14)就简化为

$$\begin{aligned}\hat{K}\rho\equiv\rho(t_0)&+\frac{1}{2\pi\mathrm{i}}\int_L\rho(t)\mathrm{d}\log\frac{t-t_0}{\bar{t}-\bar{t}_0}-\frac{1}{2\pi\mathrm{i}}\int_L\overline{\rho(t)}\mathrm{d}\frac{t-t_0}{\bar{t}-\bar{t}_0}\\&+\sum_j B_j\left\{\frac{t_0}{t_0^n-z_j^n}+\frac{t_0}{\bar{t}_0^n-\bar{z}_j^n}-\frac{nt_0\,\bar{t}_0^n}{(\bar{t}_0^n-\bar{z}_j^n)^2}+\frac{\bar{t}_0^{n-1}}{\bar{t}_0^n-\bar{z}_j^n}\right\}\\&+B_{m+1}\left(2t_0+\frac{1}{\bar{t}_0}\right)-C(t_0)\\&=f^0(t_0),\end{aligned}\tag{2.19}$$

因为易证

$$\sum_k \xi_{jk}(z) = \frac{z}{z^n - z_j^n}, \quad \sum_k \eta_{jk}(z) = \frac{z^{n-1}}{z^n - z_j^n}. \tag{2.20}$$

现在，由(2.11)，并注意(2.12)得知

$$B_j = n\mathrm{i}\int_{l_j} \rho(t)\mathrm{d}\,\bar{t} - \overline{\rho(t)}\mathrm{d}t, \quad j = 1,\cdots,m+1; \tag{2.21}$$

再由(2.15),(2.7)知，

$$C(t_0)=\begin{cases}0, & \text{当 } t_0 \overline{\in} L_0+L_{m+1},\\ -n\omega^k\int_{l_j}\rho(t)|t|^{n-1}\mathrm{d}s, & \text{当 } t_0\in L_{jk}.\end{cases} \tag{2.22}$$

这时，(2.14)′则成为

$$\hat{\boldsymbol{K}}_0\rho\equiv\hat{\boldsymbol{K}}\rho+B_0\frac{\overline{t_0}^{n-1}}{\overline{t_0}^n-\overline{z_0}^n}=f^0(t_0), \tag{2.19$'$}$$

其中 B_0 可理解为

$$B_0 = \frac{n}{2\pi\mathrm{i}}\int_{l_0}\frac{\rho(t)}{t^2}\mathrm{d}t + \frac{\overline{\rho(t)}}{\bar{t}^2}\mathrm{d}\bar{t}. \tag{2.23}$$

这样，当(2.5)满足时，方程(2.14)′的(唯一)解也必是(2.19)′的解，且满足(2.12). 反过来，如果 $\rho(t)$ 是满足条件(2.12)的(2.19)′的解，则也必是(2.14)′的解，因为，当(2.12)被满足时，(2.14)′与(2.19)′的左边是一致的. 总之，第一基本问题归结为求(2.19)′的解，但要解满足条件(2.12)；而且由以上论证，可知这种解唯一. ①

求出 $\rho(t)$ 后(从而 B_j, C_j 也求得了)，所求函数 $\varphi(z), \psi(z)$ 由(2.8),(2.9)现在就可写成

$$\varphi(z) = \frac{1}{2\pi\mathrm{i}}\int_L\frac{\rho(t)}{t-z}\mathrm{d}t + \sum_{j=1}^m\frac{B_jz}{z^n-z_j^n} + B_{m+1}z,$$

$$\psi(z) = \frac{1}{2\pi\mathrm{i}}\int_L\frac{\overline{\rho(t)}}{t-z}\mathrm{d}t - \frac{1}{2\pi\mathrm{i}}\int_L\frac{\bar{t}\rho'(t)}{t-z}\mathrm{d}t + \sum_{j=1}^m\frac{B_jz^{n-1}}{z^n-z_j^n} + \frac{B_{m+1}}{z};$$

或者，注意到

$$\frac{1}{2\pi\mathrm{i}}\int_L\frac{\rho(t)}{t-z}\mathrm{d}t=\sum_{k=0}^{n-1}\frac{1}{2\pi\mathrm{i}}\int_l\rho(t)\frac{\omega^{2k}}{\omega^kt-z}\mathrm{d}t=\frac{nz}{2\pi\mathrm{i}}\int_l\rho(t)\frac{t^{n-2}}{t^n-z^n}\mathrm{d}t$$

等等，上二式又可改写为

$$\varphi(z) = \frac{nz}{2\pi\mathrm{i}}\int_l\rho(t)\frac{t^{n-2}\mathrm{d}t}{t^n-z^n} + \sum_{j=1}^m\frac{B_jz}{z^n-z_j^n} + B_{m+1}z, \tag{2.24}$$

$$\psi(z) = \frac{nz^{n-1}}{2\pi\mathrm{i}}\int_l\frac{\overline{\rho(t)}\mathrm{d}t}{t^n-z^n} - \frac{nz^{n-1}}{2\pi\mathrm{i}}\int_l\frac{\bar{t}\rho'(t)}{t^n-z^n}\mathrm{d}t + \sum_{j=1}^m\frac{B_jz^{n-1}}{z^n-z_j^n} + \frac{B_{m+1}}{z}. \tag{2.25}$$

同时，由于

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_L\rho(t)\mathrm{d}\log\frac{t-t_0}{\bar{t}-\bar{t}_0} &= \sum_{k=0}^{n-1}\frac{1}{2\pi\mathrm{i}}\int_l\rho(t)\left\{\frac{\omega^{2k}\mathrm{d}t}{\omega^kt-t_0} - \frac{\mathrm{d}\,\bar{t}}{\overline{\omega}^k\,\bar{t}-\bar{t}_0}\right\}\\ &= \frac{nt_0}{2\pi\mathrm{i}}\int_l\rho(t)\frac{t^{n-2}\mathrm{d}t}{t^n-t_0^n} - \frac{n}{2\pi\mathrm{i}}\int_l\rho(t)\frac{\bar{t}_0^{n-1}\mathrm{d}\,\bar{t}}{\bar{t}^n-\bar{t}_0^n},\end{aligned} \tag{2.26}$$

---

① 注意，如果不要求条件(2.12)成立，不能断定方程(2.19)′有唯一解.

$$\frac{1}{2\pi\mathrm{i}}\int_L\overline{\rho(t)}\mathrm{d}\frac{t-t_0}{\bar{t}-\bar{t}_0}=-\frac{1}{2\pi\mathrm{i}}\int_L\frac{t-t_0}{\bar{t}-\bar{t}_0}\mathrm{d}\overline{\rho(t)}=-\sum_{k=0}^{n-1}\frac{1}{2\pi\mathrm{i}}\int_l\frac{\omega^k t-t_0}{\bar{t}-\omega^k\bar{t}_0}\mathrm{d}\overline{\rho(t)}$$

$$=\frac{n}{2\pi\mathrm{i}}\int_l\frac{t_0\bar{t}^{n-1}-t\cdot\bar{t}_0^{n-1}}{\bar{t}^n-\bar{t}_0^n}\mathrm{d}\overline{\rho(t)}, \tag{2.27}$$

所以方程(2.19),(2.19)′可相应改变. 但为了使积分路径以及自变量 t_0 均限在 l 上，我们作变换

$$\rho_1(t)=\frac{\rho(t)}{t},$$

于是 $\rho_1(\omega t)=\rho_1(t)$. 又令

$$\varphi_1(z)=\frac{\varphi(z)}{z},\quad \psi_1(z)=\frac{\psi(z)}{z^{n-1}}, \tag{2.28}$$

则它们也以 ω 为循环周期. 再作保形变换

$$\zeta=z^n,$$

并记 $\tau=t^n$，$\tau_0=t_0^n$. 设 l 的像为 γ，S 或 S_0 的像为 Σ，于是 $\varphi_1(z)=\varphi_*(\zeta)$，$\psi_1(z)=\psi^*(\zeta)$在 Σ 中全纯，$\rho_1(t)=\rho_*(\tau)$也是 γ 上的单值函数. 这时，由(2.24),(2.25)，并注意到 $\rho'(t)=\rho_*(\tau)+n\tau\rho_*'(\tau)$，则有

$$\varphi_*(\zeta)=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{\rho_*(\tau)\mathrm{d}\tau}{\tau-\zeta}+\sum_{j=1}^m\frac{B_j}{\zeta-\zeta_j}+B_{m+1}, \tag{2.29}$$

$$\psi^*(\zeta)=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{[\overline{\rho^*(\tau)}-\rho_*(\tau)]|\tau|^{\frac{2}{n}}}{\tau(\tau-\zeta)}\mathrm{d}\tau-\frac{n}{2\pi\mathrm{i}}\int_\gamma\frac{\rho_*'(\tau)|\tau|^{\frac{2}{n}}}{\tau-\zeta}\mathrm{d}\tau$$
$$+\sum_{j=1}^m\frac{B_j}{\zeta-\zeta_j}+\frac{B_{m+1}}{\zeta}, \tag{2.30}$$

这里已令 $\zeta_j=z_j^n$. 这时，由(2.26),(2.27)，方程(2.19)成为

$$\begin{aligned}\boldsymbol{K}_*\rho_*\equiv{}&\rho_*(\tau_0)+\frac{1}{2\pi\mathrm{i}}\int_\gamma\rho_*(\tau)\mathrm{d}\log\frac{\tau-\tau_0}{\bar{\tau}-\bar{\tau}_0}+\frac{1}{2\pi\mathrm{i}}\int_\gamma\rho_*(\tau)\left(1-\frac{\bar{\tau}_0|\tau|^{\frac{2}{n}}}{\bar{\tau}|\tau_0|^{\frac{2}{n}}}\right)\frac{\mathrm{d}\bar{\tau}}{\bar{\tau}-\bar{\tau}_0}\\&-\frac{n\bar{\tau}_0}{|\tau_0|^{\frac{2}{n}}}\frac{1}{2\pi\mathrm{i}}\int_\gamma\overline{\rho_*(\tau)}\mathrm{d}\frac{|\tau|^{\frac{2}{n}}-|\tau_0|^{\frac{2}{n}}}{\bar{\tau}-\bar{\tau}_0}\\&+\frac{\bar{\tau}_0}{|\tau_0|^{\frac{2}{n}}}\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{|\tau|^{\frac{2}{n}}-|\tau_0|^{\frac{2}{n}}}{\bar{\tau}(\bar{\tau}-\bar{\tau}_0)}\overline{\rho_*(\tau)}\mathrm{d}\bar{\tau}-\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{\overline{\rho_*(\tau)}}{\bar{\tau}}\mathrm{d}\bar{\tau}\\&+\sum_{j=0}^m B_j\left(\frac{1}{\tau_0-\zeta_j}-\frac{1}{\bar{\tau}-\bar{\zeta}_j}-\frac{n\bar{\tau}_0}{(\bar{\tau}_0-\bar{\zeta}_j)^2}+\frac{\bar{\tau}_0}{(\bar{\tau}_0-\bar{\zeta}_j)|\tau_0|^{\frac{2}{n}}}\right)\\&+B_{m+1}\left(2+\frac{1}{|\tau_0|^{\frac{2}{n}}}\right)-D_*(\tau_0)\\={}&f_*(\tau_0),\end{aligned} \tag{2.31}$$

这里已令 $f_*(\tau)=f_1(t)=\dfrac{f^0(t)}{t}$，而

$$D_*(\tau_0)=\begin{cases}0, & \text{当 } \tau_0\in\gamma_0+\gamma_{m+1},\\ -\dfrac{1}{\tau_0^{\frac{1}{n}}}\displaystyle\int_{\gamma_j}\rho_*(\tau)\tau^{\frac{1}{n}}\mathrm{d}\sigma, & \text{当 } \tau_0\in\gamma_j\ (j=1,\cdots,m),\end{cases} \tag{2.32}$$

其中 γ_j 是 l_j 的像，σ 是 γ_j 上的弧长，且 $\tau^{\frac{1}{n}}$，$\tau_0^{\frac{1}{n}}$的辐角都取在 $-\pi$ 与 π 之间. 而 B_j 由(2.21)

要理解为

$$B_j = \mathrm{i}\int_{\gamma_j} \rho_*(\tau)\frac{|\tau|^{\frac{2}{n}}}{\tau}\mathrm{d}\tau + \overline{\rho^*(\tau)}\frac{|\tau|^{\frac{2}{n}}}{\overline{\tau}}\mathrm{d}\,\overline{\tau}. \tag{2.33}$$

这时，方程(2.19)′就成为

$$\hat{\boldsymbol{K}}_*\rho_* \equiv \boldsymbol{K}_*\rho_* + B_0\frac{\overline{\tau_0}}{(\overline{\tau_0}-\overline{\zeta_0})\,|\tau_0|^{\frac{2}{n}}} = f_*(\tau_0),\quad \tau_0\in\gamma, \tag{2.31$'$}$$

其中

$$B_0 = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_0}\frac{\rho_*(\tau)}{\tau}\mathrm{d}\tau + \frac{\overline{\rho_*(\tau)}}{\overline{\tau}}\mathrm{d}\,\overline{\tau}. \tag{2.34}$$

这样，问题就化为了求解 Fredholm 积分方程(2.31)′(其解一定存在唯一). 求出$\rho_*(\tau)$后通过(2.29),(2.30)便得所要求的函数. ①

如果L_{m+1}不存在，只要在以上讨论中去掉有关B_{m+1}的项便可. 如果L_0不存在，则可先化为$\Gamma=\Gamma'=0$的情况，然后去掉上面有关B_0的项及其讨论，结论便成立.

§3 第二基本问题

第二基本问题(已知边界位移)可以直接用 Шерман 的表达式，因此我们只作简单说明. 这时边值条件为

$$\kappa\varphi(t) - t\,\overline{\varphi'(t)} - \overline{\psi(t)} = g(t),\quad t\in L, \tag{3.1}$$

中

$$g(t) = 2\mu[u(t)+\mathrm{i}v(t)]$$

为已知函数，设$g'(t)\in H$，且$g(\omega t)=\omega g(t)$.

暂时仍设L_0, L_{m+1}都存在. 令

$$\left.\begin{aligned}
\varphi(z) &= \frac{1}{2\pi\mathrm{i}}\int_L\frac{\rho(t)\mathrm{d}t}{t-z} + \sum_{j,k}A_{jk}\log(z-\omega^k z_j) + A_0\log z,\\
\psi(z) &= -\frac{\kappa}{2\pi\mathrm{i}}\int_L\frac{\overline{\rho(t)}\mathrm{d}t}{t-z} - \frac{1}{2\pi\mathrm{i}}\int_L\frac{\bar{t}\rho'(t)\mathrm{d}t}{t-z}\\
&\quad - \sum_{j,k}\overline{A_{jk}}\log(z-\omega^k z_j) + \overline{A_0}\log z,
\end{aligned}\right\} \tag{3.2}$$

其中A_{jk}, A_0为待定常数，且已令

$$A_{jk} = n\int_{L_{jk}}\rho(t)\,|t|^{n-1}\mathrm{d}s,\quad A_0 = \int_{L_0}\rho(t)\mathrm{d}s.\text{②} \tag{3.3}$$

于是就有关于$\rho(t)$的 Fredholm 方程:

$$\boldsymbol{K}\rho \equiv \kappa\rho(t_0) + \frac{\kappa}{2\pi\mathrm{i}}\int_L\rho(t)\mathrm{d}\log\frac{t-t_0}{\bar{t}-\bar{t}_0} + \frac{1}{2\pi\mathrm{i}}\int_L\overline{\rho(t)}\,d\,\frac{t-t_0}{\bar{t}-\bar{t}_0}$$

---

① 注意，也可求解(2.31)；它也一定可解，但解不一定唯一. 求出其一个解$\rho_*^0(\tau)$后，则显然$\rho_*^0(\omega^k\tau)$也是解，于是$\rho(\tau)=\sum_{k=0}^{n-1}\rho_*^0(\omega^k\tau)$便是所求的解.

② 显然$A_0=0$，但我们暂不作此假设.

$$+2\kappa\sum_{j,k}A_{jk}\ln|t_0-\omega^k z_j|+2\kappa A_0\ln|t_0|$$
$$-t_0\sum_{j,k}\frac{\overline{A_{jk}}}{\overline{t_0}-\omega^k\overline{z_j}}-\frac{t_0\overline{A_0}}{\overline{t_0}}=g(t_0),\quad t_0\in L. \tag{3.4}$$

由 Шерман 的原来结果，这方程有唯一解. 由于

$$A_{j,k+1}=n\int_{L_{j,k+1}}\rho(t)|t^{n-1}\mathrm{d}s=n\int_{L_{jk}}\rho(\omega t)|t^{n-1}|\mathrm{d}s,$$
$$A_0=\int_{L_0}\rho(\omega t)\mathrm{d}s,$$

所以显然，如果 $\rho(t)$ 是(3.4)的解，则 $\frac{\rho(\omega t)}{\omega}$ 也是它的解. 于是这里(2.12)仍告满足. 因此:

$$A_{jk}=\omega^k A_j,\quad A_j=A_{j0}=n\int_{L_j}\rho(t)|t|^{n-1}\mathrm{d}s,$$

而 $A_0=\omega A_0$，从而 $A_0=0$. 这样，方程(3.4)成为

$$\begin{aligned}K_0\rho\equiv\kappa\rho(t_0)&+\frac{\kappa}{2\pi\mathrm{i}}\int_L\rho(t)\mathrm{d}\log\frac{t-t_0}{\overline{t}-\overline{t_0}}+\frac{1}{2\pi\mathrm{i}}\int_L\overline{\rho(t)}\mathrm{d}\frac{t-t_0}{\overline{t}-\overline{t_0}}\\&+2\kappa\sum_{j,k}A_j\omega^k\ln|t_0-\omega^k z_j|-t_0\sum_j\frac{\overline{A_j}\,\overline{t_0}^{\,n-1}}{\overline{t_0}^{\,n}-\overline{z_j}^{\,n}}\\&=g(t_0).\end{aligned} \tag{3.5}$$

再就可像§2中那样，把方程(3.5)化为 γ 上的 Fredholm 积分方程.

如果 L_0 不存在，则可按[6],定理 3.12.2 的方法加以改变，不在此详述. 如果 L_{m+1} 不存在，则不会引起任何困难.

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ON MATHEMATICAL PROBLEMS OF ELASTIC PLANES WITH CYCLIC SYMMETRY

Summary

Here we discuss the first and the second fundamental problems of elastic planes with

cyclic symmetry (or, cyclic periodicity). For the shape of the region of elasticity and the boundary values assigned to it, we make no restriction except cyclic symmetry. The Sherman-Lauricella's equation is simplified so that the number of terms in it, representing the undetermined constants, is reduced to the minimum. Thereby the lines of integration of the equation may be reduced to those boundaries of the elastic region presented in a periodic angular sector.

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关于周期应力平面弹性基本问题

提　要

本文考虑了一般的平面弹性问题，只假定应力是周期的而且有界. 除此之外，无论对一个周期带中所含孔的个数或其边界的形状以及对孔边或无穷远处的应力，都不作其他限制. 文中把 Колосов 函数的多值部分与非周期部分分离出来，得到了它们的一般表达式；证明了这时位移必定是准周期的，并指出了第一、第二基本问题的一般提法. 对于有周期直线裂缝（在与周期方向平行的一直线上）的情况，本文利用周期 Riemann 边值问题的解法作出了解答；并对一个周期带中只有一个裂缝的特殊情况，把解写成了完全确定的有限形式.

关于周期应力平面弹性问题，过去已有过不少工作. 其中关于周期圆孔的情形，研究的人更多，例如文献 [1 ~ 3]. 对于任意形状周期孔的情况，Савин[4]最初作了研究，并把问题化为 Fredholm 积分方程（或参看文献 [5]）. 所有这些工作，或者孔形比较特殊，或者边值条件比较特殊，且在一个周期带内只假定有一个孔，对位移的可能情况也很少讨论. 这方面的工作情况可参看综合论文 [6]. 本文则只假定了应力是周期的并且有界，通过找出 Колосов 函数的一般表达式，证明了这时位移总是准周期的，并指出了第一、第二基本问题的正确提法.

关于平面中有双周期排列的孔的类似问题，过去也曾有过研究[7,3]. Koiter 指出，当一个周期趋于无穷时，问题便成为单周期情况. 但是，在双周期情况下，一个周期四边形中孔边外应力主矢量必然为零，并且也不产生无穷远处的应力问题. 因此，双周期问题并不能概括单周期情况.

此外，本文用前文 [9] 中的结果，求解了弹性平面中有周期直线裂缝时的基本问题，所用方法系由 Мусхелишвили 的方法（见文献 [10]，§ 120）转化而来.

一、Колосов 函数的一般表达式

设弹性平面中有一列以 $a\pi$ 为周期的孔，其边界为 L_j，$j=0, \pm 1, \pm 2, \cdots$，并且每一 L_j 中含有 n 条逐段光滑封闭曲线 $l_k^{(j)}$，$k=1,2,\cdots,n$（图 1）；而对于同一 k，$l_k^{(j)}$（$j=0, \pm 1, \pm 2, \cdots$）是周期地排列的. 把 $l_k^{(0)}$ 简记为 l_k. 取所有曲线的反时针方向为正向，把弹性体所占区域记作 S^-，把 $l_k^{(j)}$ 所围区域记作 $S_k^{(j)+}$. $S_k^{(0)+}$ 则简记为 S_k^+，并记 $S_0^+=\sum_{k=1}^{n} S_k^+$. 带形区域 $|x|<\frac{1}{2}a\pi$ 记作 S_0，并记 $S_0^-=S_0-S_0^+$.

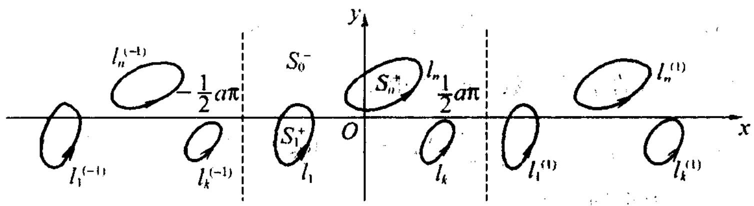

图 1

记S^-中任何点$z=x+\mathrm{i}y$处的应力状态为$\sigma_x,\sigma_y,\tau_{xy}$，位移为$D(z)=u+\mathrm{i}v$，大家知道，它们可表示成

$$\sigma_x+\sigma_y=2[\Phi(z)+\overline{\Phi(z)}],\tag{1.1}$$

$$\sigma_y-\sigma_x+2\mathrm{i}\tau_{xy}=2[\bar{z}\Phi'(z)+\Psi(z)],\tag{1.2}$$

$$2\mu D=2\mu(u+\mathrm{i}v)=\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)},\tag{1.3}$$

其中μ为弹性体的剪切弹性模数，κ是与Poisson比σ有关的一个常数($1<\kappa<3$)；而$\varphi(z)$，$\psi(z)$是S^-中(一般是多值的)解析函数，并且$\Phi(z)=\varphi'(z)$，$\Psi(z)=\psi'(z)$在S^-中全纯.

我们假定：应力是以$a\pi$为周期的，且在$z=\pm\infty\mathrm{i}$处保持有界. 根据这个假定，只需指出L_0上的外应力就足以说明所有孔边上的应力. 把l_k上的外应力主矢量记作$X_k+\mathrm{i}Y_k$，这时L_0上的外应力主矢量就是

$$X+\mathrm{i}Y=\sum_{k=1}^{n}(X_k+\mathrm{i}Y_k).\tag{1.4}$$

把$z=\pm\infty\mathrm{i}$处的应力记为

$$\sigma_\pm=\sigma_y(\pm\infty\mathrm{i}),\quad \tau_\pm=\tau_{xy}(\pm\infty\mathrm{i}),\quad h_\pm=\sigma_x(\pm\infty\mathrm{i}).\tag{1.5}$$

由应力周期性条件，考察周期带S_0^-上的平衡，立刻可知

$$\sigma_--\sigma_+=Y/a\pi,\quad \tau_--\tau_+=X/a\pi.\tag{1.6}$$

此外需注意，因为应力在$z=\pm\infty\mathrm{i}$处有界，由式(1.1)可知，$\mathrm{Re}\,\Phi(z)$在$z=\pm\infty\mathrm{i}$处有界. 设$\zeta=\tan(z/a)$时，$\Phi(z)$变成$\Phi_*(\zeta)$，于是$\mathrm{Re}\,\Phi_*(\zeta)$在$\zeta=\pm\mathrm{i}$处有界，由此可见$\zeta=\pm\mathrm{i}$是$\Phi_*(\zeta)$的正则点，从而$\Phi(z)$在$z=\pm\infty\mathrm{i}$处有界. 又因

$$\Phi'(z)=\Phi_*'(\zeta)\frac{\mathrm{d}\zeta}{\mathrm{d}z}=\frac{\Phi_*'(\zeta)}{a\cos^2\dfrac{z}{a}},$$

所以

$$\lim_{z\to\pm\infty\mathrm{i}}z\Phi'(z)=0.\tag{1.7}$$

又由式(1.2)可知，$\Psi(\pm\infty\mathrm{i})$也是有界的.

我们希望找出$\varphi(z),\psi(z)$的一般表达式，使它们的多值部分和非周期部分都分离出来.

在每一S_k^+内任取一点z_k. 大家知道，$\varphi(z)$绕过洞$S_k^{(j)+}$的多值性部分是

$$-\frac{1}{2\pi(\kappa+1)}(X_k+\mathrm{i}Y_k)\log(z-z_k-ja\pi).$$

为了分离出$\varphi(z)$的整个多值部分，这里不能直接把上式对k,j求和，因为这样会得到发散级数. 由于把上式中的对数因子改为

$$\log\left(1-\frac{z-z_k}{ja\pi}\right)\quad (j\neq 0)$$

时并不改变多值特征，于是，利用$\sin[(z-z_k)/a]$的无穷乘积表示式，立刻知道，

$$\varphi(z)=-\frac{1}{2\pi(\kappa+1)}\sum_{k=1}^{n}(X_k+iY_k)\log\sin\frac{z-z_k}{a}+\varphi^*(z),\tag{1.8}$$

其中$\varphi^*(z)$已是S^-内的全纯函数. 同理可得

$$\psi(z)=\frac{\kappa}{2\pi(\kappa+1)}\sum_{k=1}^{n}(X_k-iY_k)\log\sin\frac{z-z_k}{a}+\psi^*(z),\tag{1.9}$$

其中$\psi^*(z)$也在S^-内全纯. 从而

$$\Phi(z)=-\frac{1}{2a\pi(\kappa+1)}\sum_{k=1}^{n}(X_k+iY_k)\cot\frac{z-z_k}{a}+\Phi^*(z),\tag{1.10}$$

$$\Psi(z)=\frac{\kappa}{2a\pi(\kappa+1)}\sum_{k=1}^{n}(X_k-iY_k)\cot\frac{z-z_k}{a}+\Psi^*(z),\tag{1.11}$$

这里已令$\Phi^*(z)=\varphi^{*\prime}(z)$, $\Psi^*(z)=\psi^{*\prime}(z)$.

根据文献[5], § 55 中的结果，由应力的周期性可知，

$$\Phi(z+a\pi)=\Phi(z)+i\alpha,\quad \Psi(z+a\pi)=\Psi(z)-a\pi\Phi'(z),$$

其中α是某实数. 但由于$\Phi(\pm\infty i)$有界，立刻知道$\alpha=0$. 因此对应地便有

$$\begin{aligned}\Phi^*(z+a\pi)&=\Phi^*(z),\\ \Psi^*(z+a\pi)&=\Psi^*(z)-a\pi\Phi'(z)\\ &=\Psi^*(z)-\frac{1}{2a(\kappa+1)}\sum_{k=1}^{n}\frac{X_k+iY_k}{\sin^2\frac{z-z_k}{a}}-a\pi\Phi^{*\prime}(z).\end{aligned}$$

把这两式积分，注意φ^*, ψ^*都是单值的，便有

$$\left.\begin{aligned}\varphi^*(z+a\pi)&=\varphi^*(z)+a\pi\beta,\\ \psi^*(z+a\pi)&=\psi^*(z)-a\pi\varphi'(z)+\gamma\\ &=\psi^*(z)+\frac{1}{2(\kappa+1)}\sum_{k=1}^{n}(X_k+iY_k)\cot\frac{z-z_k}{a}-a\pi\Phi^*(z)+\gamma,\end{aligned}\right\}\tag{1.12}$$

其中β,γ是某二复数.

把式(1.3)中的z改为$z+a\pi$，再与式(1.3)相减，以式(1.9),(1.10)代入，并利用式(1.12)，便得

$$2\mu[D(z+a\pi)-D(z)]=a\pi\kappa\beta-\bar{\gamma}.$$

如果不计弹性体的刚性旋转，则还可认为上式右边是一实数. 由此可见，这时位移必定是准周期的，即

$$2\mu[u(z+a\pi)-u(z)]=a\pi q,\quad v(z+a\pi)-v(z)=0,\tag{1.13}$$

其中q为某实常数. 由前式可知，

$$\gamma=a\pi(\kappa\bar{\beta}-q).\tag{1.14}$$

这样，式(1.12)就成为

$$\begin{aligned}\varphi^*(z+a\pi)&=\varphi^*(z)+a\pi\beta,\\ \psi^*(z+a\pi)&=\psi^*(z)-a\pi\varphi'(z)+a\pi(\kappa\bar{\beta}-q).\end{aligned}$$

再令

$$\left.\begin{aligned}\varphi_0(z)&=\varphi^*(z)-\beta z,\\ \psi_0(z)&=\psi^*(z)+z\varphi'(z)-(\kappa\bar{\beta}-q)z,\end{aligned}\right\}\tag{1.15}$$

易见它们已是以$a\pi$为周期的在S^-中全纯的函数.

现在利用 $z=\pm\infty i$ 处的应力来决定 β 和 q. 由式(1.10),(1.11),

$$\Phi(z)=-\frac{1}{2a\pi(\kappa+1)}\sum_{k=1}^{n}(X_k+iY_k)\cot\frac{z-z_k}{a}+\varphi_0'(z)+\beta, \tag{1.16}$$

$$\begin{aligned}\Psi(z)=&\frac{\kappa}{2a\pi(\kappa+1)}\sum_{k=1}^{n}(X_k-iY_k)\cot\frac{z-z_k}{a}+\psi_0'(z)\\&+\frac{1}{2a\pi(\kappa+1)}\sum_{k=1}^{n}(X_k+iY_k)\left(\cot\frac{z-z_k}{a}-\frac{z}{a\sin^2\dfrac{z-z_k}{a}}\right)\\&-\varphi_0'(z)-z\varphi_0''(z)-(\beta-\kappa\bar{\beta}+q).\end{aligned} \tag{1.17}$$

由于 $\Phi(\pm\infty i),\Psi(\pm\infty i)$ 有界，所以 $\varphi_0'(z)=\Phi_0(z)$, $\psi_0'(z)=\Psi_0(z)$ 也有界. 再由式(1.8)，便知道

$$\lim_{z\to\pm\infty i}z\Phi_0'(z)=0. \tag{1.18}$$

此外，注意 $\varphi_0(z)$ 是周期函数，故

$$0=\frac{1}{a\pi}[\varphi_0(z+a\pi)-\varphi_0(z)]=\frac{1}{a\pi}\int_z^{z+a\pi}\Phi_0(z)dz=\Phi_0(\pm\infty i),$$

其中最后的等式是这样得到的：把积分路径看成从 z 到 $z+a\pi$ 的直线段，把 $\Phi_0(z)$ 的实部与虚部分开，分别应用积分中值定理，然后令 $z\to\pm\infty i$ 即得. 同理，$\Psi_0(\pm\infty i)=0$. 由此也易知 $\varphi_0(\pm\infty i),\psi_0(\pm\infty i)$ 有界.

在式(1.1),(1.2) 中消去 σ_x，并令 $z\to\pm\infty i$，由以上说明以及式(1.16),(1.17)，便得

$$\sigma_{\pm}+i\tau_{\pm}=\mp\frac{Y-iX}{2a\pi}+(\kappa+1)\bar{\beta}-q. \tag{1.19}$$

把这两式相减，便再次得到平衡条件(1.6)；把这两式相加，取共轭值，得

$$(\kappa+1)\beta-q=\frac{\sigma_-+\sigma_+}{2}-i\frac{\tau_-+\tau_+}{2}. \tag{1.20}$$

另一方面，由式(1.16),(1.17)，有

$$\Phi(\pm\infty i)=\mp\frac{X-iY}{2a\pi(\kappa+1)}+\beta.$$

于是由式(1.1) 立刻知道，

$$h_{\pm}+\sigma_{\pm}=\mp\frac{2Y}{a\pi(\kappa+1)}+2(\beta+\bar{\beta}). \tag{1.21}$$

由此式可以看出，$h_{\pm}$ 不是任意的，必须满足条件

$$h_++\sigma_++\frac{2Y}{a\pi(\kappa+1)}=h_-+\sigma_--\frac{2Y}{a\pi(\kappa+1)},$$

再由式(1.6) 可以看出，也就是必须满足条件

$$h_--h_+=\frac{3-\kappa}{\kappa+1}\frac{Y}{a\pi}. \tag{1.22}$$

最后，由式(1.20),(1.21) 便容易算出

$$q=\frac{1}{4}[(\kappa+1)h_+-(3-\kappa)\sigma_+]=\frac{1}{4}[(\kappa+1)h_--(3-\kappa)\sigma_-]. \tag{1.23}$$

再把它代回式(1.20)，便可求得

$$\beta=\frac{1}{4}(h_++\sigma_+)+\frac{Y-iX}{2a\pi(\kappa+1)}-\frac{i\tau_+}{\kappa+1}$$

$$= \frac{1}{4}(h_- + \sigma_-) - \frac{Y - iX}{2a\pi(\kappa+1)} - \frac{i\tau_-}{\kappa+1}. \tag{1.24}$$

这样，β, q 都由 $z = \pm\infty i$ 处的应力状况和一个周期带内孔边外应力主矢量决定，而与孔边外应力分布状况无关.

把以上结果代回式(1.8)，(1.9)，最终便得

$$\varphi(z) = -\frac{1}{2\pi(\kappa+1)}\sum_{k=1}^{n}(X_k + iY_k)\log\sin\frac{z - z_k}{a} + \beta z + \varphi_0(z), \tag{1.25}$$

$$\begin{aligned}\psi(z) = &\frac{\kappa}{2\pi(\kappa+1)}\sum_{k=1}^{n}(X_k - iY_k)\log\sin\frac{z-z_k}{a}\\ &+\frac{z}{2a\pi(\kappa+1)}\sum_{k=1}^{n}(X_k + iY_k)\cot\frac{z-z_k}{a}\\ &+(\kappa\bar{\beta} - \beta + q)z - z\varphi_0'(z) + \psi_0(z),\end{aligned} \tag{1.26}$$

其中 q, β 分别由式(1.23)，(1.24) 给出，而 $\varphi_0(z), \psi_0(z)$ 是两个以 $a\pi$ 为周期的在 S^- 中全纯的函数. 这样，我们便分离出了 φ, ψ 的多值部分与非周期部分.

文献[11]，§10 也给出了周期应力条件下 φ, ψ 的一般表达式，但没有分离出多值部分；又由于没有假定应力有界，表达式中一些常数间的联系也就没有了.

二、基本问题的提法

第一基本问题 已知孔边 L 上的周期应力 $X_n(t) + iY_n(t)$ 以及在 $z = -\infty i$（或 $z = +\infty i$）处的应力状态，求弹性平衡. 这是在应力为周期并且有界的条件下问题的最一般的提法. 这时，由式(1.6)，(1.22) 知，在 $z = +\infty i$（或 $z = -\infty i$）处的应力也是已知的，从而由式(1.23)，(1.24)，β 和 q 也是已知的.

这问题可化为下列边值问题：

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) + C(t), \quad t \in L, \tag{2.1}$$

其中 $C(t)$ 在各孔边上是待定常数，而

$$f(t) = -i\int_0^s (X_n + iY_n)ds^{①}, \quad t \in L.$$

把式(1.25)，(1.26) 代入式(2.1)，便得

$$\varphi_0(t) + (t - \bar{t})\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = f_0(t) + C(t), \tag{2.2}$$

其中已令

$$\begin{aligned}f_0(t) = &\frac{1}{2\pi(\kappa+1)}\sum_{k=1}^{n}(X_k + iY_k)\log\sin\frac{t-z_k}{a}\\ &-\frac{\kappa}{2\pi(\kappa+1)}\sum_{k=1}^{n}(X_k + iY_k)\overline{\log\sin\frac{t-z_k}{a}}\\ &+\frac{t-\bar{t}}{2a\pi(\kappa+1)}\sum_{k=1}^{n}(X_k + iY_k)\overline{\cot\frac{t-z_k}{a}}\\ &-(\beta + \bar{\beta})t - (\kappa\beta - \bar{\beta} + q)\bar{t} + f(t).\end{aligned} \tag{2.3}$$

① 与文献[10]，§41 相比较，等式右边多了一个负号，这是因为弹性体在 L 正向右侧的缘故.

这里$f_0(t)$显然已是L上的单值函数. 至于$C(t)$，当$t\in L_k^{(j)}$时，所取常数不都是独立的，它们间的关系如何要看式(2.3)中各对数所取的分支间的关系而定. 例如，先取定l_k上的$\log\sin[(t-z_k)/a]$的一确定分支，然后认为，当$t+ja\pi\in l_k^{(j)}$时，

$$\log\sin\frac{t+ja\pi-z_k}{a}=\log\sin\frac{t-z_k}{a}+j\pi i,$$

则由式(2.3)易见，

$$f_0(t+a\pi)-f_0(t)=-\frac{Y-iX}{2}-[(\kappa+1)\beta+q]a\pi,$$

于是，

$$\begin{aligned}C(t+a\pi)-C(t)&=\frac{Y-iX}{2}+[(\kappa+1)\beta+q]a\pi\\&=a\pi(\sigma_--i\tau_-+2q).\end{aligned}$$

总之，式(2.1)中待定常数实际上只有n个：相应于每一l_k $(k=1,\cdots,n)$上有一个. 此外，还要注意，在解(2.2)时，应该要求$\varphi_0(\pm\infty i)$,$\psi_0(\pm\infty i)$有界.

上述边值问题可化为Fredholm积分方程求解，方法见文献[5]，§55，不过那里假定了一周期带中只有一孔. 对于多孔情况，可采用文献[10]，§102处理非周期情况时的方法.

第一基本问题还有另一种提法，即除已知X_n+iY_n外，还已知σ_-,τ_-(或σ_+,τ_+)以及q，求弹性平衡. 因为由式(1.23)，立刻可算出$h_\pm$，所以结果还是一样. 例如，如果要求位移也是周期的，即$q=0$，则用不着解边值问题，便可确定$h_\pm$.

第二基本问题 已知各周期孔边的相对位移以及一个周期带内孔边的外应力主矢量$X+iY$，还知道$z=-\infty i$（或$z=+\infty i$）处的应力，求弹性平衡. 或者，已知各周期孔边的相对位移及常数q，以及$X+iY,\sigma_-,\tau_-$(或σ_+,τ_+)，求弹性平衡. 这种问题也可以像第一基本问题那样化为积分方程求解.

三、有水平裂缝的周期应力平面弹性问题

设周期裂缝都在x轴上，而在一个周期带$|x|<\frac{1}{2}a\pi$内，有n条裂缝

$$l_k:\ a_k\leqslant t\leqslant b_k\ (a_{k+1}>b_k),\quad k=1,\cdots,n-1.$$

自a_k到b_k的方向取作l_k的正向，仍记$L_0=\sum\limits_{k=1}^{n}l_k$. l_k及其周期合同裂缝$l_k^{(j)}$上的外应力主矢量记为X_k+iY_k. 现在弹性域记作S，其他记号同前.

对于一般周期曲线裂缝问题，表达式(1.25),(1.26)并不适用，其一般讨论将另文给出. 在现在的特殊情况下，可用文献[10]，§120中的方法类似地处理如下.

引进函数

$$\left.\begin{aligned}\omega(z)&=z\Phi(z)+\bar{\psi}(z),\\\Omega(z)&=\omega'(z)=\Phi(z)+z\Phi'(z)+\bar{\Psi}(z),\end{aligned}\right\}\tag{3.1}$$

易见

$$\sigma_y-i\tau_{xy}=\Phi(z)+\Omega(\bar{z})+(z-\bar{z})\overline{\Phi'(z)}.\tag{3.2}$$

由 $\Phi(z)$ 的周期性，可知 $\Omega(z)$ 也以 $a\pi$ 为周期.

在各 l_k 的端点附近，我们假定 $\Phi(z),\Omega(z)$ 至多无界可积，且

$$\lim_{z\to t} y\Phi'(z)=0\quad (z=x+\mathrm{i}y,\ z\in S,\ t\in L). \tag{3.3}$$

在 l_k(或其周期合同裂缝) 上，显然有

$$\left.\begin{aligned} X_k&=\int_{a_k}^{b_k}[\tau_{xy}^-(t)-\tau_{xy}^+(t)]\mathrm{d}t,\\ Y_k&=\int_{a_k}^{b_k}[\sigma_y^-(t)-\sigma_y^+(t)]\mathrm{d}t,\end{aligned}\quad k=1,\cdots,n,\right\} \tag{3.4}$$

其中 $\sigma_y^\pm+\mathrm{i}\tau_{xy}^\pm$ 表示在 l_k 上岸和下岸的应力. 于是，

$$\left.\begin{aligned} X&=\int_{L_0}[\tau_{xy}^-(t)-\tau_{xy}^+(t)]\mathrm{d}t,\\ Y&=\int_{L_0}[\sigma_y^-(t)-\sigma_y^+(t)]\mathrm{d}t.\end{aligned}\right\} \tag{3.5}$$

由式(1.16),(1.17) 以及 $\Phi_0(\pm\infty\mathrm{i})=\Psi_0(\pm\infty\mathrm{i})=0$①，易见

$$\left.\begin{aligned} \Phi(\pm\infty\mathrm{i})&=\mp\frac{Y-\mathrm{i}X}{2a\pi(\kappa+1)}+\beta,\\ \Psi(\pm\infty\mathrm{i})&=\mp\frac{(\kappa-1)Y+\mathrm{i}(\kappa+1)X}{2a\pi(\kappa+1)}-\beta+\kappa\bar\beta-q;\end{aligned}\right\} \tag{3.6}$$

再从式(3.1)，便知

$$\Omega(\pm\infty\mathrm{i})=\pm\frac{\kappa(Y-\mathrm{i}X)}{2a\pi(\kappa+1)}+\kappa\beta-q. \tag{3.7}$$

下面我们来考虑第一基本问题. 设已知 $\sigma_y^\pm(t),\tau_{xy}^\pm(t)$，且设它们都满足Hölder 条件. 又设 σ_-,τ_-,h_-，从而 $\sigma_+,\tau_+,h_+,\beta,q$ 均已知. 求弹性平衡. 这问题容易化为下列边值问题(见文献[10]，§120)：

$$[\Phi(t)+\Omega(t)]^+ + [\Phi(t)+\Omega(t)]^- = 2p(t), \tag{3.8}$$

$$[\Phi(t)-\Omega(t)]^+ - [\Phi(t)-\Omega(t)]^- = 2q(t), \tag{3.9}$$

其中已令

$$p(t)=\frac{1}{2}[\sigma_y^+(t)+\sigma_y^-(t)]-\frac{\mathrm{i}}{2}[\tau_{xy}^+(t)+\tau_{xy}^-(t)], \tag{3.10}$$

$$q(t)=\frac{1}{2}[\sigma_y^+(t)-\sigma_y^-(t)]-\frac{\mathrm{i}}{2}[\tau_{xy}^+(t)-\tau_{xy}^-(t)]. \tag{3.11}$$

由式(3.5) 知，

$$\int_{L_0} q(t)\mathrm{d}t=-\frac{1}{2}(Y-\mathrm{i}X). \tag{3.12}$$

先求式(3.8) 的解(参看文献[9])，得

$$\Phi(z)-\Omega(z)=\frac{1}{a\pi\mathrm{i}}\int_{L_0} q(t)\cot\frac{t-z}{a}\,\mathrm{d}t+2C, \tag{3.13}$$

其中 C 为某常数. 为了决定 C，在式(3.13) 中分别令 $z=\pm\infty\mathrm{i}$，相加，并利用式(3.6)，

① 在裂缝情况下，上节中有关 $z=\pm\infty\mathrm{i}$ 处的公式仍成立，这是因为可用封闭曲线围住裂缝，而围线上外应力主矢量与裂缝上的相同.

(3.7) 得到

$$C=\frac{1}{2}[q-(\kappa-1)\beta]. \tag{3.14}$$

再求式(3.9) 在 h_0 类中的解(按文献[9] 中记号)，得

$$\Phi(z)+\Omega(z)=\frac{X(z)}{a\pi\mathrm{i}}\int_{L_0}\frac{p(t)}{X^+(t)}\cot\frac{t-z}{a}\,\mathrm{d}t+2X(z)P_n\left(\tan\frac{z}{a}\right), \tag{3.15}$$

其中

$$P_n(\zeta)=C_0\zeta^n+\cdots+C_n \tag{3.16}$$

是系数待定的 n 次多项式，而

$$X(z)=\prod_{k=1}^{n}\left(\tan\frac{z}{a}-\tan\frac{a_k}{a}\right)^{-\frac{1}{2}}\left(\tan\frac{z}{a}-\tan\frac{b_k}{a}\right)^{-\frac{1}{2}}, \tag{3.17}$$

其中根式可任意取定一分支，例如

$$\lim_{z\to\pm\frac{a\pi}{2}}\tan^n\frac{z}{a}X(z)=1.$$

把式(3.13),(3.15) 相加相减，得

$$\left.\begin{aligned}\Phi(z)=&\frac{X(z)}{2a\pi\mathrm{i}}\int_{L_0}\frac{p(t)}{X^+(t)}\cot\frac{t-z}{a}\,\mathrm{d}t\\&+\frac{1}{2a\pi\mathrm{i}}\int_{L_0}q(t)\cot\frac{t-z}{a}\,\mathrm{d}t+X(z)P_n\left(\tan\frac{z}{a}\right)+C,\\\Omega(z)=&\frac{X(z)}{2a\pi\mathrm{i}}\int_{L_0}\frac{p(t)}{X^+(t)}\cot\frac{t-z}{a}\,\mathrm{d}t\\&-\frac{1}{2a\pi\mathrm{i}}\int_{L_0}q(t)\cot\frac{t-z}{a}\,\mathrm{d}t+X(z)P_n\left(\tan\frac{z}{a}\right)-C.\end{aligned}\right\} \tag{3.18}$$

剩下的问题是要决定 $C_0,\cdots,C_n$. 现在

$$2\mu(u+\mathrm{i}v)=\kappa\varphi(z)-\omega(\bar{z})-(z-\bar{z})\overline{\Phi(z)}+\text{const}.$$

利用位移单值性条件，便知

$$\begin{aligned}2\mu\{u+\mathrm{i}v\}_{l_k}&=\kappa\{\varphi(z)\}_{l_k}-\{\omega(\bar{z})\}_{l_k}\\&=\kappa\int_{l_k}\Phi(z)\mathrm{d}z-\int_{l_k}\Omega(\bar{z})\mathrm{d}\bar{z}=0,\quad k=1,\cdots,n.\end{aligned} \tag{3.19}$$

由此可得关于 $C_0,\cdots,C_n$ 的 n 个线性方程. 此外，利用位移的准周期性，例如，取 Λ_+ 和 Λ_- 是 z 到 $z+a\pi$ 的直线段，且 z 已分别充分接近于 $+\infty\mathrm{i}$ 和 $-\infty\mathrm{i}$，又可得两个方程：

$$\begin{aligned}2\mu\{u+\mathrm{i}v\}_z^{z+a\pi}&=\kappa[\varphi(z+a\pi)-\varphi(z)]-[\omega(\bar{z}+a\pi)-\omega(\bar{z})]\\&=\kappa\int_{\Lambda_\pm}\Phi(z)\mathrm{d}z-\int_{\Lambda_\mp}\Omega(z)\mathrm{d}z=a\pi q.\end{aligned} \tag{3.20}$$

由位移的单值性，立刻可知式(3.20) 中的一个方程可从另一个推出. 因此，式(3.19),(3.20) 中恰有 $n+1$ 个独立的方程，这就正好用来决定 $C_0,\cdots,C_n$(它们的可解性，从力学观点看很明显). 问题至此已完全解决.

作为一个特例，考察 $n=1$ 的情形：设 L_0 中只有一个线段 $-l\leqslant t\leqslant l$. 这时，式(3.18) 成为

$$\left.\begin{aligned}\Phi(z)&=\frac{1}{2a\pi\mathrm{i}\sqrt{R(z)}}\int_{-l}^{l}p(t)\sqrt{R(t)}\cot\frac{t-z}{a}\,\mathrm{d}t\\&\quad+\frac{1}{2a\pi\mathrm{i}}\int_{-l}^{l}q(t)\cot\frac{t-z}{a}\,\mathrm{d}t+\frac{C_0\tan\dfrac{z}{a}+C_1}{\sqrt{R(z)}}+C,\\\Omega(z)&=\frac{1}{2a\pi\mathrm{i}\sqrt{R(z)}}\int_{-l}^{l}p(t)\sqrt{R(t)}\cot\frac{t-z}{a}\,\mathrm{d}t\\&\quad-\frac{1}{2a\pi\mathrm{i}}\int_{-l}^{l}q(t)\cot\frac{t-z}{a}\,\mathrm{d}t+\frac{C_0\tan\dfrac{z}{a}+C_1}{\sqrt{R(z)}}-C,\end{aligned}\right\}\tag{3.18$'$}$$

其中

$$R(z)=\tan^2\frac{l}{a}-\tan^2\frac{z}{a},\tag{3.21}$$

且$\sqrt{R(z)}$已取定例如这样的一分支：

$$\lim_{z\to\pm\infty\mathrm{i}}\sqrt{R(z)}=\pm 1/\cos\frac{l}{a},\tag{3.22}$$

或即，当z在上半平面趋于$t\in L_0$时，$\sqrt{R(z)}$取正值的那一分支，亦即，在式(3.18)$'$中，$\sqrt{R(t)}$要理解为取正值.

为了决定C_0,C_1，最简单的办法是：在式(3.18)$'$的前一式中以$z=\pm\infty\mathrm{i}$代入，并相加相减，再以式(3.6),(3.12),(3.14)代入，并注意到式(1.20)，即得

$$C_0=-\frac{1}{2a\pi\mathrm{i}}\int_{-l}^{l}p(t)\sqrt{R(t)}\mathrm{d}t-\frac{(\tau_-+\tau_+)+\mathrm{i}(\sigma_-+\sigma_-)}{4\cos\dfrac{l}{a}},\tag{3.23}$$

$$C_1=\frac{\kappa-1}{\kappa+1}\,\frac{Y-\mathrm{i}X}{4a\pi\cos\dfrac{l}{a}}.\tag{3.24}$$

特别,在一个裂缝的周期情况下,如果$X=Y=0$,而记$\sigma_\infty=\sigma_\pm$,$\tau_\infty=\tau_\pm$,$h_\infty=h_\pm$,则

$$\beta=\frac{1}{4}(h_\infty+\sigma_\infty)-\frac{\mathrm{i}\tau_\infty}{\kappa+1},\quad q=\frac{\kappa+1}{4}h_\infty-\frac{3-\kappa}{4}\sigma_\infty,\tag{3.25}$$

代入式(3.14),(3.23),(3.24)后. 由式(3.18)$'$最后可得

$$\left.\begin{aligned}\Phi(z)&=-\frac{(\tau_\infty+\mathrm{i}\sigma_\infty)\sin\dfrac{z}{a}}{2\sqrt{\sin\dfrac{l+z}{a}\sin\dfrac{l-z}{a}}}+\frac{h_\infty-\sigma_\infty}{4}+\frac{\kappa-1}{\kappa+1}\,\frac{\mathrm{i}\tau_\infty}{2},\\\Omega(z)&=-\frac{(\tau_\infty+\mathrm{i}\sigma_\infty)\sin\dfrac{z}{a}}{2\sqrt{\sin\dfrac{l+z}{a}\sin\dfrac{l-z}{a}}}-\frac{h_\infty-\sigma_\infty}{4}-\frac{\kappa-1}{\kappa+1}\,\frac{\mathrm{i}\tau_\infty}{2},\end{aligned}\right\}\tag{3.26}$$

其中根式要这样理解：当z从上半平面趋于$t\in L_0$时取正值.

容易看出，要想位移是周期的，即$q=0$，由式(3.25)，应令

$$h_\infty=\frac{3-\kappa}{\kappa+1}\sigma_\infty.$$

这可这样来理解：当 $\sigma_\infty > 0$ 即在和周期方向作垂直的拉伸时，为了要保持位移的周期性，必须在 $z = \pm\ \infty\mathrm{i}$ 处作和周期方向一致的拉伸，且其强度为原拉伸强度的 $\frac{3-\kappa}{\kappa+1}(>0)$ 倍.

如果在式(3.26)中令 $a \to +\infty$，则得到一个水平裂缝时的下列公式：

$$\left.\begin{aligned}\Phi(z) &= -\frac{(\tau_\infty + \mathrm{i}\sigma_\infty)z}{2\sqrt{l^2 - z^2}} + \frac{h_\infty - \sigma_\infty}{4} + \frac{\kappa - 1}{\kappa + 1}\frac{\mathrm{i}\tau_\infty}{2}, \\ \Omega(z) &= -\frac{(\tau_\infty + \mathrm{i}\sigma_\infty)z}{2\sqrt{l^2 - z^2}} - \frac{h_\infty - \sigma_\infty}{4} - \frac{\kappa - 1}{\kappa + 1}\frac{\mathrm{i}\tau_\infty}{2},\end{aligned}\right\} \tag{3.26$'$}$$

其中当 z 从上半平面趋于 $t \in L_0$ 时根式取正值. 不难验证，如果不计刚性旋转，此结果与文献[10]，§120 中的结果一样.

这个周期问题($n = 1$ 的情况)在文献[12]中曾讨论过，但是，那是在裂缝上无外应力情况下得到解答的.

对于水平周期裂缝第二基本问题，也可类似地求解. 我们在下面只叙述 L_0 只含一个线段 $-l \leqslant t \leqslant l$ 时的结果.

设已知 L_0 上下岸位移的导数为 $u^{\pm\prime}(t) + \mathrm{i}v^{\pm\prime}(t)$，它们以 $a\pi$ 为周期，满足Hölder条件. 在 L_0 上的外应力主矢量 $X + \mathrm{i}Y$ 也已知，又已知 $z = -\infty\mathrm{i}$（从而 $z = +\infty\mathrm{i}$）处的应力，得

$$\begin{aligned}\Phi(z) = {} & \frac{1}{2a\pi\kappa\mathrm{i}\sqrt{R(z)}}\int_{-l}^{l} f(t)\sqrt{R(t)}\cot\frac{t-z}{a}\,\mathrm{d}t \\ & + \frac{1}{2a\pi\kappa\mathrm{i}}\int_{-l}^{l} g(t)\cot\frac{t-z}{a}\,\mathrm{d}t + \frac{C_0\tan\frac{z}{a} + C_1}{\kappa\sqrt{R(z)}} + \beta - \frac{q}{2\kappa}, \\ \Omega(z) = {} & -\frac{1}{2a\pi\mathrm{i}\sqrt{R(z)}}\int_{-l}^{l} f(t)\sqrt{R(t)}\cot\frac{t-z}{a}\,\mathrm{d}t \\ & + \frac{1}{2a\pi\mathrm{i}}\int_{-l}^{l} g(t)\cot\frac{t-z}{a}\,\mathrm{d}t - \frac{C_0\tan\frac{z}{a} + C_1}{\sqrt{R(z)}} + \kappa\beta - \frac{q}{2},\end{aligned}$$

其中

$$C_0 = -\frac{1}{2a\pi\mathrm{i}}\int_{-l}^{l} f(t)\sqrt{R(t)}\,\mathrm{d}t - \frac{\mathrm{i}q}{2\cos\frac{l}{a}},$$

$$C_1 = -\frac{\kappa(Y - \mathrm{i}X)}{2(\kappa+1)a\pi\cos\frac{l}{a}},$$

这里已令

$$\begin{aligned}f(t) &= \mu\{[u^{+\prime}(t) + u^{-\prime}(t)] + \mathrm{i}[v^{+\prime}(t) + v^{-\prime}(t)]\}, \\ g(t) &= \mu\{[u^{+\prime}(t) - u^{-\prime}(t)] + \mathrm{i}[v^{+\prime}(t) - v^{-\prime}(t)]\}.\end{aligned}$$

这问题($n = 1$ 时)在文献[13]中曾有不完全的讨论.

对于在文献[10]，§120 中曾讨论过的 Шерман 的混合问题，用类似方法也可推广到周期应力情况，这里就不写出具体结果了.

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ON FUNDAMENTAL PROBLEMS OF PLANE ELASTICITY WITH PERIODIC STRESSES

Abstract

In this paper, we consider the general fundamental problems of plane elasticity with simply periodic stresses. We assume that the stresses are periodic and bounded, but do not make any other restrictions either on the number of holes in a strip of periodicity or the form of their boundaries and on the stresses at infinity or those on the boundaries of holes. Here, we find out the general expressions of Kolosov functions, in which the multi-valued parts and the non-periodic parts are separated. We show that the displacements now must be quasi-periodic and give the general formulations of the first and the second fundamental problems in such cases, which are easily reduced to Fredholm integral equations.

On generalizing N. I. Muskhelishvili's method of solving the fundamental problems of an elastic plane with horizontal cracks lying on the same straight line, we solve the similar problems in periodic cases. The solutions in closed form have been obtained in the particular case when there is only one crack in a periodic strip.

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不同材料拼接平面裂纹中的数学问题

在弹性力学中裂纹的研究是极其重要的. 具一条直裂纹的平面问题是经典的[1]. 无限平面中具任意条裂纹的基本问题可化为唯一可解的奇异积分方程[2]. 周期裂纹的类似问题也可类似地解决[3]. 另一方面，不同材料的焊接问题我们在[4]中已有研究. 两拼接半平面中具一条直裂纹的问题在某些特定的边界条件下在 [5, 6, 7] 中用 Mellin 变换研究过，但方法非常复杂，且不能推广到裂纹个数和形状都任意的一般情形，即使多个直裂纹位置任意时也是如此，更不必说拼接线是封闭曲线的情况.

本文将讨论平面中二不同材料拼接在一起时的断裂问题，其中裂纹的个数、位置和形状都可任意. 这里把这类问题直接化为某种奇异积分方程，并证明其唯一可解. 我们将详细讨论两个半平面拼接的情况，因为这里要讨论到无穷直线上 Cauchy 型积分的某些特殊性质，因此情况较为复杂. 当拼接线是一封闭曲线时方法完全可用，而且更为简单. 这里的方法比[7]中的简单得多. 这一方法极为有效，关于数字例子以及应力强度因子的求法当在另文给出. 在[10]中曾讨论类似问题，但裂纹只能出现在同一个半平面中；所用方法也和这里的不同.

§1 沿无穷直线上的 Cauchy 型积分的某些性质

这种积分在[8]与[9]中已有简短的讨论. 这里将给出其后面要用到的一些性质. 引进 $g(x)\in\hat{H}(X)$ 或 $\hat{H}$，如 $g(x)$ 是定义在 x 轴上的一复函数，满足下列条件：(i) 在 x 轴的任何闭区间上，$g(x)\in H$；(ii) 对于充分大的 $|x_1|$ 和 $|x_2|$，有

$$|g(x_1)-g(x_2)|\leqslant A\left|\frac{1}{x_1}-\frac{1}{x_2}\right|^{\mu}\quad (A>0,\ 0<\mu\leqslant 1). \tag{1.1}$$

易见 $g(x)\in\hat{H}$ 隐含 $g(\pm\infty)=c$ 存在，且

$$|g(x)-c|\leqslant\frac{A}{|x|^{\mu}}\quad (当\ |x|\ 充分大)$$

或即 $g(x)=c+O\left(\dfrac{1}{|x|^{\mu}}\right)$ (当 $x\to\pm\infty$).

$\hat{H}$ 的概念可以明显的方式推广到以下各种情况：$g(z)$ 定义于闭上半平面 $\overline{Z}^{+}$ 或闭下半平面 $\overline{Z}^{-}$；或定义于包含 x 轴在内部的某一区域 D；甚或定义在无穷远点的一邻域内. 在所有的情况下，均有

$$g(z)=c+O\left(\frac{1}{|z|^{\mu}}\right)\quad (z\to\infty).$$

如果 $g(x)\in H$ 于所考虑的某无限区域 D 的任何有界部分中，且在无穷远点的邻域内解析，则显然 $g(z)\in\hat{H}(D)$.

现设 $g(x)\in\hat{H}$. 对于积分

$$G(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(\xi)}{\xi-z}d\xi=\lim_{R\to+\infty}\frac{1}{2\pi i}\int_{-R}^{R}\frac{g(\xi)}{\xi-z}d\xi,\quad z\overline{\in}X. \tag{1.2}$$

以及

$$G(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(\xi)}{\xi-x}d\xi=\lim_{\substack{R\to+\infty\\ \epsilon\to+0}}\frac{1}{2\pi i}\left(\int_{-R}^{-\epsilon}\frac{g(\xi)d\xi}{\xi-x}+\int_{\epsilon}^{R}\frac{g(\xi)d\xi}{\xi-x}\right),\quad x\in X, \tag{1.3}$$

Plemelj 公式

$$G^{\pm}(x)=\pm\frac{1}{2}g(x)+G(x) \tag{1.4}$$

成立，且我们还有

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-z}=\pm\frac{1}{2},\quad 当 z\in Z^{\pm}, \tag{1.5}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-x}=0,\quad 当 x\in X, \tag{1.6}$$

$$G^{\pm}(\infty)=\pm\frac{1}{2}g(\infty),\quad G(\pm\infty)=0.$$

用变换

$$T:\sigma+i=\frac{-1}{z+i}\ 或即\ \sigma=\frac{-iz}{z+i}$$

$$\left(t+i=\frac{-1}{x+i}\ 或即\ t=\frac{-ix}{x+i}\right)\quad (T^{-1}=T)$$

把 x 轴映射到圆周 Γ: $\left|t+\frac{i}{2}\right|=\frac{1}{2}$，而 Z^{+} 映射到 Γ 的内部. 如果记 $g^{*}(t)=g(x)$（从而 $g(\pm\infty)=g^{*}(-i)$），则 $g(x)\in\hat{H}(x)$ 当且仅当 $g^{*}(t)\in H(\Gamma)$.

我们来建立以下诸引理.

引理 1 如 $g(x)\in\hat{H}$，则 $G(z)\in\hat{H}(\overline{Z}^{\pm})$，$G(x)\in\hat{H}$，$G(\pm\infty)=0$.

证 设 $z\overline{\in}X$，则 $\sigma\overline{\in}\Gamma$，且

$$G(z)\equiv G^{*}(\sigma)=\frac{1}{2\pi i}\int_{\Gamma}\frac{g^{*}(\tau)(\sigma+i)}{(\tau-\sigma)(\tau+i)}d\tau=\tilde{G}^{*}(\sigma)-\tilde{G}^{*}(-i),$$

其中

$$\tilde{G}^{*}(\sigma)=\frac{1}{2\pi i}\int_{\Gamma}\frac{g^{*}(\tau)}{\tau-\sigma}d\tau,\quad \sigma\overline{\in}\Gamma.$$

因为 $g^{*}(\tau)\in H(\Gamma)$，则由 Привалов 定理[8] 知，$\tilde{G}^{*}(\sigma)\in H(\overline{S}^{*})$，这里 $\overline{S}^{\pm}$ 分别是 Γ 所围的内闭域与外闭域；由此立得 $G(z)\in\hat{H}(\overline{Z}^{\pm})$. 由(1.4)也可得 $G(x)\in\hat{H}$. $G(\pm\infty)=0$ 显然.

引理 2 如 $g(x)\in\hat{H}$，$g'(x)\in\hat{H}$，① 则

$$G'(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g'(\xi)}{\xi-z}d\xi,\quad z\overline{\in}X, \tag{1.7}$$

$$G'(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g'(\xi)}{\xi-x}d\xi,\quad x\in X, \tag{1.8}$$

① 一般说来，$g'(x)\in\hat{H}$ 不隐含 $g(x)\in\hat{H}$. 例如 $g(x)=x\overline{\in}\hat{H}$，但 $g'(x)=1\in\hat{H}$.

且 $G'(z) \in \hat{H}(\overline{Z}^{\pm}), G'(x) \in \hat{H}$. 此外，$G'^{\pm}(x) = [G^{\pm}(x)]'$.

证 注意 $g(\pm\infty) = c$ 有限，故得

$$G'(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g(\xi)}{(\xi - z)^2}d\xi = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g'(\xi)}{\xi - z}d\xi, \quad z \overline{\in} X,$$

因而 $G'^{\pm}(z) \in \hat{H}(Z^{\pm})$. 且

$$G'^{\pm}(x) = \pm\frac{1}{2}g'(x) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g'(\xi)}{\xi - x}d\xi. \tag{1.9}$$

现证 $G^{\pm\prime}(x) = G'^{\pm}(x)$. 如 $z_0, z \in Z^+$，则

$$G(z) = \int_{z_0}^{z} G'(\zeta)d\zeta = G(z_0),$$

其中积分路径取在 Z^+ 内. 令 z 在 Z^+ 内趋于 x，便得

$$G^{+}(x) = \int_{z_0}^{x} G'(\zeta)d\zeta - G(z_0);$$

因为我们已证 $G'(\zeta)$ 在 $\overline{Z}^+$ 上连续，故 $G^{+\prime}(x) = G'^{+}(x)$. 同理可证 $G^{-\prime}(x) = G'^{-}(x)$.

由(1.4)，得到

$$G^{\pm\prime}(x) = \pm\frac{1}{2}g'(x) + G'(x). \tag{1.10}$$

与(1.9)相比较，便得(1.8). 由引理1知 $G'(x) \in \hat{H}$.

引理3 如 $g(x) \in \hat{H}$，$g'(x) \in H$，$x^2g'(x) \in \hat{H}$，则 $z^2G(z), zG'(z), G'(z)$ 均 $\in \hat{H}(\overline{Z}^{\pm})$，且 $G'(z) = O\left(\frac{1}{|z|^2}\right)$，$G^{\pm}(z) = \pm\frac{1}{2}g(\infty) + O\left(\frac{1}{|z|}\right)$ $(z \to \infty)$. 此外，$x^2G'(x)$，$xG'(x), G'(x)$ 均 $\in \hat{H}$ 且 $G'(x) = O\left(\frac{1}{|x|^2}\right)$，$G(x) = O\left(\frac{1}{|x|}\right)$ $(x \to \pm\infty)$.

证 我们先证 $xg'(x) \in \hat{H}$. 当 x 属于一确定的有限区间时，因 $g'(x) \in H$，故 $xg'(x) \in H$. 由于 $h(x) = x^2g'(x) \in \hat{H}$，故 $|h(x)| \leqslant C_1$ 有界. 因之对于充分大的 $|x_1|, |x_2|$，有

$$\begin{aligned}|x_1g'(x_1) - x_2g'(x_2)| &= \left|\frac{h(x_1)}{x_1} - \frac{h(x_2)}{x_2}\right| \\ &\leqslant |h(x_1)|\left|\frac{1}{x_1} - \frac{1}{x_2}\right| + \frac{1}{|x_2|}|h(x_1) - h(x_2)| \\ &\leqslant C_1\left|\frac{1}{x_1} - \frac{1}{x_2}\right| + C_2\left|\frac{1}{x_1} - \frac{1}{x_2}\right|^{\mu} \\ &\leqslant C_3\left|\frac{1}{x_1} - \frac{1}{x_2}\right|^{\mu}.\end{aligned}$$

同样，由于 $g'(x) \in H$ 且 $xg'(x) \in \hat{H}$，推知 $g'(x) \in \hat{H}$. 又因

$$g^{*\prime}(t) = g'(x)(x + i)^2 \in \hat{H},$$

故 $g^{*\prime}(t) \in H(\Gamma)$. 当然亦有 $g^{*}(t) \in H(\Gamma)$.

由引理2，$G'(z) \in \hat{H}(\overline{Z}^{\pm})$，$G'(x) \in \hat{H}$. 因为

$$zG'(z) = -i\sigma(\sigma + i)G^{*\prime}(\sigma),$$

其中 $G^{*\prime}(\sigma) = \tilde{G}^{*\prime}(\sigma) \in H(\overline{S}^{\pm})$（由 Привалов 定理），故 $zG'(z) \in \hat{H}(\overline{Z}^{\pm})$.

同样，可证 $z^2G'(z) \in \hat{H}(\overline{Z}^{\pm})$.

当 z 分别在 $\overline{Z}^+$ 和 $\overline{Z}^-$ 中趋于 ∞ 时，我们来估计 $G(z)$. 暂设 $g(\infty) = g^{*}(-i) = 0$，故

$$\tilde{G}^{*\pm}(-i) = \tilde{G}^{*}(-i),$$

$$|G^{*\pm}(z)| = |\widetilde{G}^{*\pm}(\sigma) - \widetilde{G}^{*}(-\mathrm{i})| = \left|\int_{-\mathrm{i}}^{\sigma} \widetilde{G}^{*\prime}(\tau)\mathrm{d}\tau\right|,$$

其中积分路径视 $z \in \overline{Z}^+$ 或 $\overline{Z}^-$ 应分别取在 $\overline{S}^+$ 或 $\overline{S}^-$ 中. 如 $z \in \overline{Z}^+$，则它可取为自 $-\mathrm{i}$ 至 σ 的直线段，故

$$|G^{*+}(z)| \leqslant K|\sigma + \mathrm{i}| = \frac{K}{|z+\mathrm{i}|} = O\left(\frac{1}{|z|}\right), \quad K = \max_{\sigma \in \overline{S}^+} |\widetilde{G}^{*\prime}(\sigma)|.$$

现设 $z \in \overline{Z}^-$. 如 z 的像 σ 位于 Γ 在 $-\mathrm{i}$ 处切线的下侧，则上述推理仍有效；否则，积分路径可取为自 $-\mathrm{i}$ 沿水平方向前进然后沿垂直方向到达 σ 的折线段，其长小于 $2|\sigma+\mathrm{i}|$，故仍有 $G^{*-}(z) = O\left(\frac{1}{|z|}\right)$.

如若 $g(\infty) \neq 0$，则由(1.5)，

$$G(z) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{g(\xi) - g(\infty)}{\xi - z}\mathrm{d}\xi \pm \frac{1}{2}g(\infty),$$

因而 $G(z) = \pm\frac{1}{2}g(\infty) + O\left(\frac{1}{|z|}\right)$.

最后我们来估计 $|G'(z)|$：

$$|G'(z)| = \left|\widetilde{G}^{*\prime}(\sigma)\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right| \leqslant K'|\sigma + \mathrm{i}|^2 = \frac{K'}{|z+\mathrm{i}|^2} = O\left(\frac{1}{|z|^2}\right),$$

因为 $\widetilde{G}^{*\prime}(\sigma)$ 在 $\overline{S}^+$ 与 $\overline{S}^-$ 中均有界.

有关 $G(x)$ 与 $G'(x)$ 的所要求性质可由(1.4)，(1.10)以及 $G(x)$，$G'^{\pm}(x)$ 与 $g'(x)$ 的相应性质推得.

§2 问题的数学处理

设弹性平面由上半平面 Z^+ 与下半平面 Z^- 拼接而成，其弹性常数分别为 κ^+，μ^+ 与 κ^-，μ^-. 设在平面中有 p 条互不相交的曲裂纹 $L_j = \overset{\frown}{a_j b_j}(j = 1,\cdots,p)$，且已取定正向自 a_j 至 b_j，其中某些在 Z^+ 内，另一些在 Z^- 内. 记 $L = \sum_{j=1}^{p} L_j$，$S^{\pm} = Z^{\pm}\backslash L$，把 L_j 的正、负侧分别记作 L_{j+}，L_{j-}，并记 $L_{\pm} = \sum_{j=1}^{p} L_{j\pm}$. 我们还假定每一 L_j 其切线倾角作为弧长的函数时其导数属于 H.

设作用在 L_j 上的外力密度矢量为 $X_{j\pm}(t) + \mathrm{i}Y_{j\pm}(t) \in H$，这里 t 是 L 上点的复坐标. 并设在每一 L_j 两侧的外力合力为零. 令

$$f_{\pm}(t) = f_{j\pm}(t) = \mathrm{i}\int_{a_j}^{t} [X_{j\pm}(t) + \mathrm{i}Y_{j\pm}(t)]\mathrm{d}s_j, \quad t \in L_j,\ j = 1,\cdots,p,$$

这里 s_j 是 L_j 上自 a_j 量起的弧长. 在上述假定下，

$$f_{j+}(t) + f_{j-}(t) = 0, \quad j = 1,\cdots,p.$$

此外，还假定 $f_{j\pm}(t) \in H$，$f'_{j\pm}(t) \in H^*$（记号见[8]）. 我们还设在无穷远处无应力和位移.

根据平面弹性的一般理论（参看[1]），我们要寻求二分区全纯函数 $\varphi(z)$，$\psi(z)$（由上述条件，它们肯定单值），要求在无穷远处

$$\varphi(z),\psi(z) = O\left(\frac{1}{|z|}\right), \quad \varphi'(z),\psi'(z) = O\left(\frac{1}{|z|^2}\right) \quad (z \to \infty), \tag{2.1}$$

在 $L_\pm$ 上分别有边值 $\varphi_\pm(t),\psi_\pm(t)$，在 $X^\pm$ 上分别有边值 $\varphi^\pm(x),\psi^\pm(x)$ （$X^\pm$ 分别表示 x 轴的上、下侧），要求它们满足边值条件

$$\left.\begin{aligned}\varphi_+(t)+t\,\overline{\varphi_+'(t)}+\overline{\psi_+(t)}&=f_+(t)+C_j,\\ \varphi_-(t)+t\,\overline{\varphi_-'(t)}+\overline{\psi_-(t)}&=-f_-(t)+C_j,\end{aligned}\right\}\ t\in L_j, \quad \begin{matrix}(2.2)\\(2.3)\end{matrix}$$

$$\left.\begin{aligned}&\varphi^+(x)+x\,\overline{\varphi'^+(x)}+\overline{\psi^+(x)}=\varphi^-(x)+x\,\overline{\varphi'^-(x)}+\overline{\psi^-(x)},\\ &\alpha^+\varphi^+(x)-\beta^+[x\,\overline{\varphi'^+(x)}+\overline{\psi^+(x)}]=\alpha^-\varphi^-(x)-\beta^-[x\,\overline{\varphi'^-(x)}+\overline{\psi^-(x)}],\end{aligned}\right\}x\in X, \quad \begin{matrix}(2.4)\\(2.5)\end{matrix}$$

其中已令

$$\alpha^\pm=\kappa^\pm/\mu^\pm,\quad \beta^\pm=1/\mu^\pm,$$

而 $C_j(j=1,\cdots,p)$ 为待定常数，以下有时写成 $C(t)=C_j$ 当 $t\in L_j$ 时. 条件(2.2),(2.3)表示沿 $L_{j_\pm}$ 的外荷载，(2.4),(2.5)表示沿 x 轴上应力与位移的连续性.

一旦找出 $\varphi(z),\psi(z)$ 后，则在任何点 z 处的应力和位移可由下列式子给出：

$$\sigma_x+\sigma_y=4\mathrm{Re}\,\varphi'(z), \tag{2.6}$$

$$\sigma_y-\sigma_x+2\mathrm{i}\tau_{xy}=2\{\bar{z}\varphi''(z)+\psi'(z)\}, \tag{2.7}$$

$$2\mu^\pm(u+\mathrm{i}v)=\kappa^\pm\varphi(z)-z\,\overline{\varphi'(z)}-\overline{\psi(z)},\quad z\in S^\pm. \tag{2.8}$$

当 $z=t\in L$ 或 $z=x\in X$ 时，$\varphi(z),\psi(z),\varphi'(z)$ 的值必须理解为相应的极限值，但 $\varphi''(z)$，$\psi'(z)$ 的极限值可能并不存在，而只能期望当 z 从 L 的任一侧趋于 t 或从 x 轴的任一侧趋于 τ 时 $\bar{z}\varphi''(z)+\psi'(z)$ 的极限值存在.

为了求解边值问题(2.2)～(2.5)，引进未知函数 $\omega(\zeta)$，$\zeta\in L+X$，使得

$$\left.\begin{aligned}\varphi(z)&=\frac{1}{2\pi\mathrm{i}}\int_{L+X}\frac{\omega(\zeta)}{\zeta-z}\mathrm{d}\zeta,\\ \psi(z)&=-\frac{1}{2\pi\mathrm{i}}\int_{L+X}\frac{\overline{\omega(\zeta)}+\bar{\zeta}\omega'(\zeta)}{\zeta-z}\mathrm{d}\zeta+\frac{1}{2\pi\mathrm{i}}\int_L\frac{\overline{f_+(\tau)}+\overline{f_-(\tau)}}{\tau-z}\mathrm{d}\tau,\end{aligned}\right\}z\overline{\in}L+X. \quad \begin{matrix}(2.9)\\(2.10)\end{matrix}$$

这里我们暂时假定：

当 $t\in L$ 时，$\omega(t)\in H$，$\omega'(t)\in H^*$，

$$\omega(a_j)=\omega(b_j)=0,\quad j=1,\cdots,p; \tag{2.11}$$

当 $x\in X$ 时 $\omega(x),\omega'(x),x\omega'(x),x^2\omega'(x)\in\hat{H}$，

$$\omega(x)=O\left(\frac{1}{|x|}\right),\ \omega'(x)=O\left(\frac{1}{|x|^2}\right)\quad(x\to\pm\infty). \tag{2.12}$$

$\omega(\zeta)$ 的存在和唯一，其性质(2.11)与(2.12)，$\varphi(z)$ 与 $\psi(z)$ 的性质(2.1)以及当 z 从 L 的任一侧趋于 t 或从 x 轴的任一侧趋于 x 时 $\bar{z}\varphi''(z)+\psi'(z)$ 的极限值的存在，均将在下一节中证明.

把(2.9)与(2.10)代入(2.2)，利用 Plemelj 公式，可得

$$\begin{aligned}&\frac{1}{\pi\mathrm{i}}\int_{L+X}\frac{\omega(\zeta)}{\zeta-t}\mathrm{d}\zeta+\frac{1}{\pi\mathrm{i}}\int_{L+X}\frac{\overline{\omega(\zeta)}}{\bar{\zeta}-\bar{t}}\mathrm{d}\bar{\zeta}+\frac{1}{\pi\mathrm{i}}\int_{L+X}\frac{\zeta-t}{\bar{\zeta}-\bar{t}}\overline{\omega'(\zeta)}\,\mathrm{d}\,\bar{\zeta}\\ &=f_+(t)-f_-(t)+\frac{1}{\pi\mathrm{i}}\int_L\frac{f_+(\tau)+f_-(\tau)}{\bar{\tau}-\bar{t}}\mathrm{d}\bar{\tau}+C(t),\quad t\in L.\end{aligned} \tag{2.13}$$

注意到 $\dfrac{x-t}{x-t}\overline{\omega(x)}$ 当 $x\to\pm\infty$ 时对 t 一致地趋于零，且 $\omega(a_j)=\omega(b_j)=0$，故(2.13)可改写为

$$\frac{1}{\pi i}\int_{L+X}\frac{\omega(\zeta)}{\zeta-t}d\zeta+\frac{1}{2\pi i}\int_{L+X}\omega(\zeta)d\log\frac{\bar{\zeta}-\bar{t}}{\zeta-t}-\frac{1}{2\pi i}\int_{L+X}\overline{\omega(\zeta)}d\frac{\zeta-t}{\bar{\zeta}-\bar{t}}$$
$$=\frac{f_+(t)-f_-(t)}{2}+\frac{1}{2\pi i}\int_L\frac{f_+(\tau)+f_-(\tau)}{\bar{\tau}-\bar{t}}d\bar{\tau}+\frac{1}{2}C(t),\quad t\in L. \tag{2.14}$$

如把(2.9),(2.10)代入(2.3)，我们仍得(2.14)；如代入(2.4)，可发现其恒能满足. 把它们代入(2.5)，则得另一方程

$$(\alpha^++\alpha^-+\beta^++\beta^-)\omega(x)+\frac{\alpha^+-\alpha^--\beta^++\beta^-}{\pi i}\int_{L+X}\frac{\omega(\zeta)}{\zeta-x}d\zeta$$
$$+\frac{\beta^+-\beta^-}{\pi i}\int_L\omega(t)d\log\frac{t-x}{\bar{t}-x}+\frac{\beta^+-\beta^-}{\pi i}\int_L\overline{\omega(t)}d\frac{t-x}{\bar{t}-x} \tag{2.15}$$
$$=-\frac{\beta^+-\beta^-}{\pi i}\int_L\frac{f_+(t)+f_-(t)}{\bar{t}-x}d\bar{t},\quad x\in X.$$

(2.14),(2.15)一起事实上是 $L+X$ 上的一个奇异积分方程：

$$a(\zeta_0)\omega(\zeta_0)+\frac{b(\zeta_0)}{\pi i}\int_{L+X}\frac{\omega(\zeta)}{\zeta-\zeta_0}d\zeta+(k\omega)(\zeta_0)=F(\zeta_0,C_1,\cdots,C_p),\quad \zeta_0\in L+X, \tag{2.16}$$

其中

$$a(\zeta)=\begin{cases}0, & 当\ \zeta=t\in L,\\ \alpha^++\alpha^-+\beta^++\beta^-, & 当\ \zeta=x\in X;\end{cases}$$
$$b(\zeta)=\begin{cases}1, & 当\ \zeta=t\in L,\\ \alpha^+-\alpha^--\beta^++\beta^-, & 当\ \zeta=x\in X;\end{cases} \tag{2.17}$$

k 是一 Fredholm 算子，而 F 是一已知函数. (2.16)是一正则型方程，且其在 h_{2p} 类(记号见[8])中的指标为 $-p$.

于是，我们的问题化为：在适当选取常数 $C_1,\cdots,C_p$ 后，求解方程(2.16)，要求在 L 上解 $\omega(t)\in h_{2p}$，即在 L 的所有端点附近，它保持有界(实际上等于零).

§3 对于解的研究

现在我们来证明，在适当选取 $C_1,\cdots,C_p$ 后，方程(2.16)的解存在且唯一，并且前面对于解的性质所作的各个假设均为真.

首先，设对于取定的某组 $\{C_j\}$，(2.16)有一个解 $\omega(\zeta)\in h_{2p}$；我们来研究其性质. 对于 $\zeta_0=t\in L$，由(2.14)显然有

$$\frac{1}{\pi i}\int_L\frac{\omega(\tau)}{\tau-t}d\tau=-\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\omega(\xi)}{\xi-t}d\xi+\cdots+C(t)\equiv F_1(t)\in H.$$

由反演公式(参看[1],(87.41)),立得

$$\omega(t)=\frac{\sqrt{R(t)}}{\pi i}\int_L\frac{F_1(\tau)d\tau}{\sqrt{R(\tau)}(\tau-t)},$$

其中

$$R(t)=\prod_{j=1}^{p}(t-a_j)(b_j-t),$$

且 $\sqrt{R(t)}$ 可取作当复平面用 L 剖开后 $\sqrt{R(z)}$ 的任一确定分支. 因为上式右边的积分在 a_j,

b_j 处有不到$\frac{1}{2}$阶的奇异性，故 $\omega(a_j)=\omega(b_j)=0$，从而 $\omega(t)\in H$. 又由 $f'(t)\in H^*$，故知 $\omega'(t)\in H^*$(参看[1]，§ 115)，因此(2.11)得证.

对于 $\zeta_0=x\in X$，由(2.15)，有

$$(\alpha^++\alpha^-+\beta^++\beta^-)\omega(x)+\frac{\alpha^+-\alpha^--\beta^++\beta^-}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(\xi)}{\xi-x}d\xi$$
$$=-\frac{\alpha^+-\alpha^--\beta^++\beta^-}{\pi i}\int_L\frac{\omega(t)}{t-x}dt+\cdots\equiv F_2(x),$$

其中 $F_2(x)$ 可解析延拓到包含 x 轴在其内部的一区域，且 $F_2(\infty)=0$. 于是 $F_2(x)\in\hat{H}$. 由类似于闭回路积分情况时的推理，可得

$$\omega(x)=A_0F_2(x)-\frac{B_0}{\pi i}\int_{-\infty}^{\infty}\frac{F_2(\xi)}{\xi-x}d\xi,$$

其中 A_0,B_0 为常数，由此立得 $\omega(\pm\infty)=0$. 因 $F_2(z)$ 在 x 轴的邻域中全纯，故 $F_2'(z)$，$zF_2'(z),z^2F_2'(z)$ 也在此全纯，从而 $F_2'(x),xF_2'(x),x^2F_2'(x)\in\hat{H}$. 由引理 3，(2.12) 得证.

现在写

$$\varphi(z)=\frac{1}{2\pi i}\int_L\frac{\omega(t)}{t-z}dt+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\omega(x)}{x-z}dx\equiv\varphi_1(z)+\varphi_2(z),$$

其中 $\varphi_1(z)=O\left(\frac{1}{|z|}\right)$，$\varphi_1'(z)=O\left(\frac{1}{|z|^2}\right)$是显然的. 因为已经证得 $\omega'(x)\in\hat{H},x^2\omega'(x)\in\hat{H}$，再由引理 3，便知 $\varphi_2(z)=O\left(\frac{1}{|z|}\right)$，$\varphi_2'(z)=O\left(\frac{1}{|z|^2}\right)$. 类似地写出

$$\psi(z)=-\frac{1}{\pi i}\int_L\frac{\overline{\omega(t)}+\bar{t}\omega'(t)}{t-z}dt-\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\overline{\omega(x)}+x\omega'(x)}{x-z}dx$$
$$+\frac{1}{\pi i}\int_L\frac{\overline{f_+(t)}+\overline{f_-(t)}}{t-z}dt=\psi_1(z)+\psi_2(z)+\psi_3(z),$$

其中 $\psi_1(z),\psi_3(z)$ 有和 $\varphi(z)$ 相似的性质. 注意到 $F_2(x),F_2'(x)\in\hat{H}$，由引理 2，我们有

$$\omega'(x)=A_0F_2'(x)+\frac{B_0}{\pi i}\int_{-\infty}^{\infty}\frac{F_2'(\xi)}{\xi-x}d\xi\in\hat{H};$$

重复以上推理，可知 $\omega''(x),x\omega''(x),x^2\omega''(x)\in\hat{H}$，从而

$$\overline{\omega(x)}+x\omega'(x)\in\hat{H},\quad[\overline{\omega(x)}+x\omega'(x)]'\in\hat{H}.$$

所以 $\psi_2(z)=O\left(\frac{1}{|z|}\right)$，$\psi_2'(z)=O\left(\frac{1}{|z|^2}\right)$. 于是，(2.1) 得证.

$\varphi(z),\varphi'(z)$ 与 $\psi(z)$ 的边值存在，且

$$\left.\begin{aligned}\varphi_\pm(t)&=\pm\frac{1}{2}\omega(t)+\frac{1}{2\pi i}\int_{L+X}\frac{\omega(\zeta)}{\zeta-t}d\zeta, &(3.1)\\ \varphi_\pm'(t)&=\pm\frac{1}{2}\omega'(t)+\frac{1}{2\pi i}\int_{L+X}\frac{\omega'(\zeta)}{\zeta-t}d\zeta, &(3.2)\end{aligned}\right\}\quad t\in L,$$

$$\left.\begin{aligned}\varphi^\pm(x)&=\pm\frac{1}{2}\omega(x)+\frac{1}{2\pi i}\int_{L+X}\frac{\omega(\zeta)}{\zeta-x}d\zeta, &(3.3)\\ \varphi^{\pm\prime}(x)&=\pm\frac{1}{2}\omega'(x)+\frac{1}{2\pi i}\int_{L+X}\frac{\omega'(\zeta)}{\zeta-x}d\zeta, &(3.4)\end{aligned}\right\}\quad x\in X,$$

这是由于已知 $\omega(t)\in H$，$\omega'(t)\in H^*$，$\omega(x)\in\hat{H}$，$\omega'(x)\in\hat{H}$. 对 $\psi_\pm(t),\psi^\pm(x)$ 也有类

似的公式，因为由于$f'_{\pm}(t) \in H^*$，故(2.10)中最后一项的边值存在. 这些边值显然应满足边值条件(2.2)～(2.5)，因为把它们代入这些式子时得到的正是$\omega(\zeta)$所满足的奇异积分方程.

这样，如果(2.16)有一个解$\omega(\zeta) \in h_{2p}$，则§2中所作各假设均真，且由此求得的$\varphi(z),\psi(z)$即所提问题的解.

现在求证，当$C_1,\cdots,C_p$适当选择后，这种解确实存在，而且唯一. 我们先证(2.16)的相应齐次方程($F\equiv 0$)只有零解. 设$\omega_0(\zeta)$是它在h_{2p}类中的一个解，而把(2.9)与(2.10)中相应的函数分别记为$\varphi_0(z)$与$\psi_0(z)$. 仍进行如前，我们将得到关于$\varphi_0(z)$与$\psi_0(z)$类似于(2.2)～(2.5)的等式，但(2.2)与(2.3)中右边应为零. 这些等式表示零边界条件下以及无穷远处无应力和位移时的弹性静力平衡方程. 由(2.6)，我们有

$$\operatorname{Re}\varphi_0'(z) = 0,^{①} \quad z \in S^+ + S^-,$$

且$\varphi_0^{\pm}(\infty)$有限. 因此$\varphi_0^{\pm}(z)$在S^+与S^-中均为常数. 但因

$$\varphi_0^{\pm}(x) = \pm\frac{1}{2}\omega_0(x) + \frac{1}{2\pi i}\int_{L+X}\frac{\omega_0(\zeta)}{\zeta - x}d\zeta, \quad x \in X,$$

故由引理1，

$$\lim_{x\to\pm\infty}\frac{1}{2\pi i}\int_{L+X}\frac{\omega_0(\zeta)}{\zeta - x}d\zeta = \lim_{x\to\pm\infty}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\omega_0(\xi)}{\xi - x}d\xi = 0,$$

于是$\varphi_0^{\pm}(\infty) = \pm\frac{1}{2}\omega_0(\pm\infty)$. 但已证$\omega_0(x) = O\left(\frac{1}{|x|}\right)$，故$\varphi_0^{\pm}(\infty) = 0$，此即隐含$\varphi_0^{\pm}(z) \equiv 0$，这样，

$$\omega_0(\zeta) = \varphi_0^+(\zeta) - \varphi_0^-(\zeta) = 0, \quad \zeta \in X + L.$$

根据像(2.16)那样方程的一般理论(见[8]，§112)，其齐次相联方程有$2p$个(在实域中)线性无关的解，故$C_1,\cdots,C_p$的实部与虚部可如此选择，使得$F(\zeta_0,C_1,\cdots,C_p)$与所有这些解正交，因而对于这样选择的$C_j$，它有唯一解$\omega(\zeta)\in h_{2p}$.

§4 一些说明

1. 在实际应用中，我们常常只感兴趣于应力分布而置位移于不顾. 为了确定$\sigma_x,\sigma_y,\tau_{xy}$，只要求出$\omega'(\zeta)$就行而不必求出$\omega(\zeta)$本身. $\omega'(\zeta)$所应满足的方程可将(2.16)亦即(2.14)与(2.15)求导后得出. 这样一来，诸C_j也消失了，因而求解时极方便；但要在L上h_0类中求解，且应要求$\int_{L_j}\omega'(t)dt=0$.

2. 如果拼接线x轴换作一光滑封闭曲线Γ，显然前面所有的讨论仍成立；甚至论证还要简单些，因为这时不涉及无穷直线上的Cauchy型积分.

3. 当拼接线上要焊接一位移差h时也完全可同样地讨论，所不同的只是条件(2.5)的右边还要添加一项$2h$，从而方程(2.15)右边也要添加一项$4h$. 只要h满足一定的光滑条件(拼接线为x轴时，h在无穷远处也要满足一定有界性条件)，本文所论完全成立.

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① 用类似于[1]第五章中的方法，这也可用纯数学的方法证明.

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THE MATHEMATICAL PROBLEMS OF BONDED PLANE MATERIALS WITH CRACKS

Abstract

The plane crack problem of an elastic plane bonded by two half-planes with different isotropic materials is solved by means of method of analytic functions and singular integral equations. Both the number and the shape of these smooth cracks are arbitrary, some of them to be located in one half-plane while the others in the other.

[原载于武汉大学学报（自然科学版），1982，(2)：1～10.]

平面弹性第二基本问题的新提法[①]

摘　要

本文给出了平面弹性第二基本问题的一般提法，这里在多连通弹性区域的各个闭边界围线上给出的位移只是相对的，各允许有一不同的刚性移动. 在这种情况下，为了解的唯一性，证明了必须另外给出每一边界围线上外力的主矢量与主力矩. 还给出了求解的方法以及一些例题.

一、引　言

对于平面弹性的第二基本问题，亦即已知弹性区域各边界点上的位移求弹性平衡的问题，在［1］中已有精辟的论述，尤其对于多连通无限域情况更是如此（有些作者把这问题称作第一基本问题，例如[2]），曾特别指出，除给出各边界点的位移以及无穷远处的应力与转动外，还必须给出整个边界上外力的主矢量 $X+\mathrm{i}Y$ 才能保证解的唯一性（参看[1]，§40）. 例如，若弹性体是由边界为 m 条光滑围线 $L_1,\cdots,L_m$ 所构成的洞所削弱的无限平面，则每一 L_j 上作用的外力主矢量 $X_j+\mathrm{i}Y_j$ 必须与 Колосов-Мусхелишвили 函数 $\varphi(z),\psi(z)$ 同时求出，但要受一约束条件

$$\sum_{j=1}^{m}(X_j+\mathrm{i}Y_j)=X+\mathrm{i}Y, \tag{1.1}$$

其中 $X+\mathrm{i}Y$ 已预先给定. 这样，在求解过程中我们有 $m-1$ 个待定复常数.

在实际问题中，对于一个多连通的有限或无限域的第二基本问题，常在每一边界围线 L_j 上只给出各点的相对位移，而真实位移可与给出的相差一个刚性移动 R_j；但对于不同的 j，各 R_j 可以各不相同(其中之一可认为是恒等的). 相反，在每一 L_j 上，外力的主矢量 $X_j+\mathrm{i}Y_j$ 与主力矩 M_j 却是事先给定的. 例如，当一些稍大的刚性楔子插入诸洞中而彼此间并无联系，且每个楔子上无外力作用(因而主矢量与主力矩均为零）而平衡，就是这种情况. 这种提法在作者所知的文献(例如[3]～[5]）中尚未见到讨论过.

当各楔子作为一整体刚性地连在一起时，就是经典提法中出现的问题.

本文将证明新的提法是合理的，亦即这种问题的解存在且唯一，并给出其解法. 先将讨论已知位移只相对于一些平动的问题，然后在此基础上讨论一般情况. 还在文末给出一些简单例子以验证我们所论.

① 中国科学院科学基金资助的课题.

二、当所给位移相对于平动时解的唯一性

我们来讨论多连通有限域 D 的情况. D 由光滑围线 $L_0,L_1,\cdots,L_m$ 所围成，其中 L_0 是外面的围线. 设 $L=\sum_{j=0}^{m}L_j$ 的正向已取定，使 D 位于其正(左)侧.

在我们的问题中，对于 $t\in L_j(j=0,\cdots,m)$，位移函数 $g_j(t)=u_j(t)+\mathrm{i}v_j(t)$ 已给，但相对于平动. 这就是说，L_j 上 t 点的真实位移为 $g_j(t)+c_j$，这里 c_j 是待定复常数. 此外，L_j 上外力的主矢量 $X_j+\mathrm{i}Y_j$ 也是已知的，但要受平衡条件

$$\sum_{j=0}^{m}(X_j+\mathrm{i}Y_j)=0 \tag{2.1}$$

的约束. 这时 L_j 上外力的主力矩 M_j 事先并不知道.

我们可以仿照[1]之 §40 中所述方法证明此问题解的唯一性. 记 $e_{xx}(z),e_{xy}(z),e_{yy}(z)$ 是 D 中在 z 处的形变分量，$X_n(t)+\mathrm{i}Y_n(t)$ 是 D 的边界 L 上 t 处的外应力，则有

$$\int_L(X_nu+Y_nv)\mathrm{d}s=\iint_D[\lambda(e_{xx}+e_{yy})^2+2\mu(e_{xx}^2+2e_{xy}^2+e_{yy}^2)]\mathrm{d}x\mathrm{d}y, \tag{2.2}$$

其中 s 为 L 上的弧长参数，而 $\lambda>0$，$\mu>0$ 为弹性材料的 Lamé 常数.

设若问题有两组解 $X_n'(t)+\mathrm{i}Y_n'(t)$，$g_j(t)+c_j'$ 与 $X_n''(t)+\mathrm{i}Y_n''(t)$，$g_j(t)+c_j''$. 记

$$X_n(t)=X_n''(t)-X_n'(t),\quad Y_n(t)=Y_n''(t)-Y_n'(t),\quad c_j=c_j''-c_j'=A_j+\mathrm{i}B_j,$$

则 L_j 上的外应力 $X_n(t)+\mathrm{i}Y_n(t)$ 与位移 c_j 建立一弹性平衡，因而满足(2.2). 这时(2.2)左端成为

$$\sum_{j=0}^{m}\int_{L_j}[A_jX_n(t)+B_jY_n(t)]\mathrm{d}s=\sum_{j=0}^{m}A_j\int_{L_j}X_n(t)\mathrm{d}s+\sum_{j=0}^{m}B_j\int_{L_j}Y_n(t)\mathrm{d}s.$$

因为在每一 L_j 上外力的主矢量是已知的，由此故两解之差构成的外力主矢量在 L_j 上为零，即

$$\int_{L_j}X_n(t)\mathrm{d}s=0,\quad \int_{L_j}Y_n(t)\mathrm{d}s=0\quad (j=0,\cdots,m). \tag{2.3}$$

由此可见(2.2)左端为零. 但其右端被积函数是一正定二次型，故必

$$e_{xx}=e_{xy}=e_{yy}=0.$$

这就是说，D 中所有形变分量从而所有应力分量都恒等于零. 这就证明了上述两解描述了弹性体的同一平衡状态.

在以上讨论中，整个弹性体仍可允许一任意平动. 为了避免这一任意性，我们可令 c_j 中之一，例如，$c_0=0$.

现再考虑无限域情况. 这时在上述讨论中不出现 L_0. 提法与上面完全一样，只是在整个边界 L 上外力主矢量不必为零，亦即不再有如(2.1)的约束. 此外，在无穷远处的应力 $X_\infty+\mathrm{i}Y_\infty$ 与转角 ε_∞ 当然也要给定. 如不允许整个弹性体的刚性平动，我们可指定例如 $c_1=0$.

在这种情况下证明解的唯一性可用类似于有限域时的上述方法，只要在无穷远处作适当的处理即可(参看[1]之 §40 的 3°)，这里不赘述.

三、当所给位移相对于平动时的解法

对所提问题的解法，和源于Д. И. Шерман[6] 的关于第一基本问题的解法相仿，这里只作简略概述. 设 κ,μ 为材料的弹性常数.

对于有限域 D 的情况，Колосов-Мусхелишвили 函数可写成

$$\left.\begin{aligned}\varphi(z)&=-\frac{1}{2\pi(\kappa+1)}\sum_{j=1}^{m}(X_j+\mathrm{i}Y_j)\log(z-z_j)+\varphi_0(z),\\ \psi(z)&=\frac{\kappa}{2\pi(\kappa+1)}\sum_{j=1}^{m}(X_j-\mathrm{i}Y_j)\log(z-z_j)+\psi_0(z),\end{aligned}\right\}\quad(z\in D)\qquad(3.1)$$

其中 $\varphi_0(z),\psi_0(z)$ 是 D 中的全纯函数，z_j 位于 L_j 所围的孔中，且 $\log(z-z_j)$ 对各个 j 可任意取定一支. 现在 $X_j+\mathrm{i}Y_j$ 是已知的(因(2.1)的关系，$X_0+\mathrm{i}Y_0$ 不出现). 我们要求 $\varphi_0(z)$. $\psi_0(z)$ 满足下列边值条件:

$$\kappa\varphi_0(t)-t\overline{\varphi_0'(t)}-\overline{\psi_0(t)}=2\mu g^*(t)+c_j\quad(t\in L_j,j=0,\cdots,m),\qquad(3.2)$$

其中

$$\begin{aligned}2\mu g^*(t)=2\mu g_j(t)&+\frac{\kappa}{\pi(\kappa+1)}\sum_{k=1}^{m}(X_k+\mathrm{i}Y_k)\log|t-z_k|\\&-\frac{1}{2\pi(\kappa+1)}\sum_{k=1}^{m}(X_k-\mathrm{i}Y_k)\frac{t-z_k}{\bar t-\bar z_k}\quad(t\in L_j,j=0,\cdots,m),\end{aligned}\qquad(3.3)$$

这里 $g_j(t)$ 已给于 L_j 上，c_j 待定.

为要求解(3.2)，在 L 上引进新的未知函数 $\omega(t)$ 使得

$$\left.\begin{aligned}\varphi_0(z)&=\frac{1}{2\pi\mathrm{i}}\int_L\frac{\omega(t)}{t-z}\mathrm{d}t,\\ \psi_0(z)&=-\frac{1}{2\pi\mathrm{i}}\int_L\frac{\overline{\omega(t)}}{t-z}\mathrm{d}\bar t-\frac{1}{2\pi\mathrm{i}}\int_L\frac{\bar t\omega'(t)}{t-z}\mathrm{d}t,\end{aligned}\right\}\quad(z\in D)\qquad(3.4)$$

并令

$$c_j=-\int_{L_j}\omega(t)\mathrm{d}s\quad(j=0,\cdots,m).\qquad(3.5)$$

问题(3.2)就化为 $\omega(t)$ 的一个 Fredholm 积分方程

$$\kappa\omega(t)+\frac{\kappa}{2\pi\mathrm{i}}\int_L\omega(\tau)\mathrm{d}\log\frac{\tau-t}{\bar\tau-\bar t}+\frac{1}{2\pi\mathrm{i}}\int_L\overline{\omega(\tau)}\mathrm{d}\frac{\tau-t}{\bar\tau-\bar t}+\int_{L_j}\omega(t)\mathrm{d}s=2\mu g^*(t)$$
$$(t\in L_j;\ j=0,\cdots,m).\qquad(3.6)$$

可以证明，(3.6)对任何 $g^*(t)$ 解存在且唯一.

对于无限域情况十分类似. 这时 $\varphi_0(z),\psi_0(z)$ 的表达式(3.4)右边分别要添加项 Γz 与 $\Gamma' z$，这里 Γ,Γ' 是已知常数，相应于无穷远处的应力与转角. 于是我们仍得方程(3.6)，但在 $2\mu g^*(t)$ 的表达式(3.3)中要添加某些已知项. 也可证明此方程解唯一存在.

四、主力矩的表达式

熟知，在 D 内或其边界上的任何闭路 γ 上的主力矩 M 可表为

$$M = \mathrm{Re}[\chi(z) - z\psi(z) - |z|^2\varphi'(z)]_\gamma \tag{4.1}$$

其中 $\chi'(z) = \psi(z)$，且$[\cdots]_\gamma$ 表示当 z 沿 γ 正向环行一周时方括号中函数的改变量(此处主力矩作用在 γ 的左侧部分)．我们要简化(4.1) 以备后用．

因 $\varphi'(z)$ 在 D 中单值，即使 D 是多连通时也是如此，故(4.1) 可简化为

$$M = \mathrm{Re}[\chi(z) - z\psi(z)]_\gamma = \mathrm{Re}\int_\gamma \frac{\mathrm{d}}{\mathrm{d}z}[\chi(z) - z\psi(z)]\mathrm{d}z = -\mathrm{Re}\int_\gamma z\psi'(z)\mathrm{d}z. \tag{4.2}$$

我们感兴趣的是沿 D 的边界围线 L_j 上的主力矩 M_j．设 D 是有限多连通区域．由(3.1)，我们有

$$\psi'(z) = \frac{\kappa}{2\pi(\kappa+1)}\sum_{k=1}^{m}\frac{X_k - \mathrm{i}Y_k}{z - z_k} + \psi_0'(z).$$

在(4.2) 中取 $\gamma = L_j(j = 1,\cdots,m)$，并以上式代入，便得

$$\begin{aligned} M_j &= \mathrm{Re}\left\{-\int_{L_j} t\psi_0'(t)\mathrm{d}t - \frac{\kappa}{2\pi(\kappa+1)}\sum_{k=1}^{m}(X_k - \mathrm{i}Y_k)\int_{L_j}\frac{t\mathrm{d}t}{t - z_k}\right. \\ &= \mathrm{Re}\left\{\int_{L_j}\psi_0(t)\mathrm{d}t + \frac{\kappa\mathrm{i}}{\kappa+1}(X_j - \mathrm{i}Y_j)z_j\right\} \quad (j = 1,\cdots,m).^{①} \end{aligned} \tag{4.3}$$

因为 $\psi_0(t)$ 在各 L_j 上都是单值的．记

$$M_j^* = M_j - \frac{\kappa}{\kappa+1}\mathrm{Im}\{(X_j - \mathrm{i}Y_j)z_j\} \quad (j = 1,\cdots,m), \tag{4.4}$$

则可写

$$\mathrm{Re}\int_{L_j}\psi_0(t)\mathrm{d}t = M_j^* \quad (j = 1,\cdots,m). \tag{4.5}$$

如果 $X_j + \mathrm{i}Y_j$ 与 M_j 均已知，则 M_j^* 也已知．

更进一步，如果 $\psi_0(z)$ 以(3.4) 式用 $\omega(t)$ 表示，则 M_j^* 也可通过 $\omega(t)$ 表出．在(3.4) 中用 Plemelj 公式，得到

$$\psi_0(t) = -\frac{\kappa}{2}\overline{\omega(t)} - \frac{\kappa}{2\pi\mathrm{i}}\int_L\frac{\overline{\omega(\tau)}}{\tau - t}\mathrm{d}\tau - \frac{1}{2}\bar{t}\omega'(t) - \frac{1}{2\pi\mathrm{i}}\int_L\frac{\bar{\tau}\omega'(\tau)}{\tau - t}\mathrm{d}\tau \quad (t\in L,\ j=1,\cdots,m).$$

两边乘以 $\mathrm{d}t$ 并沿 L_j 积分，便得

$$\int_{L_j}\psi_0(t)\mathrm{d}t = -\kappa\int_{L_j}\overline{\omega(t)}\mathrm{d}t - \int_{L_j}\bar{t}\omega'(t)\mathrm{d}t = -\kappa\int_{L_j}\overline{\omega(t)}\mathrm{d}t + \int_{L_j}\omega(t)\mathrm{d}\bar{t} \quad (j=1,\cdots,m),$$

这里我们作了累次积分次序的交换并利用了下列事实：

$$\frac{1}{2\pi\mathrm{i}}\int_{L_j}\frac{\mathrm{d}t}{t - \tau} = \begin{cases} 0 & (\text{当 } \tau \in L_k,\ k \neq j), \\ -\dfrac{1}{2} & (\text{当 } \tau \in L_j) \end{cases} \qquad (j > 0).$$

于是，由(4.3)，得

$$\mathrm{Re}\int_{L_j}\psi_0(t)\mathrm{d}t = -(\kappa - 1)\mathrm{Re}\int_{L_j}\overline{\omega(t)}\mathrm{d}t \quad (j = 1,\cdots,m). \tag{4.6}$$

这样，(4.5) 就成为

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① 我们不再写出 L_0 上 M_0 的式子，因为 $M_0 = -\sum_{j=1}^{m}M_j$.

$$\mathrm{Re}\int_{L_j}\overline{\omega(t)}\mathrm{d}t=-\frac{M_j^*}{\kappa-1}\quad(j=1,\cdots,m).\tag{4.7}$$

注意，易见(4.4)～(4.7)诸式对无限域D也成立，不论前述Γ与Γ'如何.

五、一般情况时的解法

如前，仍设D是多连通有限域. 在每一边界围线$L_j(j=0,\cdots,m)$上位移$g_j(t)=u_j(t)+\mathrm{i}v_j(t)$只是给出相对于刚性移动，在$L_j$上的真实位移是$g_j(t)+\mathrm{i}\alpha_j t+c_j'$，其中$\alpha_j$与$c_j'$分别是待定的实的与复的常数，$\alpha_j$表示$L_j$绕原点的转角，$c_j'$表示$L_j$的平动. 为确定起见，可设$\alpha_0=0$. 此外，在每一$L_j$处还给出了外力主矢量$X_j+\mathrm{i}Y_j$和主力矩$M_j$，满足平衡条件:

$$\sum_{j=0}^{m}(X_j+\mathrm{i}Y_j)=0,\quad \sum_{j=0}^{m}M_j=0.\tag{5.1}$$

此问题可化为边值问题

$$\kappa\varphi_0(t)-t\overline{\varphi_0'(t)}-\overline{\psi_0(t)}=2\mu g^*(t)+\mathrm{i}\nu_j t+c_j\quad(t\in L_j,j=0,\cdots,m),\tag{5.2}$$

其中$g^*(t)$由(3.3)给出，而

$$\nu_j=2\mu\alpha_j\quad(j=1,\cdots,m;\ \nu_0=0),\tag{5.3}$$

以及$c_j=2\mu c_j'$又是待定的. 根据题设，我们还应补充要求(4.5)，其中M_j^*由(4.4)给出.

为要求解(5.2)，我们先令所有$\nu_j=0$. 如第三节中所示，我们可得唯一的解组$\varphi_*(z)$，$\psi_*(z),c_j^*$满足

$$\kappa\varphi_*(t)-t\overline{\varphi_*'(t)}-\overline{\psi_*(t)}=2\mu g^*(t)+c_j^*\quad(t\in L_j,j=0,\cdots,m).\tag{5.4}$$

其次，对于固定的k $(1\leqslant k\leqslant m)$，我们来求解边值问题

$$\kappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}=\mathrm{i}\delta_{kj}t+c_j\quad(t\in L_j,j=0,\cdots,m),$$

其中δ_{jk}是Kronecker记号(当$j\neq k$时$\delta_{kj}=0$，而当$k=j$时$\delta_{kk}=1$). 这又是第三节中讨论过的问题. 故又可获得其唯一解组$\varphi_k(z),\psi_k(z),c_{jk}$，满足

$$\kappa\varphi_k(t)-t\overline{\varphi_k'(t)}-\overline{\psi_k(t)}=\mathrm{i}\delta_{kj}t+c_{jk}\quad(t\in L_j,j=0,\cdots,m).\tag{5.5}$$

注意$\nu_1,\cdots,\nu_m$是实常数，我们立即可知

$$\left.\begin{aligned}&\varphi_0(z)=\varphi_*(z)+\sum_{k=1}^{m}\nu_k\varphi_k(z),\\&\psi_0(z)=\psi_*(z)+\sum_{k=1}^{m}\nu_k\psi_k(z),\\&c_j^0=c_j^*+\sum_{k=1}^{m}c_{jk}\quad(j=0,\cdots,m)\end{aligned}\right\}\ (z\in D)\tag{5.6}$$

是(5.2)的唯一解组，只要$\nu_1,\cdots,\nu_m$已取定.

为了满足(4.5)，必须选取ν_j使得

$$\mathrm{Re}\int_{L_j}\left[\psi_*(t)+\sum_{k=1}^{m}\nu_k\psi_k(t)\right]\mathrm{d}t=M_j^*\quad(j=1,\cdots,m),$$

亦即

$$\sum_{k=1}^{m}A_{jk}\nu_k=M_j^*-B_j^*\quad(j=1,\cdots,m),\tag{5.7}$$

这里已令

$$A_{jk}=\mathrm{Re}\int_{L_j}\psi_k(t)\mathrm{d}t \quad (j,k=1,\cdots,m), \tag{5.8}$$

$$B_j^*=\mathrm{Re}\int_{L_j}\psi^*(t)\mathrm{d}t \quad (j=1,\cdots,m). \tag{5.9}$$

注意，A_{jk},B_j^* 均为已知常数，且 A_{jk} 与原问题的边界条件无关.

现证矩阵(A_{jk})是满秩的. 事实上，我们来考察零边界条件的问题，亦即，对每一j，设 $g_j(t)=0$, $X_j+\mathrm{i}Y_j=0$, $M_j=0$ $(j=0,\cdots,m)$. 对这个问题显然D只可能作一刚性移动，这是因为，这时在 L_j 上的真实位移只可能是 $\mathrm{i}\nu_j t+c_j$. 如记 $c_j=\alpha_j+\mathrm{i}\beta_j$，则易证

$$\begin{aligned}J&=\int_L(uX_n+vY_n)\mathrm{d}s=\sum_{j=0}^m\int_{L_j}\{(-\nu_j y+\alpha_j)X_n+(\nu_j x+\beta_j)Y_n\}\mathrm{d}s\\&=\sum_{j=0}^m\left\{-\nu_j\int_{L_j}(xY_n-yX_n)\mathrm{d}s+\alpha_j\int_{L_j}X_n\mathrm{d}s+\beta_j\int_{L_j}Y_n\mathrm{d}s\right\}\\&=\sum_{j=0}(-\nu_jM_j+\alpha_jX_j+\beta_jY_j).\end{aligned}$$

在所设条件下，$J=0$. 因此，用类似于第二节中的讨论，便可得出上述结论. 但我们已设 $\nu_0=0$，所以不会有转动，因而所有ν_j等于零. 另一方面，这时 $M_j^*=0$, $B_j^*=0$（因为这时 $\psi_0(z)$ 只可能是一常数)，故(5.7)成为齐次的. 这样，相应于(5.7)的齐次方程组只有平凡解. 这就证明了我们的论断.

所以，(5.7)对任何 M_j^*,B_j^* 有唯一解组. 这就表明原问题唯一可解.

想到 A_{jk} 与 B_j^* 要分别通过$\{\psi_j(t)\}$与 $\psi^*(t)$ 确定，而这些函数本身又要通过某些 Fredholm 积分方程的解 $\omega_k(t)$ 与 $\omega^*(t)$ 的奇异积分表示. 这在实用上极不方便. 但我们可容易地得到 A_{jk} 与 B_j^* 分别通过 $\omega_k(t)$ 与 $\omega^*(t)$ 的表示式. 由(4.6)与(5.8),(5.9)，我们有

$$A_{jk}=-(\kappa-1)\mathrm{Re}\int_{L_j}\overline{\omega_k(t)}\mathrm{d}t \quad (j,k=1,\cdots,m),$$

$$B_j^*=-(\kappa-1)\mathrm{Re}\int_{L_j}\overline{\omega^*(t)}\mathrm{d}t \quad (j=1,\cdots,m).$$

于是，如令

$$\alpha_{jk}=\mathrm{Re}\int_{L_j}\overline{\omega_j(t)}\mathrm{d}t \quad (j,k=1,\cdots,m), \tag{5.10}$$

$$b_j^*=\mathrm{Re}\int_{L_j}\overline{\omega^*(t)}\mathrm{d}t \quad (j=1,\cdots,m), \tag{5.11}$$

则方程组可写成

$$\sum_{k=1}^m\alpha_{jk}\nu_k=-\frac{M_j^*}{\kappa-1}-b_j^* \quad (j=1,\cdots,m). \tag{5.12}$$

它也恒唯一可解.

对多连通无限域情况，只要略作修改，可同样地进行讨论.

六、例　　题

为了验证我们的论述，举两个简单例子.

例 1 设 D 是由一圆孔削弱的无限平面，圆的半径为 r，中心在原点 O 处. 设在边界围线 $L(=L_1)$ 上既无相对位移也无外力主矢量，但已知作用于 L 上外力的主力矩为 $M(=M_1)$. 另外还设在无穷远处既无应力也无转动.

现在我们有 $g_1(t)=0$，$X_1+\mathrm{i}Y_1=0$，$M_1^*=M$. 边值问题(5.2)现成为

$$\kappa\varphi_1(t)-t\overline{\varphi_1'(t)}-\overline{\psi_1(t)}=\mathrm{i}\nu_1 t+c_1\quad(t\in L),\tag{6.1}$$

其中 $\nu_1=2\mu\alpha$，而 α 是沿 L 的转角. 注意，对 $t\in L$，有 $t\bar{t}=r^2$. (6.1)的唯一解组当 $\nu_1=1$ 时显然是

$$\varphi_1(z)=0,\quad \psi_1(z)=\frac{\mathrm{i}r^2}{z},\quad c=0,$$

而

$$A_{11}=\mathrm{Re}\int_L\psi_1(t)\mathrm{d}t=\mathrm{Re}\left\{\mathrm{i}r^2\int_L\frac{\mathrm{d}t}{t}\right\}=2\pi r^2.$$

因此

$$\nu_1=\frac{M}{A_{11}}=\frac{M}{2\pi r^2},\tag{6.2}$$

所以

$$\alpha=\frac{\nu_1}{2\mu}=\frac{M}{4\pi\mu r^2}.\tag{6.3}$$

现在 Колосов-Мусхелишвили 函数是

$$\varphi(z)=0,\quad \psi(z)=\nu_1\psi_1(z)=\frac{\mathrm{i}M}{2\pi z}.$$

由此容易算出，对 $z=\rho e^{\mathrm{i}\theta}$ 点，

$$u+\mathrm{i}v=\frac{\mathrm{i}Me^{\mathrm{i}\theta}}{4\pi\mu\rho}.\tag{6.4}$$

例 2 设 D 是圆环域 $(0<)r_1\leqslant|z|\leqslant r_2$. 设在 $L_1(|t|=r_1)$ 与 $L_2(|t|=r_2)$ 上都无相对位移与外力主矢量，而在 L_1 与 L_2 上分别作用有外力主力矩 M 与 $-M$. 此外，还假定 L_2 保持不动.

我们可利用例 1 结果求解本题. 在上例中，由(6.4)，L_2 上的点 $t=r_2e^{\mathrm{i}\theta}$ 处的位移是

$$u+\mathrm{i}v=\frac{\mathrm{i}Me^{\mathrm{i}\theta}}{4\pi\mu r_2}.$$

如写 $u+\mathrm{i}v=\varepsilon e^{\mathrm{i}\tau}$，则 $\varepsilon=\dfrac{M}{4\pi\mu r_2}\left(\tau=\theta+\dfrac{\pi}{2}\right)$，故在 L_2 上的转角为

$$\alpha_2=\frac{\varepsilon}{r_2}=\frac{M}{4\pi\mu r_2^2}.$$

将整个圆环域作角 $-\alpha_2$ 的转动，就可得到 L_1 上的转角

$$\alpha=\frac{M}{4\pi\mu}\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right),\tag{6.5}$$

这里已保持 L_2 不动.

当圆环域作 $-\alpha_2$ 角转动时，其 Колосов-Мусхелишвили 函数易知为

$$\varphi(z)=-\frac{2\mu\alpha_2\mathrm{i}z}{\kappa+1}=-\frac{\mathrm{i}Mz}{2\pi(\kappa+1)r_2^2},\quad \psi(z)=0,$$

因此所提问题的解为

$$\varphi(z) = -\frac{\mathrm{i}Mz}{2\pi(\kappa+1)r_2^2}, \quad \psi(z) = \frac{\mathrm{i}M}{2\pi z}. \tag{6.6}$$

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New Formulations of the Second Fundamental Problem in Plane Elasticity

Abstract

Some general formulations of the second fundamental problem in plane elasticity are proposed here when the displacements given on the closed boundary contours of a multi-connected elastic region are relative to certain rigid motions which are different to each other for different boundary contours. In such case, it is proved that, for the unique existence of solution, there must be given in addition the principal vector and the principal moment of the external forces on each boundary contour. A method of solution is also given together with some illustrated examples.

［原载于应用数学和力学，1985，6 (3)：223～230；同时以英文稿载于 Appl. Math. & Mech.，1985，6 (3)：223～231.］

双周期平面弹性理论中的复 Airy 函数*

对于平面弹性理论中的复 Airy 函数或称复应力函数，在一般非周期情况下，其表达式早已熟知[1]. 对于单周期情况，这也已清楚[2]. 当弹性区域中有裂纹时，无论非周期或单周期情况，其一般表达式也已获得[3,4]. 对于双周期情况，当基本周期四边形中只有一个洞时，其复应力函数的表达式以及第一基本问题的提法和求解，曾由 W. T. Koiter[5]讨论过；但当其中含有若干个洞时，复应力函数将是多值的，因而复杂得多. 又当其中不是含一些空洞而是含一些裂纹时，情况又有所不同. 本文将就这些一般情况给出复应力函数的一般表达式. 这样就给求解双周期的第一和第二基本问题提供了数学依据.

本文所用记号见[2]和[6]，不再一一解释.

(一) 一般说明

我们将设双周期为 $2\omega_1, 2\omega_2$，其中 $\operatorname{Im}(\omega_2/\omega_1)>0$. 以 $0, 2\omega_1, 2\omega_1+2\omega_2, 2\omega_2$ 为顶点的平行四边形 P_0 称为基本胞腔. 设弹性体在 P 内部有 p 个空洞，其边界为 $L_1, \cdots, L_p$；也可能是含有 p 条裂纹，仍记为 $L_1, \cdots, L_p$(因此是一些开口弧段). 设它们彼此不相交，且都是光滑曲线. 整个弹性区域 S 则是由上述图像作双周期延拓的结果. 用 X_j+iY_j 记 L_j 上外应力的(合)主矢量. 且记 $L=\sum_{j=1}^{p} L_j$.

所谓双周期弹性问题是指：整个 S 中各点的应力都是双周期的，即

$$\sigma_x(z+2\omega_k)=\sigma_x(z),\ \sigma_y(z+2\omega_k)=\sigma_y(z),\ \tau_{xy}(z+2\omega_k)=\tau_{xy}(z),\quad k=1,2. \tag{1.1}$$

由广义胡克定律，由(1.1)立刻可知位移 $g(z)=u(z)+iv(z)$ 是(加法)双准周期的：

$$g(z+2\omega_k)=g(z)+g_k,^{①} \tag{1.2}$$

其中 g_1, g_2 为常数，称为位移的加数.

由于整个弹性体可相差一刚性位移，而平动不改变位移加数，但旋转位移 $i\varepsilon z$ (ε 为实数) 的加数为 $2i\varepsilon\omega_k$. 因此，当 g_k 同时改变这样的项时，并不影响弹性体内各点间的相对位移，当然也不会改变应力分布.

由双周期性，在 P_0 两对对边上的外应力主矢量应相互抵消，故由平衡条件，必有

$$\sum_j (X_j+iY_j)=0. \tag{1.3}$$

所以，如果 P_0 内只有一个空洞或一条裂纹，则其上外应力(合) 主矢量 $X_1+iY_1=0$.

下面对 S 是带洞区域和带裂纹区域的情况分别进行讨论.

* 国家自然科学基金资助课题.

① 本文中下标 k 总是指 $k=1,2$；下标 j 总是指 $j=1,\cdots,p$；不一一标明.

我们恒设 S 是各向同性的，并记其弹性常数为 κ,μ.

（二）带洞区域情况

先考虑 P_0 中带有 p 个空洞的情况. 设各 L_j 取定顺时针向为正向，p_0 的边界 Γ 取定反时针向为正向. Γ 由四边 $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4$ 组成，如图 1 所示. L 和 Γ 所围区域记为 S_0^+. 因此弹性区域 $S=S^+$ 是 $S_0^++\Gamma$ 作双周期延拓的结果.

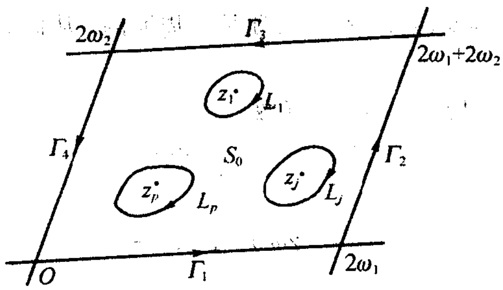

图 1

今在 L_j 所围内域 S_j^- 中任意取定一点 z_j. 根据弹性一般理论，并考虑到双周期性，复应力函数 $\varphi(z),\psi(z)$ 有如下的表达式：

$$\varphi(z)=-\frac{1}{2\pi(\kappa+1)}\sum_j(X_j+\mathrm{i}Y_j)\log\sigma(z-z_j)+\varphi_0(z),\tag{2.1}$$

$$\psi(z)=\frac{\kappa}{2\pi(\kappa+1)}\sum_j(X_j-\mathrm{i}Y_j)\log\sigma(z-z_j)+\psi_1(z)\tag{2.2}$$

$(z\in S)$，其中 $\varphi_0(z),\psi_1(z)$ 在 S^+ 内全纯，$\sigma(z)$ 则是 Weierstrass σ 函数[6]，对数可任意取定分支. 这样，就把 $\varphi(z),\psi(z)$ 的多值部分分离了出来. 将此二式求导，并记 $\Phi_0(z)=\varphi_0'(z)$，$\Psi_1(z)=\psi_1'(z)$ 等等，则

$$\Phi(z)=-\frac{1}{2\pi(\kappa+1)}\sum_j(X_j+\mathrm{i}Y_j)\zeta(z-z_j)+\Phi_0(z),\tag{2.3}$$

$$\Psi(z)=\frac{\kappa}{2\pi(\kappa+1)}\sum_j(X_j-\mathrm{i}Y_j)\zeta(z-z_j)+\Psi_1(z),\tag{2.4}$$

其中 $\zeta(z)=\sigma'(z)/\sigma(z)$ 是 Weierstrass ζ 函数，具有双准周期性

$$\zeta(z+2\omega_k)=\zeta(z)+2\eta_k,\quad \eta_k=\zeta(\omega_k),\tag{2.5}$$

且有性质

$$\omega_2\eta_1-\omega_1\eta_2=\frac{1}{2}\pi\mathrm{i}.\tag{2.6}$$

由(1.3)知，(2.3),(2.4)中求和部分已是双周期的.

熟知，应力和位移具有下列表达式：

$$\sigma_x(z)+\sigma_y(z)=4\mathrm{Re}\Phi(z),\tag{2.7}$$

$$\sigma_y(z)-\sigma_x(z)+2\mathrm{i}\tau_{xy}(z)=2[\bar{z}\Phi'(z)+\Psi(z)],\tag{2.8}$$

$$2\mu g(z)=\kappa\varphi(z)-z\,\overline{\varphi'(z)}-\overline{\psi(z)}.\tag{2.9}$$

记 $\Phi_0(z)=P+\mathrm{i}Q$. 我们将证明 $\Phi_0(z)$ 是双周期的. 由 σ_x,σ_y 的双周期性，故知 P 是双

周期的. 但由 Cauchy-Riemann 方程，因此$\frac{\partial Q}{\partial x},\frac{\partial Q}{\partial y}$也是双周期的，从而 Q 本身是双准周期的. 设$Q(z+2\omega_k)=Q(z)+\nu_k$，其中$\nu_k$为实常数. 因此，$\Phi_0(z+2\omega_k)=\Phi_0(z)+\mathrm{i}\nu_k$. 选取常数$A_0,B_0$，使得$\Phi_*(z)=\Phi_0(z)-A_0z-B_0\zeta(z-z_1)$为双周期的（这一定做得到，见[7]），将上式乘以$\mathrm{d}z/2\pi\mathrm{i}$，并沿$\Gamma$积分，注意到$\varphi_0(z)$已是单值的，立即可知$B_0=0$. 于是，

$$\Phi_0(z+2\omega_k)-\Phi_0(z)=2A_0\omega_k=\mathrm{i}\nu_k.$$

如果$A_0\neq 0$，则$2A_0=\mathrm{i}\nu_k/\omega_k$，因而$\omega_2/\omega_1=\nu_2/\nu_1$是实数，矛盾. 所以$A_0=0$，亦即$\Phi_0(z)$是双周期的.

由(2.8)立即可知，$\bar{z}\Phi'(z)+\Psi(z)$必须是双周期的，从而$\bar{z}\Phi'(z)+\Psi_1(z)$也是双周期的.

选取常数A,B，使得S^+内的全纯函数

$$D(z)=Az+B\zeta(z-z_1) \tag{2.10}$$

是双准周期的，且加数为$2\overline{\omega_k}$；

$$D(z+2\omega_k)=D(z)+2\overline{\omega_k}, \tag{2.11}$$

于是函数

$$m(z)=\bar{z}-D(z) \tag{2.12}$$

就在S^+内是双周期的[5]，利用(2.6)易知

$$A=\frac{2}{\pi\mathrm{i}}(\overline{\omega_2}\eta_1-\overline{\omega_1}\eta_2),\quad B=\frac{2}{\pi\mathrm{i}}(\overline{\omega_2}\omega_1-\overline{\omega_1}\omega_2), \tag{2.13}$$

可见B是实数. 再令$\Psi_0(z)=\Psi_1(z)+D(z)\Phi'(z)=\Psi_1(z)+\bar{z}\Phi'(z)-m(z)\Phi'(z)$，它已是双周期的，且在$S$内全纯. 因此，(2.4)可改写为

$$\Psi(z)=\frac{\kappa}{2\pi(\kappa+1)}\sum_j(X_j-\mathrm{i}Y_j)\zeta(z-z_j)-D(z)\Phi'(z)+\Psi_0(z). \tag{2.14}$$

(2.3),(2.14)便是$\Phi(z),\Psi(z)$的一般表达式，其中$\Phi_0(z),\Psi_0(z)$已都是S^+内的双周期全纯函数.

回到(2.2)，我们可以写

$$\psi(z)=\frac{\kappa}{2\pi(\kappa+1)}\sum_j(X_j-\mathrm{i}Y_j)\log\sigma(z-z_j)-D(z)\Phi(z)+\psi_0(z), \tag{2.15}$$

其中$\psi_0(z)$和$\varphi_0(z)$一样，可相差一常数项，且

$$\psi_0'(z)=\Psi_0(z)+D'(z)\Phi(z) \tag{2.16}$$

是S^+内的双周期全纯函数. (2.1),(2.15)便是$\varphi(z),\psi(z)$的一般表达式. 由于它们的多值部分已分离出来，故$\varphi_0(z),\psi_0(z)$都是S^+内的全纯函数，且由于$\varphi_0'(z)=\Phi_0(z)$，$\psi_0'(z)$都是双周期的，所以它们本身是双准周期的. 记

$$\varphi_0(z+2\omega_k)=\varphi_0(z)+\varphi_k,\quad \psi_0(z+2\omega_k)=\psi_0(z)+\psi_k, \tag{2.17}$$

其中φ_k,ψ_k是常数.

（三）带裂纹区域情况

现设P_0中带有p条(开口)裂纹$L_j=\overset{\frown}{a_jb_j}$，均取定自$a_j$到$b_j$为正向，如图 2. 这时$L_j$两

侧各有外应力，其主矢量分别记为 $X_j^{\pm}+\mathrm{i}Y_j^{\pm}$，而其上合主矢量

$$X_j+\mathrm{i}Y_j=(X_j^{+}+\mathrm{i}Y_j^{+})+(X_j^{-}+Y_j^{-}).$$

这时条件(1.3)仍成立. 如果 P_0 内只有一条裂纹 L_1，则其上合主矢量为 0，或即 $X_1^{+}=-X_1^{-}$，$Y_1^{+}=-Y_1^{-}$.

易见，代替(2.1)，(2.2). 现在应有表达式

$$\varphi(z)=-\frac{1}{4\pi(\kappa+1)}\sum_j(X_j+\mathrm{i}Y_j)\log\sigma(z-a_j)\sigma(z-b_j)+\varphi_0^*(z), \tag{3.1}$$

$$\psi(z)=\frac{\kappa}{4\pi(\kappa+1)}\sum_j(X_j-\mathrm{i}Y_j)\log\sigma(z-a_j)\sigma(z-b_j)+\psi_1^*(z), \tag{3.2}$$

这里已把 $\varphi(z)$,$\psi(z)$的多值部分分离了出来，而 $\varphi_0^*(z)$,$\psi_1^*(z)$已在 S 内全纯. 但由于现在上二式中求和部分在各裂纹端点处有对数奇异性，其导数将有一阶奇异性，因此运用起来极不方便. 为了克服这一缺点，我们引进函数

$$H_j(z)=\int_{L_j}h_j(t)\zeta(t-z)\mathrm{d}t, \tag{3.3}$$

其中

$$h_j(t)=\frac{2t-a_j-b_j}{b_j-a_j}. \tag{3.4}$$

由于 $\zeta(t-z)$在 $t=z$ 处的主部为$\frac{1}{t-z}$，而 $h_j(a_j)=-1$,$h_j(b_j)=+1$，故知[8]，$H_j(z)$在 $z=a_j$ 和 b_j 处，分别有 $\log(z-a_j)$和 $\log(z-b_j)$的奇异性. 因此 $\log\sigma(z-a_j)\sigma(z-b_j)-H_j(z)$在 $z=a_j$,b_j 附近就保持有界了. 现在，我们可将(3.1)，(3.2)改写为

$$\varphi(z)=-\frac{1}{4\pi(\kappa+1)}\sum_j(X_j+\mathrm{i}Y_j)[\log\sigma(z-a_j)\sigma(z-b_j)-H_j(z)]+\varphi_0(z), \tag{3.5}$$

$$\psi(z)=\frac{\kappa}{4\pi(\kappa+1)}\sum_j(X_j-\mathrm{i}Y_j)[\log\sigma(z-a_j)\sigma(z-b_j)-H_j(z)]+\psi_1(z); \tag{3.6}$$

由于 $H_j(z)$在 S 内单值，所以这样的改写并不影响式中多值部分的分离，因而 $\varphi_0(z)$，$\psi_1(z)$仍是 S 内的全纯函数，且在各端点附近，它们的奇异情况分别仍和 $\varphi(z)$,$\psi(z)$的相同. 顺便注意，易证 $H_j(z)$是双周期的.

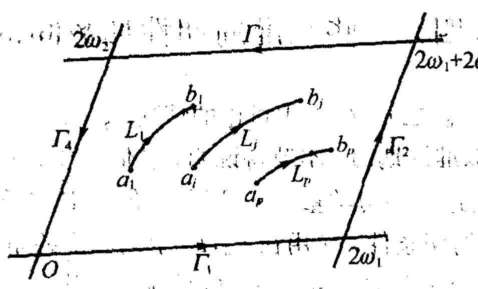

图 2

对于 $\Phi(z)$,$\Psi(z)$，我们有

$$\Phi(z)=-\frac{1}{4\pi(\kappa+1)}\sum_j(X_j+\mathrm{i}Y_j)[\zeta(z-a_j)+\zeta(z-b_j)-H_j'(z)]+\Phi_0(z), \tag{3.7}$$

$$\Psi(z)=\frac{\kappa}{4\pi(\kappa+1)}\sum_j(X_j-\mathrm{i}Y_j)[\zeta(z-a_j)+\zeta(z-b_j)-H_j'(z)]+\Psi_1(z), \tag{3.8}$$

而

$$H_j'(z)=-\int_{a_j}^{b_j}h_j(t)\zeta''(t-z)\mathrm{d}t \tag{3.9}$$

当然也是双周期的.

和(二)中相似，立即可证$\Phi_0(z)$是双周期的. 再令

$$\zeta_1(z)=\frac{6}{(b_1-a_1)^3}\int_{L_1}(b_1-t)(t-a_1)\zeta(t-z)\mathrm{d}t, \tag{3.10}$$

它在S内全纯，在a_j,b_j附近有界，且

$$\zeta_1(z+2\omega_k)=\zeta_1(z)+2\eta_k, \tag{3.11}$$

亦即$\zeta_1(z)$和$\zeta(z)$有相同的加数. 代替$D(z)$，现在改用函数

$$D_1(z)=Az+B\zeta_1(z); \tag{3.12}$$

其中A,B仍由(2.13)给出. 故$D_1(z)$也以$2\overline{\omega_k}$为加数，而

$$m_1(z)=\bar{z}-D_1(z) \tag{3.13}$$

就是双周期的. 这样代替(2.14)，现在可写

$$\Psi(z)=\frac{\kappa}{4\pi(\kappa+1)}\sum_j(X_j-\mathrm{i}Y_j)[\zeta(z-a_j)+\zeta(z-b_j)-H_j(z)]$$
$$-D_1(z)\Phi'(z)+\Psi_0(z), \tag{3.14}$$

其中$\Psi_0(z)$已是S内的双周期全纯函数. 由于$D_1(z)$在各裂纹端点附近保持有界，因此它和上节中的$D(z)$起着同样的作用，并不改变$\Psi(z)$或$\Psi_0(z)$的奇异性态. (3.7),(3.14)便是带裂纹情况时$\Phi(z),\Psi(z)$的一般表达式，其中$\Phi_0(z),\Psi_0(z)$都是双周期的全纯函数.

回到$\psi(z)$，和(2.15)相像，(3.6)现在可改写为

$$\psi(z)=\frac{\kappa}{4\pi(\kappa+1)}\sum_j(X_j-\mathrm{i}Y_j)[\log\sigma(z-a_j)\sigma(z-b_j)-H_j'(z)]$$
$$-D_1(z)\Phi(z)+\psi_0(z), \tag{3.15}$$

这时(2.16)仍成立. (3.5),(3.15)就是现在情况下的双准周期全纯函数，加数仍分别记为φ_k,ψ_k.

(四) 一些双准周期性加数间的关系

前面已看到，不论是(二)或(三)中讨论的情况，$\varphi_0(z),\psi_0(z)$都是双准周期的；我们也知道，位移$g(z)$也是双准周期的. 本节将说明，它们的加数φ_k,ψ_k,g_k间有些什么关系.

首先，我们还要引进一些力学量. 作用在S_0^+的边Γ_k上的外应力主矢量记为$F_k=X_{\Gamma_k}+\mathrm{i}Y_{\Gamma_k}(k=1,2)$. 由双周期条件，作用在$\Gamma_3,\Gamma_4$上的相应的量自然分别为$-F_1,-F_2$. 如果用$X_n(\tau)+\mathrm{i}Y_n(\tau)$ $(\tau\in\Gamma)$表示Γ上对于S_0^+的外应力，则

$$F_k=\int_{\Gamma_k}[X_n(\tau)+\mathrm{i}Y_n(\tau)]\mathrm{d}\sigma, \tag{4.1}$$

其中σ为Γ上的弧长参数. 又令

$$f(\tau)=\mathrm{i}\int_0^{\tau}[X_n(\tau)+\mathrm{i}Y_n(\tau)]\mathrm{d}\sigma,\quad \tau\in\Gamma, \tag{4.2}$$

则因外应力在整个Γ上的主矢量为0，故$f(\tau)$是Γ上的单值函数，且易见

$$F_k = -\mathrm{i}[f(\tau)]_{\Gamma_k}, \tag{4.3}$$

这里$[f(\tau)]_{\Gamma_k}$表示 $f(\tau)$当 τ 沿 Γ_k 正向描过一遍时的改变量（以下将常用这种类似记号），亦即

$$[f(\tau)]_{\Gamma_1} = f(2\omega_1) - f(0) = f(2\omega_1), \quad [f(\tau)]_{\Gamma_2} = f(2\omega_1 + 2\omega_2) - f(2\omega_1).$$

熟知[1]，

$$\varphi(\tau) + \tau\overline{\varphi'(\tau)} + \overline{\psi(\tau)} = f(\tau) + \mathrm{const}, \quad \tau \in \Gamma, \tag{4.4}$$

因此由(4.3)，

$$[\varphi(\tau) + \tau\overline{\varphi'(\tau)} + \overline{\psi(\tau)}]_{\Gamma_k} = \mathrm{i}F_k. \tag{4.5}$$

先考虑带空洞情况．利用$\sigma(z)$的性质[6]：

$$\sigma(z - z_j + 2\omega_k) = -e^{2\eta_k(z - z_j + \omega_k)}\sigma(z - z_j), \tag{4.6}$$

以(2.1)，(2.15)代入上式，并注意到(1.3)，便得

$$\frac{1}{\pi(\kappa+1)}\sum_j (X_j + \mathrm{i}Y_j)(\eta_k z_j - \kappa\,\overline{\eta_k}\,\overline{z_j}) + \varphi_k + \overline{\psi_k} = \mathrm{i}F_k. \tag{4.7}$$

另一方面，以(2.1)，(2.15)代入(2.9)，则考虑$[g(\tau)]_{\Gamma_k}$时，便得

$$\frac{2\kappa}{\pi(\kappa+1)}\sum_j (X_j + \mathrm{i}Y_j)\mathrm{Re}\{\eta_k z_j\} + \kappa\varphi_k - \overline{\psi_k} = 2\mu g_k. \tag{4.8}$$

由(4.7)，(4.8)，立即可得出

$$\left.\begin{aligned} \varphi_k &= -\frac{\eta_k}{\pi(\kappa+1)}\sum_j (X_j + \mathrm{i}Y_j)z_j + \frac{1}{\kappa+1}(\mathrm{i}F_k + 2\mu g_k), \\ \psi_k &= \frac{\kappa\eta_k}{\pi(\kappa+1)}\sum_j (X_j - \mathrm{i}Y_j)z_j - \frac{1}{\kappa+1}(\mathrm{i}\kappa\,\overline{F_k} + 2\mu\,\overline{g_k}). \end{aligned}\right\} \tag{4.9}$$

此外，我们还用 M_Γ 表示(对于 S_0^+ 来说)Γ 上外应力的主力矩．显然作用在 $\Gamma_1+\Gamma_2$ 上的相应主力矩为$\frac{1}{2}M_\Gamma$．又若记 L 上外应力的主力矩为 M_L，则由平衡条件，$M_\Gamma = -M_L$．我们可求得 M_Γ(或 M_L)和 F_1, F_2 之间的关系式．设 $\tau = \xi + \mathrm{i}\eta$，则

$$M_\Gamma = \int_\Gamma [\xi Y_n(\tau) - \eta X_n(\tau)]\mathrm{d}\sigma = \mathrm{Im}\int_\Gamma \bar{\tau}[X_n(\tau) + \mathrm{i}Y_n(\tau)]\mathrm{d}\sigma.$$

另一方面，由应力的双周期性知，$X_n(\tau + 2\omega_k) = -X_n(\tau)$，$Y_n(\tau + 2\omega_k) = -Y_n(\tau)$．因此，由上式可知

$$M_\Gamma = 2\mathrm{Im}\{\overline{\omega_1}F_2 - \overline{\omega_2}F_1\}. \tag{4.10}$$

同时我们还可求得 M_Γ 和 ψ_k 之间的关系式．因为(参看[9])

$$M_\Gamma = -\mathrm{Re}\int_\Gamma \tau\Psi(\tau)\mathrm{d}\tau,$$

故由 $\Psi(\tau)$的双周期性，立得

$$M_\Gamma = 2\mathrm{Re}\{\omega_2[\psi(\tau)]_{\Gamma_1} - \omega_1[\psi(\tau)]_{\Gamma_2}\}.$$

以(2.15)代入，再次利用条件(1.3)，则得

$$\begin{aligned} M_\Gamma = 2\mathrm{Re}\Big\{&\omega_2\Big[\frac{\kappa\eta_1}{\pi(\kappa+1)}\sum_j (X_j - \mathrm{i}Y_j)z_j - 2\,\overline{\omega_1}\Phi(0) + \psi_1\Big] \\ &- \omega_1\Big[\frac{\kappa\eta_2}{\pi(\kappa+1)}\sum_j (X_j - \mathrm{i}Y_j)z_j - 2\,\overline{\omega_2}\Phi(0) + \psi_2\Big]\Big\}. \end{aligned}$$

再注意到(2.6)，经化简后，得

$$M_\Gamma=-\frac{\kappa}{\kappa+1}\mathrm{Im}\{\sum_j(X_j-\mathrm{i}Y_j)z_j\}+\frac{2}{\pi(\kappa+1)}\mathrm{Re}\{(\omega_1\overline{\omega_2}-\overline{\omega_1}\omega_2)\sum_j(X_j+\mathrm{i}Y_j)\zeta(z_j)\}$$
$$+2\mathrm{Re}\{\omega_2\psi_1-\omega_1\psi_2\}+4\mathrm{Re}\{(\omega_1\overline{\omega_2}-\overline{\omega_1}\omega_2)\Phi_0(0)\}. \tag{4.11}$$

但因 $\varphi_0(z)$ 可相差一项 $\mathrm{i}\varepsilon z$，因此不妨设

$$\mathrm{Im}\,\varphi_0'(0)=\mathrm{Im}\,\Phi_0(0)=0, \tag{4.12}$$

则(4.11)右端最后一项可略去而成为

$$M_\Gamma=-\frac{\kappa}{\kappa+1}\mathrm{Im}\{\sum_j(X_j-\mathrm{i}Y_j)z_j\}$$
$$+\frac{2}{\pi(\kappa+1)}\mathrm{Re}\{(\omega_1\overline{\omega_2}-\overline{\omega_1}\omega_2)\sum_j(X_j+\mathrm{i}Y_j)\zeta(z_j)\}$$
$$+2\mathrm{Re}\{\omega_2\psi_1-\omega_1\psi_2\}. \tag{4.11'}$$

注意，当条件(4.12)满足时，$g(z)$ 就只能相差一平动，因而 g_k 就一意确定了.

由(4.10)和(4.11)′，可得 ψ_k 和 F_k 之间的关系式. 例如，当(4.12)满足时，此关系式为

$$\mathrm{Re}\{\omega_2\psi_1-\omega_1\psi_2\}=\frac{\kappa}{2(\kappa+1)}\mathrm{Im}\sum_j(X_j-\mathrm{i}Y_j)z_j$$
$$-\frac{1}{\pi(\kappa+1)}\mathrm{Re}\{(\omega_1\overline{\omega_2}-\overline{\omega_1}\omega_2)\sum_j(X_j+\mathrm{i}Y_j)\zeta(z_j)\}$$
$$+\mathrm{Im}\{\overline{\omega_1}F_2-\overline{\omega_2}F_1\}. \tag{4.13}$$

特别，当每一 L_j 上的外应力主矢量 $X_j+\mathrm{i}Y_j=0$ 时，(4.9)和(4.13)分别成为

$$\varphi_k=\frac{1}{\kappa+1}(\mathrm{i}F_k+2\mu g_k),\quad \psi_k=-\frac{1}{\kappa+1}(\mathrm{i}\kappa\overline{F_k}-2\mu\overline{g_k}), \tag{4.14}$$

$$\mathrm{Re}\{\omega_2\psi_1-\omega_1\psi_2\}=\mathrm{Im}\{\overline{\omega_1}F_2-\overline{\omega_2}F_1\}. \tag{4.15}$$

由此还立即可知，在条件(4.12)之下，

$$\mathrm{Re}\{\omega_2\psi_1-\omega_1\psi_2\}=\mathrm{Im}\{\overline{\omega_1}F_2-\overline{\omega_2}F_1\}=\frac{2\mu}{\kappa+1}\mathrm{Re}\{\overline{\omega_2}g_1-\overline{\omega_1}g_2\}$$
$$=\frac{1}{2}M_\Gamma=-\frac{1}{2}M_L; \tag{4.16}$$

此外，由以上诸式，还可推知

$$\mathrm{Re}\{\overline{\omega_2}\varphi_1-\overline{\omega_1}\varphi_2\}=0. \tag{4.17}$$

现在再考虑带裂纹情况. 如上面演算，并注意 $H_j(z)$ 是双周期的，易见(4.7)，(4.8)现在成为

$$\frac{1}{2\pi(\kappa+1)}\sum_j(X_j+\mathrm{i}Y_j)[\eta_k(a_j+b_j)-\kappa\overline{\eta_k}(\overline{a_j}+\overline{b_j})]+\varphi_k+\overline{\psi_k}=\mathrm{i}F_k, \tag{4.18}$$

$$\frac{\kappa}{\pi(\kappa+1)}\sum_j(X_j+\mathrm{i}Y_j)\mathrm{Re}\{\eta_k(a_j+b_j)\}+\kappa\varphi_k-\overline{\psi_k}=2\mu g_k, \tag{4.19}$$

而(4.9)现在成为

$$\left.\begin{aligned}\varphi_k&=-\frac{\eta_k}{2\pi(\kappa+1)}\sum_j(X_j+\mathrm{i}Y_j)(a_j+b_j)+\frac{1}{\kappa+1}(\mathrm{i}F_k+2\mu g_k),\\ \psi_k&=\frac{\kappa\eta_k}{2\pi(\kappa+1)}\sum_j(X_j-\mathrm{i}Y_j)(a_j+b_j)-\frac{1}{\kappa+1}(\mathrm{i}\kappa\overline{F_k}+2\mu\overline{g_k}).\end{aligned}\right\} \tag{4.20}$$

这时，显然(4.10)仍成立. 由于 $D_1(z)$ 和 $D(z)$ 有完全相同的双准周期性，所以易见(4.11)现在成为

$$M_\Gamma=-\frac{\kappa}{2(\kappa+1)}\mathrm{Im}\{\sum_j(X_j-\mathrm{i}Y_j)(a_j+b_j)\}$$
$$+\frac{1}{\pi(\kappa+1)}\mathrm{Re}\{(\omega_1\overline{\omega_2}-\overline{\omega_1}\omega_2)\sum_j(X_j+\mathrm{i}Y_j)[\zeta(a_j)+\zeta(b_j)]\}$$
$$+2\mathrm{Re}\{\omega_2\psi_1-\omega_1\psi_2\}+4\mathrm{Re}\{(\omega_1\overline{\omega_2}-\overline{\omega_1}\omega_2)\Phi_0(0)\}. \tag{4.21}$$

同样，特别地，在(4.21)条件之下，上式右端最后一项可略去，而(4.13)成为

$$\mathrm{Re}\{\omega_2\psi_1-\omega_1\psi_2\}=\frac{\kappa}{4(\kappa+1)}\mathrm{Im}\sum_j(X_j-\mathrm{i}Y_j)(a_j+b_j)$$
$$-\frac{1}{2\pi(\kappa+1)}\mathrm{Re}\{(\omega_1\overline{\omega_2}-\overline{\omega_1}\omega_2)\sum(X_j+\mathrm{i}Y_j)[\zeta(a_j)+\zeta(b_j)]\}$$
$$+\mathrm{Im}\{\overline{\omega_1}F_2-\overline{\omega_2}F_1\}. \tag{4.22}$$

同样，当每一裂纹 L_j 上两侧外应力的合主矢量 $X_j+\mathrm{i}Y_j=0$ 时，以上诸式仍为(4.14)～(4.17)不变.

附注 以上诸节的讨论不难推广到弹性区域中既有空洞又有裂纹的情况，从略.

(五) 有关基本问题的提法

从前面的讨论，可以说明双周期弹性静力平衡基本问题的提法. 今就带洞区域问题作一些说明.

首先，就第一基本问题而言，即已知 L 上的外应力 $X_n(t)+\mathrm{i}Y_n(t)$ 求弹性平衡. 但显然这是不够的. 还应知道例如 Γ_k 上的外应力主矢量 F_k. 有了这些，则问题解的唯一性立刻可以证明. 因为，如果 $X_n(t)+\mathrm{i}Y_n(t)=0$ 于 L 上，且 $F_k=0$，则立刻可以看出

$$J=\int_L(X_nu+Y_nv)\mathrm{d}s+\int_\Gamma(X_nu+Y_nv)\mathrm{d}\sigma=\mathrm{Re}\int_\Gamma(X_n+\mathrm{i}Y_n)g(\tau)\mathrm{d}\sigma$$
$$=2\mathrm{Re}\{\overline{g_1}F_2-\overline{g_2}F_1\}=0, \tag{5.1}$$

从而可知 S^+ 内各点应力为 0[1]. 不过需注意，由于这时

$$M_L=\int_L(yX_n-xY_n)\mathrm{d}s$$

是已知数，故由(4.10)知，

$$\mathrm{Im}\{\overline{\omega_1}F_2-\overline{\omega_2}F_1\}=-\frac{1}{2}M_L. \tag{5.2}$$

由此可见，F_1,F_2 不能随意给定，而必须要求满足(5.2)式.

这时，第一基本问题的边值条件为

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f(t)+C_j(m,n),\quad t\in L_j(m,n)=L_j+\Omega_{mn}, \tag{5.3}$$

其中 $\Omega_{mn}=2m\omega_1+2n\omega_2$，而 m,n 为整数；又

$$f(t)=\mathrm{i}\int_{t_j}^{t}[X_n(t)+\mathrm{i}Y_n(t)]\mathrm{d}s,\quad t_j,t\in L_j, \tag{5.4}$$

且已作双周期延拓于 $L_j(m,n)$ 上；$C_j(m,n)$ 为待定常数. 以(2.1)、(2.15)代入，便得 $\varphi_0(z)$，

$\psi_0(z)$所应满足的边值条件：

$$\varphi_0(t)+m(t)\overline{\varphi_0'(t)}+\overline{\psi_0(t)}=f_0(t)+C_j(m,n),\quad t\in L_j(m,n).\tag{5.5}$$

这时，$f_0(t)$已是单值的. 这就相当于化为了所有 $X_j+\mathrm{i}Y_j=0$ 的情况，且 $f_0(t)$是双周期的. 我们以后恒作此假定. 求解此问题，实际上只需求解

$$\varphi_0(t)+m(t)\overline{\varphi_0'(t)}+\overline{\psi_0(t)}=f_0(t)+C_j,\quad t\in L_j,\tag{5.6}$$

其中 $C_j=C_j(0,0)$为待定常数，而其他 $C_j(m,n)$可自然地由 $\varphi_0(z),\psi_0(z)$的加数确定.

我们恒要求(4.12)成立. 现因 $X_j+\mathrm{i}Y_j=0$，故

$$\varphi_k+\overline{\psi_k}=\mathrm{i}F_k;\tag{5.7}$$

又因 $f_0(t)$已在各 L_j 上单值，故(4.16)中的

$$M_L=\mathrm{Re}\left\{\int_L f_0(t)\mathrm{d}\bar{t}\right\}.\tag{5.8}$$

如已求出 $\varphi_0(z),\psi_0(z)$，用两种不同方式考虑 M_Γ，易证(4.15)成立；当然(4.17)也成立.

总结起来，双周期第一基本问题可归结为：不失一般性，在假定每一 L_j 上外应力主矢量 $X_j+\mathrm{i}Y_j=0$ 的前提下，寻求 S^+内的两个双准周期全纯函数 $\varphi_0(z),\psi_0(z)$，使其满足边值条件(5.6)（其中 C_j 为待定常数)，且已给 Γ_1,Γ_2 上对 S_0^+ 来说的外应力主矢量 F_1,F_2，满足条件

$$\mathrm{Im}\{\overline{\omega_1}F_2-\overline{\omega_2}F_1\}=-\frac{1}{2}\mathrm{Re}\left\{\int_L f_0(t)\mathrm{d}\bar{t}\right\}\tag{5.9}$$

者，且 $\varphi_0(z),\psi_0(z)$的加数要满足(5.7). 这里已设(4.12)成立. 此外，为了解的唯一性，应补充要求例如 $\varphi_0(0)=\psi_0(0)=0$；如事先已取定 $C_1=0$，则只要补充要求 $\varphi_0(0)=0$ 或 $\psi_0(0)=0$.

条件(4.17)和(4.15)可作为所得解 $\varphi_0(z),\psi_0(z)$的加数 φ_k,ψ_k 的检验公式.

上述问题解的唯一性易于证明，而其存在性可通过化为积分方程（例如，将 Шерман 方法[1]推广)求解时讨论.

对于第二基本问题，即已知各 L_j 上的位移 $g(t)$及其加数 g_k，求弹性平衡. 这时 $X_j+\mathrm{i}Y_j$ 是待定的，问题可化为求解双准周期边值问题

$$\kappa\varphi_0(t)-m(t)\overline{\varphi_0'(t)}-\overline{\psi_0(t)}=g_0(t),\quad t\in L_j,\tag{5.10}$$

这里 $g_0(t)$中除含 $g(t)$的项外，还含有以 $X_j+\mathrm{i}Y_j$ 为待定系数的项，且这时对 g_k 没有像类似(5.9)那样的附加要求. 为了 $\varphi_0(z),\psi_0(z)$的唯一性，应补充要求 $\varphi_0(0)=0$ 或 $\psi_0(0)=0$. 这易于证明，而其存在性也要在求解中证实.

当然，也可考虑各 L_j 上已知相对位移的第二基本问题[9]，从略.

对于带裂纹区域的情况，也可类似地讨论，但允许 $\varphi_0(z),\psi_0(z)$在各裂纹端点附近可有不到一阶的奇异性. 这些无原则困难，也从略.

附注 对非周期和单周期的带裂纹区域，复应力函数中多值部分的分离，除[3,4]中的方法外，也可用类似于这里提供的方法完成.

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ON THE COMPLEX AIRY FUNCTIONS IN DOUBLY PERIODIC PLANE ELASTICITY

Abstract

The general expressions for complex Airy functions are deduced when both the elastic region and the stresses are doubly periodic. The shape and the number of holes or cracks in a fundamental periodic parallelogram are arbitrary. The correct formulations of the first and the second basic problems of elastostatics are also given.

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THE MATHEMATICAL PROBLEMS OF COMPOUND MATERIALS WITH CRACKS IN PLANE ELASTICITY*

The problems of fracture of compound materials are very important in practical applications. In case of infinite plane, when the interface is a straight line and the cracks are rectilinear, there were many investigations under special boundary conditions, e. g., [1,2]. When the interface is a straight line but the cracks may be curvilinear, they were discussed in [3,4]; when the interface is a circle and the crack is a segment, they were discussed in [5]. In this paper, we shall consider the case when the shape of cracks and interface is arbitrary. In particular, the case of the region of compound materials to be finite is also considered. The problems are analyzed in detail and are reduced to certain singular integral equations, the unique solvability of which is proved mathematically.

We shall mainly discuss the first fundamental problem, i. e., given the external stresses on the boundary and on both sides of the cracks, which is most important in practice; in the mean time, it is allowed there is a given displacement difference on the interface for welding. The method used here is also effective for the second fundamental problem, i. e., given the displacements along the boundary and both sides of the cracks; it is also effective even for some kind of mixed boundary problems. These will be remarked at the end of this paper.

§ 1 The Case of Infinite Plane

Let the elastic body be the infinite plane consisting of two different isotropic materials, the interface of which is a closed contour L and both of which contain some cracks, p total in number: $\gamma_j=\widehat{a_jb_j}$, $j=1,\ldots,p$. Assume L and the γ_j's are non-intersecting to each other and are *Liapunov* curves, i. e., the angle of inclination at the point of which, as the function of arc-length, satisfies Hölder condition ($\in H$). The positive direction of L is taken to be clockwise, while that of γ_j is from a_j to b_j. The elastic region (with cracks) located in the positive and the negative sides of L are denoted by S^+ and S^- re-

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spectively (Fig. 1). Denote $S=S^{+}+S^{-}$ and $\gamma \doteq \sum_{j=1}^{p} \gamma_j$. The elastic constants of $S^{\pm}$ are $\kappa^{\pm}$ and $\mu^{\pm}$. The regions (without cracks) located in the positive and the negative sides of L are denoted by S_0^{+} and S_0^{-} respectively.

By X_j+iY_j we denote the external resultant principal vector of γ_j. Γ, Γ' are constants related to the stresses and rotation at infinity. Analogous to the method used in [6], the stress functions in the present case, have the following expressions:

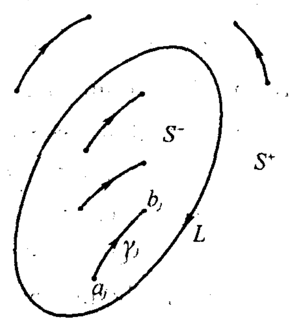

Fig. 1

$$\varphi(z)=-\frac{1}{2\pi(\kappa^{\pm}+1)}\sum_{j=1}^{p}(X_j+iY_j)\log\zeta_j(z)+\Gamma z+\varphi_0(z),\quad z\in S^{\pm}, \tag{1.1}$$

$$\psi(z)=\frac{\kappa^{\pm}}{2\pi(\kappa^{\pm}+1)}\sum_{j=1}^{p}(X_j-iY_j)\log\zeta_j(z)+\Gamma' z+\psi_0(z),\quad z\in S^{\pm}, \tag{1.2}$$

where

$$\zeta_j(z)=\left(\sqrt{\frac{z-a_j}{z-b_j}}+1\right)\Bigg/\left(\sqrt{\frac{z-a_j}{z-b_j}}-1\right),\quad j=1,\ldots,p,$$

where the square root is taken to be a definite branch in the plane cut by γ_j (the logarithms remain multi-valued there), and $\varphi_0(z)$, $\psi_0(z)$ are sectionally holomorphic functions in S with line of jump L and are finite at infinity.

Let us consider the first fundamental problem: given external stress functions $X_n^{\pm}(\tau)+iY_n^{\pm}(\tau)$ along the positive and the negative sides of γ_j and the constants Γ,Γ', determine the elastic equilibrium. Assume these functions $\in H$. Given also the function of displacement difference along the two sides of L. This means: if we denote $u^{\pm}(t)+i\,v^{\pm}(t)$ as the displacements at the point $t\in L$ on the positive and the negative sides respectively, then the function

$$g(t)=[u^{+}(t)+i\,v^{+}(t)]-[u^{-}(t)+i\,v^{-}(t)] \tag{1.3}$$

is given. Assume $g(t)\in H$ too. We want to eliminate this displacement difference by welding.

Denote

$$X_j^{\pm}+iY_j^{\pm}=\int_{\gamma_j}[X_n^{\pm}(\tau)+iY_n^{\pm}(\tau)]ds,\quad j=1,\ldots,p. \tag{1.4}$$

They are respectively the principal vectors of the external stresses along the positive and the negative sides of γ_j, so that

$$X_j+iY_j=(X_j^{+}+iY_j^{+})+(X_j^{-}+iY_j^{-}). \tag{1.5}$$

If we set

$$\left.\begin{aligned} f_j^{+}(\tau)&=i\int_{a_j}^{\tau}[X_n^{+}(\tau)+iY_n^{+}(\tau)]ds,\\ f_j^{-}(\tau)&=i(X_j+iY_j)-i\int_{a_j}^{\tau}[X_n^{-}(\tau)+iY_n^{-}(\tau)]ds,\end{aligned}\right\} \tag{1.6}$$

then, similar to [7], we have the boundary conditions on γ_j:

$$\varphi_j^{\pm}(\tau) + \tau\overline{\varphi'^{\pm}(\tau)} + \overline{\psi^{\pm}(\tau)} = f_j^{\pm}(\tau) + C_j, \quad \tau \in \gamma_j, \quad j = 1,\ldots,p, \tag{1.7}$$

where the C_j's are undetermined constants.

The external stresses on the two sides of L must be in equilibrium, hence we have

$$\varphi^{+}(t) + t\overline{\varphi'^{+}(t)} + \overline{\psi^{+}(t)} = \varphi^{-}(t) + t\overline{\varphi'^{-}(t)} + \overline{\psi^{-}(t)}, \quad t \in L. \tag{1.8}$$

Moreover, by the expressions of displacements of the two sides of L:

$$2\mu^{\pm}[u^{\pm}(t) + \mathrm{i}\,v^{\pm}(t)] = \kappa^{\pm}\varphi^{\pm}(t) - \overline{\varphi'^{\pm}(t)} - \overline{\psi^{\pm}(t)}; \quad t \in L,$$

we have another boundary condition

$$\alpha^{+}\varphi^{+}(t) - \beta^{+}[t\overline{\varphi'^{+}(t)} + \overline{\psi^{+}(t)}] = \alpha^{-}\varphi^{-}(t) - \beta^{-}[t\overline{\varphi'^{-}(t)} + \overline{\psi^{-}(t)}], \quad t \in L, \tag{1.9}$$

in which we have put

$$\alpha^{\pm} = \kappa^{\pm}/\mu^{\pm}, \quad \beta^{\pm} = 1/\mu^{\pm}. \tag{1.10}$$

Substituting (1.1), (1.2) into (1.7) ~ (1.9), we may obtain the corresponding boundary conditions of $\varphi_0(z)$ and $\psi_0(z)$; the forms of (1.7), (1.9) remain unchanged with certain new given functions in their right-hand members, while (1.8) becomes

$$\begin{aligned}
&\varphi_0^{+}(t) + t\overline{\varphi_0'^{+}(t)} + \overline{\psi_0^{+}(t)} - \frac{1}{2\pi(\kappa^{+}+1)}\sum_{j=1}^{p}(X_j + \mathrm{i}Y_j)\log\zeta_j(t) \\
&\quad - \frac{t}{2\pi(\kappa^{+}+1)}\sum_{j=1}^{p}(X_j - \mathrm{i}Y_j)\Big/\overline{\sqrt{R_j(t)}} \\
&\quad + \frac{\kappa^{+}}{2\pi(\kappa^{+}+1)}\sum_{j=1}^{p}(X_j + \mathrm{i}Y_j)\,\overline{\log\zeta_j(t)} \\
&= \varphi_0^{-}(t) + t\overline{\varphi_0'^{-}(t)} + \overline{\psi_0^{-}(t)} - \frac{1}{2\pi(\kappa^{-}+1)}\sum_{j=1}^{p}(X_j + \mathrm{i}Y_j)\log\zeta_j(t) \\
&\quad - \frac{t}{2\pi(\kappa^{-}+1)}\sum_{j=1}^{p}(X_j - \mathrm{i}Y_j)\Big/\overline{\sqrt{R_j(t)}} \\
&\quad + \frac{\kappa^{-}}{2\pi(\kappa^{-}+1)}\sum_{j=1}^{p}(X_j + \mathrm{i}Y_j)\,\overline{\log\zeta_j(t)}, \quad t \in L,
\end{aligned}$$

in which we have denoted $R_j(t) = (t-a_j)(t-b_j)$, $\sqrt{R_j(t)} = \sqrt{\dfrac{t-a_j}{t-b_j}}(t-b_j)$ and have noted that $[\log\zeta_j(t)]' = 1/\sqrt{R_j(t)}$. The above equality may be rewritten as

$$\varphi_0^{+}(t) + t\overline{\varphi_0'^{+}(t)} + \overline{\psi_0^{+}(t)} = \varphi_0^{-}(t) + t\overline{\varphi_0'^{-}(t)} + \overline{\psi_0^{-}(t)} + h(t), \quad t \in L, \tag{1.8$'$}$$

where

$$\begin{aligned}
h(t) = -\frac{\kappa^{+}-\kappa^{-}}{2\pi(\kappa^{+}+1)(\kappa^{-}+1)}\Big\{&\sum_{j=1}^{p}(X_j - \mathrm{i}Y_j)\ln|\zeta_j(t)| \\
&+ \frac{1}{2}\sum_{j=1}^{p}(X_j - \mathrm{i}Y_j)\Big/\overline{\sqrt{R_j(t)}}\Big\},
\end{aligned} \tag{$*$}$$

which is already a single-valued function on L.

Therefore, if we replace $\psi_0(z)$ in (1.2) by

$$\psi_0(z) - \frac{1}{2\pi\mathrm{i}}\int_L \frac{\overline{h(t)}}{t-z}\,\mathrm{d}t,$$

then, by Plemelj's formula, the last term $h(t)$ in the right-hand member of (1.8)$'$ will disappear and we will return to (1.8), while the forms of (1.7), (1.9) still remain unchanged (of course the known functions in their right-hand members are changed anew). Thus, without loss of generality, we may assume each $X_j+iY_j=0$ at the very beginning as well as $\Gamma=\Gamma'=0$. Then $\varphi(z)=\varphi_0(z)$, $\psi(z)=\psi_0(z)$ are sectionally holomorphic and finite at infinity. In this case, (1.6) becomes

$$f_j^\pm(\tau)=\pm\,\mathrm{i}\int_{a_j}^{\tau}[X_n^\pm(\tau)+\mathrm{i}Y_n^\pm(\tau)]\mathrm{d}s, \tag{1.11}$$

and thereby $f_j^\pm(a_j)=0, f_j^+(b_j)=f_j^-(b_j)$. Let

$$F(\tau)=f_j^+(\tau)-f_j^-(\tau),\quad G(\tau)=f_j^+(\tau)+f_j^-(\tau),\quad \tau\in\gamma_j, \tag{1.12}$$

so that $F(a_j)=F(b_j)=0$.

In order to solve the boundary value problem (1.7)~(1.9), we introduce a new unknown function $\omega(\zeta)\in H$, $\zeta\in L+\gamma$, assuming $\omega'(\zeta)\in H^*$ on γ (for the notation, cf. [8]) and $\in H$ on L, such that

$$\varphi(z)=\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\omega(\zeta)}{\zeta-z}\mathrm{d}\zeta,\quad z\in S, \tag{1.13}$$

$$\psi(z)=-\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\overline{\omega(\zeta)}+\bar\zeta\omega'(\zeta)}{\zeta-z}\mathrm{d}\zeta+\frac{1}{2\pi\mathrm{i}}\int_{\gamma}\frac{\overline{F(\tau)}}{\tau-z}\mathrm{d}\tau,\quad z\in S. \tag{1.14}$$

We suppose temporarily

$$\omega(a_j)=\omega(b_j)=0,\quad j=1,\ldots,p. \tag{1.15}$$

Of course, the existence (also uniqueness) of $\omega(\zeta)$ and the equalities (1.15) ought to be proved later.

Substituting (1.13),(1.14) into (1.7), either for positive or negative boundary value, we obtain the same equation

$$(\boldsymbol{K}\omega)(\tau)=\frac{1}{2}G(\tau)+\frac{1}{2\pi\mathrm{i}}\int_{\gamma}\frac{F(\tau_1)}{\bar\tau_1-\bar\tau}\mathrm{d}\,\bar\tau_1,\quad \tau\in\gamma_j,\quad j=1,\ldots,p, \tag{1.16}$$

where we have defined the operator $\boldsymbol{K}$ as

$$\begin{aligned}(\boldsymbol{K}\omega)(\zeta_0)=&\frac{1}{\pi\mathrm{i}}\int_{L+\gamma}\frac{\omega(\zeta)}{\zeta-\zeta_0}\mathrm{d}\zeta-\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\omega(\zeta)\mathrm{d}\log\frac{\zeta-\zeta_0}{\bar\zeta-\bar\zeta_0}\\&-\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\overline{\omega(\zeta)}\,\mathrm{d}\frac{\zeta-\zeta_0}{\bar\zeta-\bar\zeta_0},\quad \zeta_0\in L+\gamma.\end{aligned} \tag{1.17}$$

Substituting (1.13),(1.14) into (1.8), it is observed that it is identically satisfied. At last, substituting them into (1.9), we get an equation on L:

$$\begin{aligned}\boldsymbol{L}\omega\equiv&\ A\omega(t)+\frac{B}{\pi\mathrm{i}}\int_{L+\gamma}\frac{\omega(\zeta)}{\zeta-t}\mathrm{d}\zeta\\&+\frac{\beta^+-\beta^-}{\pi\mathrm{i}}\left\{\int_{L+\gamma}\omega(\zeta)\mathrm{d}\log\frac{\zeta-t}{\bar\zeta-\bar t}+\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\overline{\omega(\zeta)}\,\mathrm{d}\frac{\zeta-t}{\bar\zeta-\bar t}\right\}\\=&\ 4g(t)-\frac{\beta^+-\beta^-}{\pi\mathrm{i}}\int_{\gamma}\frac{F(\tau)}{\bar\tau-\bar t}\mathrm{d}\bar\tau,\quad t\in L,\end{aligned} \tag{1.18}$$

where we have set

$$A = \alpha^+ + \alpha^- + \beta^+ + \beta^-, \quad B = \alpha^+ - \alpha^- - \beta^+ + \beta^-. \tag{1.19}$$

(1.16),(1.18) as a whole constitute a singular integral equation of normal type along $L+\gamma$ (since $A \pm B \neq 0$). We want to find its solution in class h_{2p}, i.e., $\omega(a_j)$, $\omega(b_j)$ to be finite. Since $F(a_j)=F(b_j)=0$, it is easy to verify that, if this equation has a solution in h_{2p}, then (1.15) is really valid and $\omega(\zeta)\in H$, $\omega'(t)\in H$, $\omega'(\tau)\in H^*$.

Now we shall prove: when C_j's ($j=1,\ldots,p$) are suitably chosen (also uniquely), equations (1.16),(1.18) are uniquely solvable in h_{2p}. Analogous to [7], we first prove: when $f_j^{\pm}(\tau)=0$, $g(t)=0$, the corresponding equation has a solution $\omega_0(\zeta)$ after $C_j=C_j^0$ ($j=1,\ldots,p$) is taken, then $\omega_0(\zeta)=0$ on $L+\gamma$ (and hence $C_j^0=0$ necessarily).

Let the corresponding functions given by (1.1),(1.2) through the substitution $\omega_0(\zeta)$ be $\varphi_0(z)$, $\psi_0(z)$ respectively. It is easy to verify that they satisfy the corresponding boundary conditions (1.7)~(1.9), which form the first fundemental problem under zero conditions. By the uniqueness theorem①, the elastic region may only have a rigid motion. Since $\varphi_0(\infty)=\psi_0(\infty)=0$, so $\varphi_0(z)=\psi_0(z)=0$ in S^+, while in S^-, we have

$$\varphi_0(z) = \mathrm{i}\varepsilon z + c,\ \psi_0(z) = d, \quad z\in S^-, \tag{1.20}$$

where ε is a real constant and c, d are complex constants. Thus, we have, whether γ_j is in S^+ or in S^-,

$$\omega_0(\tau) = \varphi_0^+(\tau) - \varphi_0^-(\tau) = 0, \quad \tau\in\gamma.$$

Hence $\varphi_0(z)=\psi_0(z)=0$ in S_0^+ and (1.20) is valid in S_0^-. Then the boundary conditions (1.8),(1.9) on L become

$$\left.\begin{aligned} &\varphi_0^-(t) + t\,\overline{\varphi_0'^-(t)} + \overline{\psi_0^-(t)} = 0,\\ &\kappa^-\varphi_0^-(t) + t\,\overline{\varphi_0'^-(t)} - \overline{\psi_0^-(t)} = 0, \end{aligned}\right\}\quad t\in L. \tag{1.21}$$

which immediately follow $\varphi_0^-(t)=\psi_0^-(t)=0$ on L. Hence $\omega_0(t)=\varphi_0^+(t)-\varphi_0^-(t)=0$, $t\in L$.

Therefore, we have proved $\omega_0(\zeta)=0$ on $L+\gamma$.

Since the index of (1.16),(1.18) related to class h_{2p} is $-p$, then, by the results in [8], §§112,113, its adjoint equation has $2p$ (complex) solutions $\sigma_1(\zeta),\ldots,\sigma_{2p}(\zeta)$ (linearly independent in the real coefficient field) in class h_0 (i.e., solutions may have integrable singularities at a_j, b_j) and (1.16),(1.18) are solvable in h_{2p} iff

$$\operatorname{Re}\sum_{j=1}^{p} C_j\int_{\gamma_j}\sigma_k(\tau)\mathrm{d}\tau = \lambda_k, \quad k=1,\ldots,2p, \tag{1.22}$$

where

$$\lambda_k = \operatorname{Re}\int_{L+\gamma} H(\zeta)\sigma_k(\zeta)\mathrm{d}\zeta, \quad k=1,\ldots,2p, \tag{1.23}$$

in which we have written the right-hand members of (1.16),(1.18) (excluding C_j) as $H(\zeta)$ in a unified form. Seperating the real and the imaginary parts of (1.22), it becomes

① The uniqueness theorem for the considered region (as well as that will be considered in the next section) remains valid, which may be proved by the method similar to that used in [9], § 40.

a system of $2p$ real linear equations with $2p$ real unknowns. It is non-degenerate, since it has only trivial solution $C_j=0$ when $H(\xi)=0$ and so $\lambda_k=0$, as shown before. Hence, (1.16), (1.18) is always uniquely solvable in h_{2p} when the C_j's are uniquely suitably chosen. Hence the problem has been completely solved.

If only the distribution of stresses are required (not necessary for displacements), we may take derivatives of both sides of (1.16), (1.18) and so obtain a singular integral equation in $\Omega(\zeta)=\omega'(\zeta)$, which has to be solved in class h_0 with additional conditions

$$\int_{\gamma_j}\Omega(\tau)\mathrm{d}\tau = 0, \quad j = 1,\dots,p. \tag{1.24}$$

The existence and uniqueness of such a solution may be proved as in [7], which will not be repeated here. The obtained equations do not contain any undetermined constants, which much simplifies the process of solution.

Remark. We may suppose $f_j^{\pm}(\tau)$ and $g(t)\in H$ only. To illustrate this, we should replace (1.14) by

$$\begin{aligned}\psi(z) = &-\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\overline{\omega(\zeta)}}{\zeta-z}\mathrm{d}\zeta + \frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\omega(\zeta)}{\zeta-z}\mathrm{d}\bar{\zeta}\\ &-\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\bar{\zeta}\omega(\zeta)}{(\zeta-z)^2}\mathrm{d}\zeta, \quad z\in S,\end{aligned} \tag{1.14$'$}$$

which is coincident to (1.14) if $\omega'(t)\in H$, $\omega'(\tau)\in H^*$. In this case, on L, for instance, it is impossible to calculate $\varphi'^{\pm}(t)$ and $\psi^{\pm}(t)$ seperately. But it is easily seen that the boundary values $\chi^{\pm}(t)$ of

$$\chi(z) = \bar{z}\varphi'(z) + \psi(z)$$

exist. For, by (1.13) and (1.14)$'$,

$$\begin{aligned}\chi(z) = &-\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\overline{\omega(\zeta)}}{\zeta-z}\mathrm{d}\zeta + \frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\omega(\zeta)}{\zeta-z}\mathrm{d}\bar{\zeta}\\ &-\frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\bar{\zeta}-\bar{z}}{\zeta-z}\,\frac{\omega(\zeta)}{\zeta-z}\mathrm{d}\zeta,\end{aligned}$$

both the positive and the negative boundary values on L of the last term in its right-hand member exist (Plemelj's formulas remain valid in this case[8]) and they are respectively

$$\mp\frac{1}{2}\mathrm{e}^{-2i\theta} - \frac{1}{2\pi\mathrm{i}}\int_{L+\gamma}\frac{\bar{\zeta}-\bar{t}}{\zeta-t}\,\frac{\omega(\zeta)}{\zeta-t}\mathrm{d}\zeta,$$

where θ is the angle of inclination of the tangent at the point $t\in L$. The same is for the boundary values of $\chi(z)$ on γ. Hence, if we understand $t\,\overline{\varphi'^{\pm}(t)}+\overline{\psi^{\pm}(t)}$ $(t\in L)$ and $\tau\,\overline{\varphi'^{\pm}(\tau)}+\overline{\psi^{\pm}(\tau)}$ $(\tau\in\gamma)$ in (1.7)~(1.9) respectively by $\overline{\chi^{\pm}(t)}$ and $\overline{\chi^{\pm}(\tau)}$, then the above discussions remain valid and the same equations (1.16), (1.18) can be obtained without using $\omega'(\zeta)$. However, in the proof of the existence and uniqueness of the solutions, it is natural $\omega_0'(\zeta)\in H$ on account of the zero boundary conditions, so that the proof is valid at all.

This remark may be as well applied to the problem considered in next section as well as those problems discussed in [6,7].

§2 The Case of Bounded Region

In this section we assume the elastic region in consideration is bounded, the external boundary contour and the interface of which are L_0 and L_1 respectively (Fig. 2). All the notations used here are the same as in §1, but now S^+ is the region with cracks bounded between L_0 and L_1. We assume L_0 is also a Liapunov curve and take the counter-clockwise sense as its positive direction. Assume again all the curves considered are non-intersecting to each other. Assume $O\in S^+$ and denote $L=L_0+L_1$. The meaning of S_0^+ is obvious.

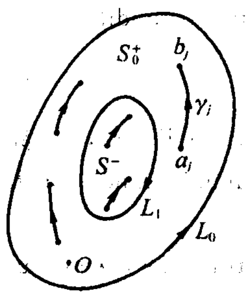

Fig. 2

In this sense, the stress functions $\varphi(z)$, $\psi(z)$ are still given by (1.1), (1.2) but without the terms $\Gamma z, \Gamma' z$.

Again consider the first fundamental problem. Besides the given functions and the assumptions subjected to them as in §1, we know also the external stress function $X_n(t)+\mathrm{i}Y_n(t)\in H$, $t\in L_0$, and denote its principal vector by $X_0+\mathrm{i}Y_0$. The given functions must satisfy the conditions of equilibrium

$$\sum_{j=0}^{p}(X_j+\mathrm{i}Y_j)=0,\quad \sum_{j=0}^{p}M_j=0,$$

where M_0 and $M_j(j>0)$ represent respectively the (resultant) principal moments of the external stresses on L_0 and on the two sides of γ_j. We want to find the elastic equilibrium.

As the same as in §1, we may assume $X_j+\mathrm{i}Y_j=0$ at beginning and so $\varphi(z)$, $\psi(z)$ are sectionally holomorphic in S. At this time, the condition $\sum_{j=0}^{p}M_j=0$ is equivalent to

$$\mathrm{Re}\left\{\int_{L_0} f_0(t)\,\mathrm{d}\bar{t}+\int_{\gamma}F(\tau)\,\mathrm{d}\bar{\tau}\right\}=0, \tag{2.1}$$

where (t_0 being fixed on L_0)

$$f_0(t)=\mathrm{i}\int_{t_0}^{t}[X_n(t)+\mathrm{i}Y_n(t)]\,\mathrm{d}s,\quad t\in L_0, \tag{2.2}$$

which is single-valued on L_0. The proof of (2.1) is similar to that given in [7]. It appears valid since the principal moments applied on the different sides of L_1 are canceled to each other on account of the conditions of equilibrium.

Now, the boundary conditions of the problem to be solved are: on L_0,

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f_0(t)+C_0,\quad t\in L_0, \tag{2.3}$$

where C_0 is a constant; the same as (1.7) on γ; the same as (1.8), (1.9) on L_1 (noting that we have written L_1 in place of L here). On solving the problem, we introduce $\omega(\zeta)$, $\zeta\in L+\gamma$, as the same as (1.13), (1.14) and suppose again (1.15) (its truth may be proved as in §1 and will not be given later). Substituting them into (2.3), we get an integral equation

$$(\boldsymbol{K}\omega)(t)=f_0(t)+\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{F(\tau)}{\bar{\tau}-\bar{t}}\mathrm{d}\bar{\tau}+C_0,\quad t\in L_0,\tag{2.4}$$

where $\boldsymbol{K}\omega$ is given by (1.17); substituting them into (1.7), (1.9), we again obtain (1.16), $(1.18)_1$[①]; (1.8) is satisfied identically again. Using the method analogous to that of D. I. Sherman (cf. [9]) and setting

$$C_0=-\int_{L_0}\omega(t)\mathrm{d}s,\tag{2.5}$$

then (2.4), (1.16), $(1.18)_1$ as a whole constitute a singular integral equation on $L+\gamma$ with undetermined constants $C_1,\ldots,C_p$, which is required to be solved in h_{2p}.

The equation is not always solvable in h_{2p} for arbitrary given functions in the right-hand member, after fixing $C_1,\ldots,C_p$ to be any constants whatever. As in [9] or [7], introduce a pure imaginary constant

$$b_0=\frac{1}{\pi\mathrm{i}}\mathrm{Re}\int_{L_0}\frac{\omega(t)}{t^2}\mathrm{d}t\tag{2.6}$$

and change equation (2.4) into

$$(\boldsymbol{K}\omega)(t)+\frac{b_0}{t}=f_0(t)+\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{F(\tau)}{\bar{\tau}-\bar{t}}\mathrm{d}\bar{\tau}+C_0,\quad t\in L_0,\tag{2.4$'$}$$

where C_0 is still given by (2.5).

We first prove: if condition (2.1) is satisfied and $(2.4)'$, (1.16), $(1.18)_1$ have a solution $\omega(\zeta)$, $\zeta\in L+\gamma$, in class h_{2p} when $C_1,\ldots,C_p$ are suitably chosen, then b_0 defined by (2.6) must be zero. Let the functions given by (1.13), (1.14) and the constant in (2.5) with this $\omega(\zeta)$ be $\varphi(z)$, $\psi(z)$ and C_0 respectively. Returning to boundary conditions, we have

$$\begin{aligned}&\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}+\frac{b_0}{t}=f_0(t)+C_0,\quad t\in L_0,\\&\varphi^\pm(\tau)+\tau\overline{\varphi'^\pm(\tau)}+\overline{\psi^\pm(\tau)}=f_j^\pm(\tau)+C_j,\quad \tau\in\gamma_j,\ j=1,\ldots,p,\\&\varphi^+(t)+t\overline{\varphi'^+(t)}+\overline{\psi^+(t)}=\varphi^-(t)+t\overline{\varphi'^-(t)}+\overline{\psi^-(t)},\quad t\in L_1\end{aligned}$$

(another equality on L_1 is omitted here since it is of no use). Multiplying both sides of these equalities by $\mathrm{d}\bar{t}$ or $\mathrm{d}\bar{\tau}$ and then integrating along L_0,γ_j or L_1 respectively, we get by integration by parts

$$\begin{aligned}&\int_{L_0}[\varphi(t)\mathrm{d}\bar{t}-\overline{\varphi(t)}\mathrm{d}t]+\int_{L_0}\overline{\psi(t)}\mathrm{d}\bar{t}-2\pi\mathrm{i}b_0=\int_{L_0}f_0(t)\mathrm{d}\bar{t},\\&\int_{\gamma_j}[\varphi^\pm(\tau)\mathrm{d}\bar{\tau}-\overline{\varphi^\pm(\tau)}\mathrm{d}\tau]+\int_{\gamma_j}\overline{\psi^\pm(\tau)}\mathrm{d}\bar{\tau}=\int_\gamma f_j^\pm(\tau)\mathrm{d}\bar{\tau}+C_j(\bar{b}_j-\bar{a}_j),\quad j=1,\ldots,p,\\&\int_{L_1}[\varphi^+(t)\mathrm{d}\bar{t}-\overline{\varphi^+(t)}\mathrm{d}t]+\int_{L_1}\overline{\psi^+(t)}\mathrm{d}\bar{t}=\int_{L_1}[\varphi^-(t)\mathrm{d}\bar{t}-\overline{\varphi^-(t)}\mathrm{d}t]+\int_{L_1}\overline{\psi^-(t)}\mathrm{d}\bar{t}.\end{aligned}$$

Taking the positive and the negative boundary equalities in the middle, subtracting, summing up with respect to j from 1 to p, adding to the first of these equalities and taking real parts of both sides, we get by (2.1),

$$\mathrm{Re}\left\{\int_{L_0}\psi(t)\mathrm{d}t+\sum_{j=1}^p\int_{\gamma_j}[\psi^+(\tau)-\psi^-(\tau)]\mathrm{d}\tau\right\}-2\pi\mathrm{i}b_0=0.$$

① Write (1.18) as $(1.18)_1$ when $t\in L_1$.

Taking the real part of the last equality above, we have

$$\mathrm{Re}\left\{\int_{L_1}\psi^+(t)\mathrm{d}t-\int_{L_1}\psi^-(t)\mathrm{d}t\right\}=0.$$

Adding the two obtained equalities together, we get, by rewriting in form,

$$\mathrm{Re}\left\{\int_{L_0}\psi(t)\mathrm{d}t+\sum_j{}^+\int_{\gamma_j}[\psi^+(\tau)-\psi^-(t)]\mathrm{d}\tau+\int_{L_1}\psi^+(t)\mathrm{d}t\right\}$$

$$+\mathrm{Re}\left\{-\sum_j{}^-\int_{\gamma_j}[\psi^+(\tau)-\psi^-(\tau)]\mathrm{d}\tau-\int_{L_1}\psi^-(t)\mathrm{d}t\right\}-2\pi\mathrm{i}b_0=0,$$

where $\sum_j{}^+$ or $\sum_j{}^-$ means summation with respect to j for $\gamma_j\in S_0^+$ or $\in S_0^-$. Since $\psi(z)$ is sectionally holomorphic and has only singularities of order less then one at the end-points a_j,b_j, we know that the expressions in the brackets are zeros by Cauchy's theorem. Hence we have proved $b_0=0$.

Let us now prove: the general equations $(2.4)'$, (1.16), $(1.18)_1$, after $C_1,\ldots,C_p$ are (uniquely) suitably chosen, are solvable in h_{2p} and the solution is unique. As in § 1, it is sufficient to prove that, if by choosing $C_j=C_j^0$ $(j=1,\ldots,p)$, they have a solution $\omega_0(\zeta)$ in h_{2p} under zero boundary conditions, then $\omega_0(\zeta)=0$ on $L+\gamma$.

Let us have corresponding $\varphi_0(z),\psi_0(z),C_0^0$ and b_0^0 for $\omega_0(\zeta)$. Since (2.1) is obviously fulfilled for zero boundary conditions, then $b_0^0=0$, i.e., by(2.6),

$$\mathrm{Re}\int_{L_0}\frac{\omega(t)}{t^2}\mathrm{d}t=0. \tag{2.6$'$}$$

Therefore, on returning to boundary conditions, we shall have the zero boundary value conditions of the first fundamental problem in S. By the uniqueness theorem, we have

$$\varphi_0(z)=\mathrm{i}\varepsilon^\pm z+c^\pm,\ \psi_0(z)=d^\pm,\quad z\in S^\pm, \tag{2.7}$$

where $\varepsilon^\pm$ are real constants and $c^\pm,d^\pm$ are complex constants. It follows immediately

$$\omega_0(\tau)=\varphi_0^+(\tau)-\varphi_0^-(\tau)=0,\quad \tau\in\gamma,$$

and $(2.7)'$ remains valid for $z\in S_0^+$. Substituting them into the boundary conditions of both sides of L_1, we have further

$$c^++\overline{d}^+=c^-+\overline{d}^-, \tag{2.8}$$

$$\alpha^+c^+-\beta^+\overline{d}^+=\alpha^-c^--\beta^-\overline{d}^-, \tag{2.9}$$

$$(\alpha^+-\beta^+)\varepsilon^+=(\alpha^--\beta^-)\varepsilon^-. \tag{2.10}$$

Now, the expressions of $\varphi_0(z),\psi_0(z)$ become respectively

$$\varphi_0(z)=\mathrm{i}\varepsilon^\pm z+c^\pm=\frac{1}{2\pi\mathrm{i}}\int_L\frac{\omega_0(t)}{t-z}\mathrm{d}t,\quad z\in S_0^\pm, \tag{2.11}$$

$$\psi_0(z)=d^\pm=-\frac{1}{2\pi\mathrm{i}}\int_L\frac{\overline{\omega_0(t)+\overline{t}\omega_0'(t)}}{t-z}\mathrm{d}t,\quad z\in S_0^\pm. \tag{2.12}$$

Let z approach to a point t on L_1 from its different sides, by Plemelj's formulas, we obtain, by (2.11),

$$\omega_0(t)=\mathrm{i}(\varepsilon^+-\varepsilon^-)t+(c^+-c^-),\quad t\in L_1.$$

Thereby, if we restrict $z\in S_0^+$ in (2.11), we get

$$i\varepsilon^+ z + c^+ = \frac{1}{2\pi i}\int_{L_0} \frac{\omega_0(t)}{t-z}dt, \quad z \in S_0^+.$$

Taking derivatives on both sides, let $z=0$ (since $O \in S_0^+$), we immediately get $\varepsilon^+ = 0$ by (2. 6)′. Then, by (2. 10), we get $\varepsilon^- = 0$ since $\alpha^- - \beta^- = (\kappa^- - 1)/\mu^- > 0$ for $\kappa^- > 1$. Thus, we know that

$$\omega_0(t) = c^+ - c^-, \quad t \in L_1, \tag{2.13}$$

and furthermore,

$$\left.\begin{aligned} c^+ &= \frac{1}{2\pi i}\int_{L_0} \frac{\omega_0(t)}{t-z}dt, \\ d^+ &= -\frac{1}{2\pi i}\int_{L_0} \frac{\overline{\omega_0(t)} + \bar{t}\omega_0'(t)}{t-z}dt, \end{aligned}\right\} \quad z \in S_0^+.$$

Put

$$\varphi_*(t) = \omega_0(t) - c^+, \ \psi_*(t) = -\omega_0(t) - \bar{t}\omega_0'(t) - d^+, \quad t \in L_0. \tag{2.14}$$

We know, as in [7], $\varphi_*(t)$ and $\psi_*(t)$ are respectively boundary values of holomorphic functions $\varphi_0(z)$ and $\psi_0(z)$ in the region exterior to L_0, which may be proved identical to zero. Hence, by (2. 14), we know that

$$\omega_0(t) = c^+, \quad t \in L_0, \tag{2.15}$$

$$c^+ + \bar{d}^+ = 0. \tag{2.16}$$

On the other hand, $\varphi_0(z), \psi_0(z)$ satisfy boundary condition

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = C_0^0, \quad t \in L_0,$$

and we have already known $\varphi_0(z) = c^+$, $\psi_0(z) = d^+$ in S_0^+, So, by (2. 16), we have $C_0^0 = 0$. But by definition,

$$C_0^0 = -\int_{L_0} \omega_0(t)ds = -c^+|L_0|,$$

where $|L_0|$ is the total length of L_0. Therefore $c^+ = 0$, and then, by (2. 15), $\omega_0(t) = 0$ on L_0. At the same time, we know also $d^+ = 0$ by (2. 16).

Moreover, by (2. 8), (2. 9), we have $c^- = d^- = 0$. Thus, (2. 11) now becomes

$$\frac{1}{2\pi i}\int_{L_1} \frac{\omega_0(t)}{t-z}dt = 0, \quad z \in S_0^\pm,$$

from which we have $\omega_0(t) = 0$ for $t \in L_1$ by Plemelj's formulas.

Hence, $\omega_0(\zeta) = 0$, $\zeta \in L + \gamma$, has been proved, which solves the proposed problem completely.

The statements at the end of § 1 before the Remark remain valid in the case considered here. As for the Remark itself, some comments are necessary in this case for the proof of $b_0 = 0$ under condition (2. 1). For example, (2. 3) must be understood here by

$$[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}]_{z=t}^+ = f_0(t) + C_0, \quad t \in L_0.$$

Particular attention should be paid that we could not take limits seperately for the last two terms in the brackets. But if we take a closed contour L_0' in S_0^+ sufficiently closing to L_0, then we have

$$\int_{L_0'}[\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}]d\bar{z}=\int_{L_0'}[\varphi(z)d\bar{z}-\overline{\varphi(z)}dz+\overline{\psi(z)}d\bar{z}].$$

Letting L_0' tend to L_0, it will tend to

$$\int_{L_0}[\varphi(z)d\bar{z}-\overline{\varphi(z)}dz+\overline{\psi(z)}d\bar{z}]^+_{z=t}$$

as limit (the existence of the latter is known) on account of the uniform convergence by the properties of Cauchy-type integrals. Taking the real parts, we have

$$\int_{L_0}[\operatorname{Re}\psi(z)\,dz]^+_{z=t}=\operatorname{Re}\int_{L_0}f_0(t)d\bar{t},$$

in spite of that the limit $[\psi(z)dz]^+_{z=t}$ itself may not exist. Similar situations will appear on both sides of L_1. On γ_j, we actually only use the difference of

$$\int_{\gamma_j}[\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}]^{\pm}_{z=\tau}d\tau.$$

The uniform convergence of the Cauchy-type integrals is not involved in at all, which could not be assured near the end-points of γ. Hence, the deductions used there remain effective. Thus, the Remark is fit for the case considered in this section.

§3 Some Comments

1. The method used here is also in effect for the second fundamental problem, i.e., given the displacements on both sides of γ (on L_0 also in case S is bounded) and the displacement difference on the interface; in case of the infinite plane, besides, given Γ,Γ' and $X+iY=\sum_{j=1}^{p}(X_j+iY_j)$. In case of bounded region, (1.13), (1.14) should be replaced by

$$\varphi(z)=\frac{1}{2\pi i}\int_{L+\gamma}\frac{\omega(\zeta)}{\zeta-z}d\zeta-\frac{1}{2\pi(\kappa^{\pm}+1)}\sum_{j=1}^{p}(X_j+iY_j)\log\zeta_j(z),\quad z\in S^{\pm},$$

$$\begin{aligned}\psi(z)=&\frac{\kappa^{\pm}}{2\pi i}\int_{L_0}\frac{\overline{\omega(t)}}{t-z}dt+\frac{\kappa^{\pm}}{2\pi i}\int_{L_1+\gamma}\frac{\overline{\omega(\zeta)}}{\zeta-z}d\zeta-\frac{1}{2\pi i}\int_{L+\gamma}\frac{\bar{\zeta}\omega'(\zeta)}{\zeta-z}d\zeta\\&+\frac{\kappa^{\pm}}{2\pi(\kappa^{\pm}+1)}\sum_{j=1}^{p}(X_j-iY_j)\log\zeta_j(z)-\frac{1}{2\pi i}\int_{\gamma}\frac{F(\tau)}{\tau-z}d\tau\\&+\frac{1}{2\pi i}\int_{L_1}\frac{h(t)}{t-z}dt,\quad z\in S^{\pm},\end{aligned}$$

where $F(\tau)=2\mu_j[g_j^+(\tau)-g_j^-(\tau)]$, $\tau\in\gamma_j$ ($\mu_j=\mu^+$ or μ^- according as $\gamma_j\in S_0^+$ or $\in S_0^-$, $g_j^{\pm}(\tau)$ are the displacements at the point $\tau\in\gamma_j$ on the different sides); $h(t)$ is given by (*) in §1; and X_j+iY_j ($j=1,\ldots,p$) are undetermined constants. In case of infinite plane, the terms involving L_0 certainly disappear and in the right-hand member of the expression $\psi(z)$ an additional undetermined constant term B should be added. Moreover, at this time, X_p+iY_p, for instance, may be given by X_j+iY_j, $j=1,\ldots,p-1$, since $X+iY$ is known. Therefore, there are still p undetermined complex constants $X_1+iY_1,\ldots,X_{p-1}$

$+iY_{p-1}$ and B. The method of its solution may be processed as in [7].

In such cases, we could not eliminate the undetermined constants by differentiation. However, if the considered problem is the second fundamental one with given relative displacements (cf. [10]), then such advantage is available.

2. If, in the elastic region, besides cracks, there are some "holes", we may establish effective method of solution by using that established here combining with that given in [9]; even if the interfaces are more than one in number and different materials are more than two in number, the method used here is also effective with reference to [11]①.

3. The method used here is also in effect for the following mixed boundary value problems: given stresses on both sides of some γ_j's and displacements on the others (in case of bounded region, given stresses or displacement on L_0). In particular, the most important case in practice for bounded region is: given stresses on both sides of each γ_j and given displacement on L_0. In this case, we may again eliminate the undetermined constants by differentiation, so that the method is more effective.

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① Note that the general expressions of $\varphi(z)$, $\psi(z)$ and of the introduced function $\omega(\zeta)$ in [11] should be modified as in this paper, though the reasoning used there is valid.

具裂纹的复合材料拼接半平面的第二基本问题*

摘　要

本文考虑由两种带裂纹的不同材料拼接时的问题，已知每一裂纹两侧相对于平动的位移和其上外应力的合主矢量．问题化为了裂纹上的奇异积分方程，并导出了应力强度因子公式．

带裂纹的两种不同材料的拼接半平面，当裂纹上有外荷载时求弹性平衡的问题已有不少研究[1~3]．对此问题的较一般情况，裂纹的个数、形状与位置除要求不和材料交接线相交外均不受限制，我们在[4]中也已解决，把问题归结为求解裂纹和交接线上的奇异积分方程，且证明了解的存在和唯一．在[5]中，并把它具体应用于双直裂纹情况，且化简为只是裂纹上的第一型奇异积分方程，从而易于求数值解．这些都是属于所谓第一基本问题（参看[6]；在[7]中则称为第二基本问题，我们不用这一称呼）．

本文将讨论第二基本问题．这里将不采用经典的提法[6]，而改用我们在[8]中的新提法，即，已给每一裂纹两侧上相对于平动的位移以及其上的外应力合主矢量，求弹性平衡，当然在无穷远处的应力和转角也要给定．这种问题在实际应用中是有意义的，例如在各裂纹中分别独立地打进一些楔子时就会出现这种情况．对于给出各裂纹上相对于刚体运动的位移，并外加给出其上合主力矩的情况，这里就不讨论了，因为如［8］中所示，可化为上述问题求解．

这里将采用与[4]中相似的方法加以适当修改．对于只有一条直裂纹的情况，将作较详细讨论，但所用方法也适用于一般情况．最后并给出求应力强度因子的数学公式．

这里所用方法还可用来求解带裂纹的圆板焊入无限平面时的类似问题，这一情况的第一基本问题已在[9]中解决．

还可注意，当两半平面交接线上有位移差需要焊接时，如[9]中那样，只要稍加修改，这里的方法也完全适用．

（一）问题的数学模型

我们将沿用[4]中的记号．设无限平面由不同的各向同性弹性材料的上、下半平面 Z^+，

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Z^- 拼接而成，交接线为 x 轴，其弹性常数分别记为 $\kappa^+,\mu^+;\kappa^-,\mu^-$. 设平面中有 p 条互不相交的裂纹 $L_1,\cdots,L_p$，它们也都不和 x 轴相交，每一 $L_j=\widehat{a_jb_j}$ 为一 Ляпунов 开口弧段，已取定正向自 a_j 到 b_j. 记 $L=\sum_{j=1}^{p}L_j$. 这些裂纹可以某些在上半平面，另一些在下半平面. 带裂纹的上、下半平面分别记为 S^+,S^-.

我们的问题是：已给诸裂纹 L_j 正(左)、负(右)侧上相对于平动的位移

$$g_\pm(t)=g_{j\pm}(t)=u_{j\pm}(t)+iv_{j\pm}(t),\quad t\in L_j,$$

且设 $g_\pm'(t)\in H$（本文所用的一些记号见[10]）①，自然还应有 $g_+(a_j)=g_-(a_j)$，$g_+(b_j)=g_-(b_j)$. 还设已知每一 L_j 上两侧的外应力合主矢量 X_j+iY_j 以及无穷远处的应力和转角；不失一般性，可假定它们都等于零. 在这些条件下，要求弹性静力平衡.

我们还将采用这样的记号：L_j 所在的半平面的弹性常数记作 κ_j,μ_j；于是，当 $L_j\in Z^+$（或 Z^-）时，$\kappa_j=\kappa^+$（或 κ^-），$\mu_j=\mu^+$（或 μ^-）.

根据所设条件，问题的 Колосов 函数 $\varphi(z),\psi(z)$ 是 S^++S^- 中的分区全纯函数，在裂纹两侧要满足位移条件

$$\kappa_j\varphi_\pm(t)-\overline{[t\varphi_\pm'(t)+\psi_\pm(t)]}=2\mu_jg_\pm(t)+C_j,\quad t\in L_j,\ j=1,\cdots,p,\tag{1.1}$$

其中 C_j 为待定常数，由它可决定 L_j 上的绝对位移，这里 $\varphi_\pm(t)$ 等等表示 $\varphi(z)$ 等等在 L 两侧的边值.

在交接线 x 轴上，由应力和位移的连续性条件，应满足

$$\varphi^+(x)+x\overline{\varphi^{+\prime}(x)}+\overline{\psi^+(x)}=\varphi^-(x)+x\overline{\varphi^{-\prime}(x)}+\overline{\psi^-(x)},\tag{1.2}$$

$$\alpha^+\varphi^+(x)-\beta^-[x\overline{\varphi^{+\prime}(x)}+\overline{\psi^+(x)}]=\alpha^-\varphi^-(x)-\beta^-[x\overline{\varphi^{-\prime}(x)}+\overline{\psi^-(x)}],\tag{1.3}$$

这里 $\varphi^\pm(x)$ 等等表示 $\varphi(z)=\varphi^\pm(z)$ 等等在 x 轴上、下侧的边值，且已令

$$\alpha^\pm=\kappa^\pm/\mu^\pm,\quad \beta^\pm=1/\mu^\pm.\tag{1.4}$$

由于当 $z\to\infty$ 时 $\varphi^\pm(z),\psi^\pm(z)$ 均有限，我们设

$$\left.\begin{aligned}&\varphi^\pm(z)=C^\pm+O\left(\frac{1}{|z|}\right),\quad \psi^\pm(z)=\overline{D}^\pm+O\left(\frac{1}{|z|}\right),\\&\Phi^\pm(z)=\varphi^{\pm\prime}(z)=O\left(\frac{1}{|z|^2}\right),\quad \Psi^\pm(z)=\psi^{\pm\prime}(z)=O\left(\frac{1}{|z|^2}\right),\end{aligned}\right\}\tag{1.5}$$

在(1.2),(1.3)中令 $x\to+\infty$，应有

$$C^++D^+=C^-+D^-,\quad \alpha^+C^+-\beta^+D^+=\alpha^-C^--\beta^-D^-$$

所以实际上我们有两个独立常数，例如 C^-,D^-，如果(1.1)中诸 C_j 事先指定一个，例如 $C_1=0$（即 L_1 上给的是绝对位移），则 C^-,D^- 中也就能事先指定一个；反之，如果 C^-,D^- 都已事先指定，则所有 C_j 均待定，需在求解过程中求出. 以下我们恒设 $C^-=D^-=0$，从而也有 $C^+=D^+=0$.

引进新的未知函数 $\omega(\zeta)$，$\zeta\in L+X$，使得

$$\varphi(z)=\frac{1}{2\pi i}\int_{L+X}\frac{\omega(\zeta)}{\zeta-z}d\zeta,\tag{1.6}$$

① 当 $g'(t)$ 在任一端点 c 附近有 $g'(t)=h_c(t)/(t-c)^\nu$，$0\leqslant\nu<\frac{1}{2}$，$h_c(t)\in H$ 时，本文结果也可证明成立.

$$\psi(z)=\sum_{k=1}^{p}\frac{\kappa_k}{2\pi i}\int_{L_k}\frac{\overline{\omega(\tau)}}{\tau-z}d\tau-\frac{1}{2\pi i}\int_{L}\frac{\bar{\tau}\omega'(\tau)}{\tau-z}d\tau$$
$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\xi)+\xi\omega'(\xi)}{\xi-z}d\xi-\sum_{k=1}^{p}\frac{\mu_k}{\pi i}\int_{L_k}\frac{g_{k+}(\tau)-g_{k-}(\tau)}{\tau-z}d\tau \tag{1.7}$$

($z\bar{\in}L+X$)，我们将假定 $\omega(a_j),\omega(b_j)$ $(j=1,\cdots,p)$ 均有界，这是由于在各裂纹端点位移有界所要求的，以后还将看出，$\omega(a_j)=\omega(b_j)=0$. 我们还假定 $\omega'(t)\in H$ 于 L 上，$\omega(x)$, $\omega'(x)$, $x\omega'(x)$, $x^2\omega'(x)\in\hat{H}$（记号见[7]），且

$$\omega(x)=O\left(\frac{1}{|x|}\right),\ \omega'(x)=O\left(\frac{1}{x^2}\right)\quad(x\to\pm\infty);$$

所有这些假定和 $\omega(\zeta)$ 的存在性，以后将证实.

将(1.6)，(1.7)代入(1.1)中，利用 Plemeli 公式，经化简后，可得同一方程

$$\frac{\kappa_j}{\pi i}\int_{L_j}\frac{\omega(\tau)}{\tau-t}d\tau+\frac{\kappa_j}{2\pi i}\int_{L_j}\omega(\tau)d\log\frac{\bar{\tau}-\bar{t}}{\tau-t}+\frac{1}{2\pi i}\int_{L}\overline{\omega(\tau)}d\frac{\tau-t}{\bar{\tau}-\bar{t}}$$
$$+\kappa_j\sum_{k\neq j}\frac{1}{2\pi i}\int_{L_k}\frac{\omega(\tau)}{\tau-t}d\tau+\sum_{k\neq j}\frac{\kappa_k}{2\pi i}\int_{L_k}\frac{\overline{\omega(\tau)}}{\bar{\tau}-\bar{t}}d\bar{\tau}$$
$$+\frac{\kappa_j}{2\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\xi)}{\xi-t}d\xi-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\xi)}{\xi-\bar{t}}d\xi+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\omega(\xi)d\frac{\xi-t}{\xi-\bar{t}}$$
$$=\mu_j[g_{j+}(t)-g_{j-}(t)]+\sum_{k=1}^{p}\frac{\mu_k}{\pi i}\int_{L_k}\frac{\overline{g_{j+}(\tau)-g_{j-}(\tau)}}{\bar{\tau}-\bar{t}}d\bar{\tau}+C_j,\quad t\in L_j. \tag{1.8}$$

把(1.6)，(1.7)代入(1.2)，可看出是恒等满足的. 最后，把它们代入(1.3)，则得另一方程

$$(\alpha^++\alpha^-+\beta^++\beta^-)\omega(x)+\frac{\alpha^+-\alpha^--\beta^++\beta^-}{\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\xi)}{\xi-x}d\xi$$
$$+\frac{\alpha^+-\alpha^-}{\pi i}\int_{L}\frac{\omega(\tau)}{\tau-x}d\tau+\frac{\beta^+-\beta^-}{\pi i}\int_{L}\overline{\omega(\tau)}d\frac{\tau-x}{\bar{\tau}-x}$$
$$+(\beta^+-\beta^-)\sum_{k=1}^{p}\frac{\kappa_k}{\pi i}\int_{L_k}\frac{\overline{\omega(\tau)}}{\bar{\tau}-x}d\bar{\tau}$$
$$=2(\beta^+-\beta^-)\sum_{k=1}^{p}\frac{\mu_k}{\pi i}\int_{L_k}\frac{\overline{g_{k+}(\tau)-g_{k-}(\tau)}}{\bar{\tau}-x}d\bar{\tau}. \tag{1.9}$$

(1.8)，(1.9)一起构成 $L+X$ 上的一正则型奇异积分方程

$$A(\zeta_0)\omega(\zeta_0)+\frac{B(\zeta_0)}{\pi i}\int_{L+X}\frac{\omega(\zeta)}{\zeta-\zeta_0}d\zeta+\int_{L+X}K_1(\zeta_0,\zeta)\omega(\zeta)d\zeta$$
$$+\int_{L+X}\overline{K_2(\zeta_0,\zeta)}\ \overline{\omega(\zeta)}d\bar{\zeta}=f(\zeta_0)+C(\zeta_0),\quad\zeta_0\in L+X, \tag{1.10}$$

其中

$$A(\zeta)=\begin{cases}0, & \zeta\in L,\\ \alpha^++\alpha^-+\beta^++\beta^-, & \zeta\in X;\end{cases}$$

$$B(\zeta)=\begin{cases}\kappa_j, & \zeta\in L_j,\\ \alpha^+-\alpha^--\beta^++\beta^-, & \zeta\in X;\end{cases}$$

$$C(\zeta)=\begin{cases}C_j, & \zeta\in L_j,\\ 0, & \zeta\in X;\end{cases}$$

$f(\zeta)\in H$ 于 L 上，$\in \hat{H}$ 于 X 上，而 $k_1(\zeta_0,\zeta)$，$k_2(\zeta_0,\zeta)$ 均 $\in H$. 我们要在 h_{2p} 类（即在各裂纹端点处要求 $\omega(\tau)$ 有界）中求解，因而其指标为 $-p$.[10]

类似于[4]，§3 中的讨论，可证明在适当选取 C_j 后，方程(1.10)有唯一解 $\omega(\zeta)$，且此解必满足 $\omega(a_j)=\omega(b_j)=0$ 以及前面所有提到的对于 $\omega(\zeta)$ 的各项假定.

（二）化为裂纹上的奇异积分方程

本节中将说明如何把方程(1.10)或即(1.8)，(1.9)化为积分曲线只在裂纹上的方程因而便于求解. 为简单起见，将假定只有一条裂纹 $L=L_1$；但所用方法对一般情况原则上是一样的. 为确定起见，设 L_1 位于下半平面 Z^- 内.

这时，(1.8)，(1.9)分别成为

$$
\begin{aligned}
&\frac{\kappa^-}{\pi i}\int_L \frac{\omega(\tau)}{\tau-t}d\tau+\frac{\kappa^-}{\pi i}\int_L \omega(\tau)d\log\frac{\bar{\tau}-\bar{t}}{\tau-t}+\frac{1}{2\pi i}\int_L \overline{\omega(\tau)}d\frac{\tau-t}{\bar{\tau}-\bar{t}}\\
&\quad+\frac{\kappa^-}{2\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\xi)}{\xi-t}d\xi-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\xi)}{\xi-\bar{t}}d\xi+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\overline{\omega(\xi)}d\frac{\xi-t}{\xi-\bar{t}}\\
&=\mu^-[g_+(t)+g_-(t)]+\frac{\mu^-}{\pi i}\int_L\frac{g_+(\tau)-g_-(\tau)}{\bar{\tau}-\bar{t}}d\bar{\tau}+C,\quad t\in L,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&(\alpha^++\alpha^-+\beta^++\beta^-)\omega(x)+\frac{\alpha^+-\alpha^--\beta^++\beta^-}{\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\xi)}{\xi-x}d\xi\\
&\quad+\frac{\alpha^+-\alpha^-}{\pi i}\int_L\frac{\omega(\tau)}{\tau-x}d\tau+\frac{\beta^+-\beta^-}{\pi i}\int_L\omega(\tau)d\frac{\tau-x}{\bar{\tau}-x}\\
&\quad+\frac{(\beta^+-\beta^-)\kappa^-}{\pi i}\int_L\frac{\omega(\tau)}{\bar{\tau}-x}d\bar{\tau}\\
&=\frac{2(\beta^+-\beta^-)\mu^-}{\pi i}\int_L\frac{g_+(\tau)-g_-(\tau)}{\bar{\tau}-x}d\bar{\tau},
\end{aligned}
\tag{2.2}
$$

其中 C 为待定常数.

将(2.2)改写为

$$
\begin{aligned}
&A\omega(x)+\frac{B}{\pi}\int_{-\infty}^{+\infty}\frac{\omega(\xi)}{\xi-x}d\xi\\
&\quad=-(\alpha^+-\alpha^-)I_1(x)+(\beta^+-\beta^-)[-\kappa^-I_2(x)+I_3(x)+2\mu^-I_4(x)],
\end{aligned}
\tag{2.3}
$$

这里已令

$$
A=\alpha^++\alpha^-+\beta^++\beta^-,\quad B=\alpha^+-\alpha^--\beta^++\beta^-,
\tag{2.4}
$$

且不论 z 属于 L 或否，已记

$$
\left.
\begin{aligned}
&I_1(z)=\frac{1}{\pi i}\int_L\frac{\omega(\tau)}{\tau-z}d\tau,\quad I_2(z)=\frac{1}{\pi i}\int_L\frac{\omega(\tau)}{\bar{\tau}-z}d\bar{\tau},\\
&I_3(z)=-\frac{1}{\pi i}\int_L\omega(\tau)d\frac{\tau-z}{\bar{\tau}-z}=\frac{1}{\pi i}\int_L\frac{\tau-z}{\bar{\tau}-z}d\omega(\tau)\\
&\qquad=\frac{1}{\pi i}\int_L\frac{\tau-\bar{\tau}}{\bar{\tau}-z}d\omega(\tau),\\
&I_4(z)=\frac{1}{\pi i}\int_L\frac{g_+(\tau)-g_-(\tau)}{\bar{\tau}-z}d\bar{\tau},
\end{aligned}
\right\}
\tag{2.5}
$$

故 $I_4(z)$ 是已知函数. 暂时把 $I_1(z)$，$I_2(z)$，$I_3(z)$ 也当作已知函数，求解(2.3)，则得

$$\omega(x)=\frac{A}{A^2-B^2}\{-(\alpha^+-\alpha^-)I_1(x)+(\beta^+-\beta^-)[-\kappa^-I_2(x)+I_3(x)+2\mu^-I_4(x)]\}$$
$$-\frac{B}{A^2-B^2}\frac{1}{\pi i}\int_L\{-(\alpha^+-\alpha^-)I_1(\xi)+(\beta^+-\beta^-)[-\kappa^-I_2(\xi)+I_3(\xi)+2\mu^-I_4(\xi)]\}\frac{d\xi}{\xi-x},\tag{2.6}$$

交换积分次序，易于验证：

$$\left.\begin{aligned}&\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{I_1(\xi)}{\xi-z}d\xi=\begin{cases}I_1(z), & z\in Z^+,\\ 0, & z\in Z^-,\end{cases}\quad \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{I_1(\xi)}{\xi-x}d\xi=I_1(x),\\ &\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{I_j(\xi)}{\xi-z}d\xi=\begin{cases}0, & z\in Z^+,\\ -I_1(z), & z\in Z^-,\end{cases}\quad \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{I_j(\xi)}{\xi-x}d\xi=-I_j(x),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j=2,3,4.\end{aligned}\right\}\tag{2.7}$$

以此代入(2.6)，则得

$$\omega(x)=-\alpha^*I_1(x)+\beta^*[-\kappa^-I_2(x)+I_3(x)+2\mu^-I_4(x)],\tag{2.8}$$

这里已令

$$\alpha^*=\frac{\alpha^+-\alpha^-}{A+B}=\frac{\alpha^+-\alpha^-}{2(\alpha^++\beta^-)},\quad \beta^*=\frac{\beta^+-\beta^-}{A-B}=\frac{\beta^+-\beta^-}{2(\alpha^-+\beta^+)}.\tag{2.9}$$

为了要将(2.8)代入(2.1)，注意到$t\in L\subset Z^-$，故

$$\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\xi)}{\xi-t}d\xi=\beta^*[\kappa^-I_2(t)-I_3(t)-2\mu^-I_4(t)],$$
$$\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\xi)}{\xi-\bar t}d\xi=-\alpha^*I_1(\bar t),$$
$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\overline{\omega(\xi)}d\frac{\xi-t}{\xi-\bar t}&=\frac{\bar t-t}{2\pi i}\int_{-\infty}^{+\infty}\overline{\omega(\xi)}d\frac{1}{\xi-\bar t}\\&=\frac{\bar t-t}{2\pi i}\overline{\int_{-\infty}^{+\infty}\omega(\xi)d\frac{1}{\xi-t}}\\&=(\bar t-t)\overline{\left[\frac{d}{dt}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\omega(\xi)}{\xi-t}d\xi\right]}\\&=\beta^*(\bar t-t)[\kappa^-\overline{I_2'(t)}-\overline{I_3'(t)}-2\mu^-\overline{I_4'(t)}].\end{aligned}$$

这样，(2.1)就成为

$$\begin{aligned}&\frac{\kappa^-}{\pi i}\int_L\frac{\omega(\tau)}{\tau-t}d\tau+\frac{\kappa^-}{2\pi i}\int_L\omega(\tau)d\log\frac{\bar\tau-\bar t}{\tau-t}+\frac{1}{2\pi i}\int_L\overline{\omega(\tau)}d\frac{\tau-t}{\bar\tau-\bar t}\\&\quad+\alpha^*I_1(\bar t)+\kappa^-\beta^*[\kappa^-I_2(t)-I_3(t)]-\beta^*(t-\bar t)[\kappa^-\overline{I_2'(t)}-\overline{I_3'(t)}]\\&=\mu^-[g_+(t)+g_-(t)+I_4(\bar t)]\\&\quad+2\mu^-\beta^*[\kappa^-I_4(t)-(t-\bar t)\overline{I_4'(t)}]+C,\quad t\in L,\end{aligned}\tag{2.10}$$

这已经是L上的一个第一型奇异积分方程，要求在h_2类中求解.

（三）直裂纹情况，待定常数的消除

对应力场来说，重要的不是$\omega(t)$本身而是$\Omega(t)=\omega'(t)$，而求$\Omega(t)$时可消除确定待定常数C的困难. 为了说明这一过程，为简明起见，现设$L=ab$在Z^-内是一直线段，其倾角为λ $(0\leqslant\lambda\leqslant\pi/2)$. 因此

$$t=-h\mathrm{i}+r\mathrm{e}^{\mathrm{i}\lambda},\quad -1\leqslant r\leqslant 1,\tag{3.1}$$

这里已设 L 的长度为 2，而其中点位于 $-h\mathrm{i}$ 处($h>\sin\lambda$)．注意 $\mathrm{d}\bar{t}/\mathrm{d}t=\mathrm{e}^{-2\mathrm{i}\lambda}$．为简单起见，还设 L 两侧的位移是对称的：

$$g_+(t)=-g_-(t)=g(t),$$

这时当然 $g(a)=g(b)=0$．在这种情况下，(2.10)成为

$$\frac{\kappa^-}{\pi\mathrm{i}}\int_L\frac{\omega(\tau)}{\tau-t}\mathrm{d}\tau+\alpha^*I_1(\bar{t})+\kappa^-\beta^*[\kappa^-I_2(t)+I_3(t)]-\beta^-(t-\bar{t})[\kappa^-\overline{I_2'(t)}-\overline{I_3'(t)}]$$
$$=\mu^-[2\kappa^-\beta^*I_4(t)+I_4(\bar{t})-2\beta^*(t-\bar{t})\overline{I_4'(t)}]+C,\tag{3.2}$$

且这时

$$I_4(z)=\frac{2}{\pi\mathrm{i}}\int_L\frac{g(\tau)}{\tau-z}\mathrm{d}\bar{\tau}\quad(z\in L\text{ 或否}).\tag{3.3}$$

将(3.2)两端对 t 求导数，便得

$$\frac{\kappa^-}{\pi\mathrm{i}}\int_L\frac{\Omega(\tau)}{\tau-t}\mathrm{d}\tau+\alpha^*\mathrm{e}^{-2\mathrm{i}\lambda}J_1(\bar{t})+\kappa^-\beta^*[\kappa^-J_2(t)-J_3(t)]$$
$$-\beta^*(1-\mathrm{e}^{-2\mathrm{i}\lambda})[\kappa^-\overline{J_2(t)}-\overline{J_3(t)}]-\beta^*\mathrm{e}^{-2\mathrm{i}\lambda}(t-\bar{t})[\kappa^-\overline{J_2'(t)}-\overline{J_3'(t)}]=G(t),\tag{3.4}$$

这里 $G(t)$ 是已知函数：

$$G(t)=\mu^-[2\kappa^-\beta^*J_4(t)+\mathrm{e}^{-2\mathrm{i}\lambda}J_4(\bar{t})-2\beta^*(1-\mathrm{e}^{-2\mathrm{i}\lambda})\overline{J_4(t)}$$
$$-2\beta^*\mathrm{e}^{-2\mathrm{i}\lambda}(t-\bar{t})\overline{J_4'(t)}],\tag{3.5}$$

且已令

$$J_1(\bar{t})=\frac{1}{\pi\mathrm{i}}\int_L\frac{\Omega(\tau)}{\tau-\bar{t}}\mathrm{d}\tau,$$
$$J_2(t)=I_2'(t)=\frac{1}{\pi\mathrm{i}}\int_L\frac{\Omega(\tau)}{\bar{\tau}-t}\mathrm{d}\tau,\quad J_2'(t)=\frac{1}{\pi\mathrm{i}}\int_L\frac{\Omega(\tau)}{(\bar{\tau}-t)^2}\mathrm{d}\tau,$$
$$J_3(t)=I_3'(t)=\frac{1}{\pi\mathrm{i}}\int_L\frac{\tau-\bar{\tau}}{(\bar{\tau}-t)^2}\overline{\Omega(\tau)}\mathrm{d}\bar{\tau},\quad J_3'(t)=\frac{2}{\pi\mathrm{i}}\int_L\frac{\tau-\bar{\tau}}{(\bar{\tau}-t)^3}\overline{\Omega(\tau)}\mathrm{d}\bar{\tau},$$
$$J_4(t)=I_4'(t)=\frac{2}{\pi\mathrm{i}}\int_L\frac{g'(\tau)}{\bar{\tau}-t}\mathrm{d}\bar{\tau},\quad J_4'(t)=\frac{2}{\pi\mathrm{i}}\int_L\frac{g'(\tau)}{(\bar{\tau}-t)^2}\mathrm{d}\bar{\tau}.$$

如前，我们已设 $g'(t)\in H$.

现在(3.4)应在 h_0 类（即在 $t=a,b$ 处 $\Omega(t)$ 可以有不到一阶的奇异性）中求解．这时其指标为 $+1$，故其一般解中含有一个任意常数．但由于现在

$$\int_\Lambda\Omega(t)\mathrm{d}t=0,\tag{3.6}$$

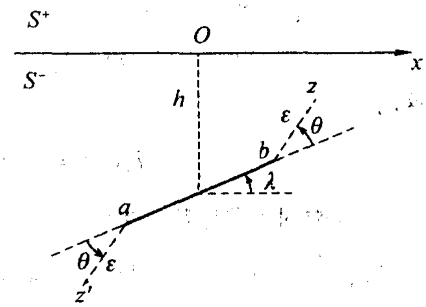
由此条件就可确定此常数．换句话说，我们要在 h_0 类中求满足附加条件(3.6)的方程(3.4)的解，而这种解是唯一存在的．用数值方法求解时，用实变量 r 极为方便，这时(3.4)可化为 $\Omega(t)$ 改写为 $\Omega(r)$ 在 $-1<r<+1$ 上的方程．注意到它是第一型的，在 h_0 类中 $\Omega(r)$ 必为下形：

$$\Omega(r)=\frac{\Omega_0(r)}{\sqrt{1-r^2}},\quad -1<r<1,$$

其中 $\Omega_0(r)\in H$ 于 $-1\leqslant r\leqslant+1$ 上．这样，我们就可用例如 Chebyshev-Lobatto 方法或别的方法用配位法求其数值解(例如，参看[11,12])．

（四）应力强度因子

在实际问题中，最感兴趣的是裂纹端点附近的应力分布，即所谓应力强度因子，亦即在裂纹端点附近应力（用极坐标记号）$\sigma_\theta,\sigma_r,\tau_{r\theta}$的极限状态．仍以上节中考虑的情况来说明，如右图．由(1.6)，



$$\Phi(z)=\varphi'(z)=\frac{1}{2\pi i}\int_{L+X}\frac{\Omega(\zeta)}{\zeta-z}d\zeta.$$

由于我们只对$z=a,b$附近（在这里$\Phi(z)$无界）的情况感兴趣，而$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\Omega(\xi)}{\xi-z}d\xi$有界，因此考虑应力强度因子时，只需考虑

$$\Phi_0(z)=\frac{1}{2\pi i}\int_L\frac{\Omega(\tau)}{\tau-z}d\tau.$$

今在$b=-hi+e^{i\lambda}$附近取一点

$$z=b+\varepsilon e^{i(\lambda+\theta)}=-hi+e^{i\lambda}(1+\varepsilon e^{i\theta}),$$

而令

$$\tau=-hi+\rho e^{i\lambda},$$

则

$$\tau-z=e^{i\lambda}(\rho-\zeta),\quad \zeta=1+\varepsilon e^{i\lambda}.$$

因此

$$\Phi_0(z)=\frac{1}{2\pi i}\int_{-1}^{+1}\frac{\Omega(\rho)}{\rho-\zeta}d\rho=\frac{1}{2\pi i}\int_{-1}^{+1}\frac{\Omega_0(\rho)d\rho}{(\rho-\zeta)\sqrt{1-\rho^2}}.$$

由于$-1<\rho<+1$时$\sqrt{1-\rho^2}>0$，所以$\rho>1$时$\sqrt{1-\rho^2}=i\sqrt{\rho+1}\sqrt{\rho-1}$，其中根式均取正值．于是（参看[10]，(22.6)式），

$$\Phi_0(z)=-\frac{e^{-\frac{1}{2}\pi i}}{2i\sin\frac{\pi}{2}}\frac{\Omega_0(1)}{-i\sqrt{2}\sqrt{\zeta-1}}=\frac{1}{\sqrt{2\varepsilon}}\cdot\frac{i}{2}e^{-\frac{i}{2}\theta}\Omega_0(1)+\cdots,\tag{4.1}$$

其中略去部分在$z=b$附近有界．因此

$$\lim_{\varepsilon\to0}\sqrt{2\varepsilon}\Phi_0(z)=\frac{i}{2}e^{-\frac{i}{2}\theta}\Omega_0(1).\tag{4.2}$$

再由(1.7)，令$\Psi(z)=\psi'(z)$，且略去在$z=b$附近的全纯部分以及对数型奇异部分后，所剩

$$\Psi_0(z)=\frac{\kappa^-}{2\pi i}\int_L\frac{\overline{\Omega(\tau)}}{\tau-z}d\bar\tau-\frac{1}{2\pi i}\int_L\frac{\bar\tau\Omega(\tau)}{(\tau-z)^2}d\tau.$$

所以

$$\begin{aligned}\bar z\Phi_0'(z)+\Psi_0(z)&=\frac{\kappa^-}{2\pi i}\int_L\frac{\overline{\Omega(\tau)}}{\tau-z}d\bar\tau-\frac{1}{2\pi i}\int_L\frac{(\bar\tau-\bar z)\Omega(\tau)}{(\tau-z)^2}d\tau\\&=\kappa^-\chi_1(z)+\chi_2(z),\end{aligned}$$

其中

$$\chi_1(z)=\frac{1}{2\pi i}\int_L\frac{\overline{\Omega(\tau)}}{\tau-z}d\bar{\tau}=\frac{e^{-2i\lambda}}{\pi i}\int_{-1}^{+1}\frac{\overline{\Omega_0(\rho)}d\rho}{(\rho-\tau)\sqrt{1-\rho^2}},$$

$$\chi_2(z)=-\frac{1}{2\pi i}\int_L\frac{(\bar{\tau}-\bar{z})\Omega(\tau)}{(\tau-z)^2}d\tau=-\frac{e^{-2i\lambda}}{2\pi i}\int_{-1}^{+1}\frac{(\rho-\bar{\zeta})\Omega(\rho)}{(\rho-\zeta)^2}d\rho$$

$$=-\frac{e^{-2i\lambda}}{2\pi i}\int_{-1}^{+1}\frac{\Omega(\rho)d\rho}{\rho-\zeta}-\frac{e^{-2i\lambda}(\zeta-\bar{\zeta})}{2\pi i}\int_{-1}^{1}\frac{\Omega(\rho)d\rho}{(\rho-\zeta)^2}$$

$$=-e^{-2i\lambda}\Big[\Phi_0(z)-2i\sin\theta\frac{d}{d\zeta}\Phi_0(z)\Big].$$

易见

$$\lim_{\varepsilon\to0}\sqrt{2\varepsilon}\chi_1(z)=\frac{i}{2}e^{-i(2\lambda+\frac{\theta}{2})}\overline{\Omega_0(1)}.$$

另一方面，由(4.1)，

$$\frac{d}{d\zeta}\Phi_0(z)=\frac{1}{\sqrt{2\varepsilon}}\cdot\frac{e^{-\frac{i}{2}\theta}}{4}\cdot\frac{e^{-i\theta}}{\varepsilon i}\Omega_0(1)+\cdots,$$

或即

$$2i\sin\theta\cdot\frac{d}{d\zeta}\Phi_0(z)=\frac{\sin\theta}{2\sqrt{2\varepsilon}}e^{-\frac{3}{2}i\theta}\Omega_0(1)+\cdots.$$

因此

$$\lim_{\varepsilon\to0}\sqrt{2\varepsilon}\chi_2(z)=-\frac{i}{2}e^{-2i\lambda}(e^{-\frac{i}{2}\theta}-i\sin\theta\cdot e^{-\frac{3}{2}i\theta})\Omega_0(1).$$

于是

$$\lim_{\varepsilon\to0}\sqrt{2\varepsilon}[z\Phi_0'(z)+\Psi_0(z)]$$

$$=\frac{i\kappa^-}{2}e^{-2i\lambda-\frac{i}{2}\theta}\overline{\Omega_0(1)}-\frac{i}{2}e^{-2i\lambda}(e^{-\frac{i}{2}\theta}-i\sin\theta\cdot e^{-\frac{3}{2}i\theta})\Omega_0(1),$$

从而

$$\lim_{\varepsilon\to0}\sqrt{2\varepsilon}[z\Phi_0'(z)+\Psi_0(z)]e^{2i(\lambda+\theta)}$$

$$=-\frac{i}{2}e^{\frac{3}{2}i\theta}[\Omega_0(1)-\kappa^-\overline{\Omega_0(1)}]-\frac{1}{2}\sin\theta\cdot e^{\frac{i}{2}\theta}\Omega_0(1). \tag{4.3}$$

所以(参看([6]，§ 39)，

$$\lim_{\varepsilon\to0}\sqrt{2\varepsilon}(\sigma_\theta+\sigma_\gamma)=4\lim_{\varepsilon\to0}\mathrm{Re}\{\sqrt{2\varepsilon}\Phi_0(z)\}$$

$$=\mathrm{Re}\{2i\,e^{-\frac{i}{2}\theta}\Omega_0(1)\}=2(G_b\sin\frac{\theta}{2}-H_b\cos\frac{\theta}{2}),$$

其中已令

$$\Omega_0(1)=G_b+iH_b, \tag{4.4}$$

而

$$\lim_{\varepsilon\to0}\sqrt{2\varepsilon}(\sigma_\theta-\sigma_r+2i\tau_{r\theta})=2\lim_{\varepsilon\to0}\sqrt{2\varepsilon}\{\bar{z}\Phi_0'(z)+\Psi_0(z)\}e^{2i(\lambda+\theta)}$$

$$=-i\,e^{\frac{3}{2}i\theta}[G_b+iH_b-\kappa^-(G_b-iH_b)]-\sin\theta\,e^{\frac{i}{2}\theta}(G_b+iH_b).$$

由以上二极限式，最后可得

$$\left.\begin{aligned}
\lim_{\varepsilon\to 0}\sqrt{2\varepsilon}\sigma_\theta &= -\left(\frac{\kappa^- - 1}{2}\sin\frac{3}{2}\theta - \sin^3\frac{\theta}{2}\right)G_b \\
&\quad + \left(\frac{\kappa^- + 1}{2}\cos\frac{3}{2}\theta - \cos^3\frac{\theta}{2}\right)H_b, \\
\lim_{\varepsilon\to 0}\sqrt{2\varepsilon}\sigma_r &= \left[\frac{\kappa^- - 1}{2}\sin\frac{3}{2}\theta + \sin\frac{\theta}{2}\left(1+\cos^2\frac{\theta}{2}\right)\right]G_b \\
&\quad - \left[\frac{\kappa^- + 1}{2}\cos\frac{3}{2}\theta + \cos\frac{\theta}{2}\left(1+\sin^2\frac{\theta}{2}\right)\right]H_b, \\
\lim_{\varepsilon\to 0}\sqrt{2\varepsilon}\tau_{r\theta} &= \left(\frac{\kappa^- - 1}{2}\cos\frac{3}{2}\theta - \sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\right)G_b \\
&\quad + \left(\frac{\kappa^- + 1}{2}\sin\frac{3}{2}\theta - \sin\frac{\theta}{2}\cos^3\frac{\theta}{2}\right)H_b.
\end{aligned}\right\} \tag{4.5}$$

在 $z=a$ 附近，取 $z=a-\varepsilon e^{i(\lambda+\theta)}$，类似地处理，并记

$$\Omega_0(-1) = G_a + iH_a, \tag{4.6}$$

则可得类似于(4.5)的式子，但 G_b,H_b 要分别改为 G_a,H_a，且整个式子要变号.

以上应力强度因子公式虽是对一条直裂纹情况进行的，但对多条直裂纹情况也完全成立，因为在某一裂纹端点附近，其他裂纹的影响均只涉及一些有界函数.

附注　如果在(1.1)中令 $\kappa_j=-1$，但在(1.2),(1.3)中的 $\alpha^\pm,\beta^\pm$ 保持原样，则这里所论就变成第一基本问题（与[4]相对照，但(1.1)中应取 $\mu_j=-\frac{1}{2}$，$g_\pm(t)=\pm f_\pm(t)$. 特别地，在(4.5),(4.7)中令 $\kappa^-=-1$，便得第一基本问题时的应力强度因子公式. 这样，第一、第二基本问题可统一处理，可以大大减轻编制计算程序时的工作量.

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ON THE SECOND FUNDAMENTAL PROBLEM OF BONDED HALF-PLANES OF DIFFERENT MEDIA WITH CRACKS

Abstract

The second fundamental problem of two bonded half-planes of different isotropic media with cracks is considered. The displacements relative to a translation on both sides of each crack as well as the resultant principal vector of external stresses on each crack are given. The problem is reduced to a singular integral equation on cracks. The formulas for the stress intensity factors are also derived.

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A CLASS OF MIXED-TYPE FUNDAMENTAL PROBLEMS ON ELASTIC HALF-PLANE*

Abstract

The problems of equilibrium of an elastic half-plane with loads on some intervals of the boundary and zero displacements on its other parts are discussed. The solutions of such problems are expressed in terms of integrals by reducing them to Riemann boundary value problems. The analytic expressions of the solutions are obtained in case of uniform loads. In particular, the solution is written in detail for the important case when uniform pressure is given on a single interval or two equal intervals.

§ 1 Introduction

The solution of an elastic half-plane with stamps is classical[1], which is a kind of mixed-type fundamental problem: given displacements on some intervals of the boundary as well as the principal vector of external stresses on them while on its other infinite part there are no external forces, to find the equilibrium. Here we consider another kind of mixed-type problems: given external forces on some intervals of the boundary while on its other infinite part zero displacements are assumed, to find the equilibrium. As same as the former, this kind of problems has significant applications too. These two kinds of problems could not be simply transferred to each other, since there would be stamps on the infinite part if the latter is transferred to the former. Moreover, the latter itself is very important and its solution may be written in analytic expressions for certain significant special cases, which are convenient in practice. In case of uniform loads on the partial boundary, especially for the case when uniform pressures are applied to a single interval or two equal intervals, the solution is expressed by elementary analytic functions and concrete formulas for finding the undetermined constants appeared are also obtained. For general problems with axis symmetry, our method is most effective (including to determine those constants).

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§ 2 General Formation and Solution of the Problem

Denote the complex plane by Z and the upper (lower) half-plane by $Z^+(Z^-)$. Assume an isotropic medium occupies Z^+ with elastic constants κ, μ. Let $L_j=[a_j, b_j](j=1,\ldots,p)$ be p intervals (from left to right) on the real axis X. Denote

$$L=\sum_{j=1}^{p} L_j, \quad L'=X\setminus L.$$

Our problem is: given the external stress function $X_n(x)+\mathrm{i}Y_n(x)$ on L (assuming to be Hölder continuous), the displacements $u(x)+\mathrm{i}v(x)=0$ on L' and without stresses and rotation at infinity, to find the equilibrium.

First note that, the principal vector of external stresses on the whole x-axis ought to be zero since the displacements on it tend to zero when $|x|\to\infty$ (Cf. [1], § 90); in other words, there must exists a principal vector of external stresses on L' exactly in equilibrium with that on L:

$$X+\mathrm{i}Y=\int_L [X_n(x)+\mathrm{i}Y_n(x)]\mathrm{d}x. \tag{2.1}$$

Thus, for the Kolosov functions, we must have

$$\varphi(z)=\varphi(\infty)+O(1/|z|), \quad \psi(z)=\psi(\infty)+O(1/|z|), \tag{2.2}$$

When $z\to\infty$, and, without loss of generality, we may assume $\varphi(\infty)=\psi(\infty)=0$.

As usual, put

$$f(x)=\mathrm{i}\int_{a_j}^{x}[X_n(t)+\mathrm{i}Y_n(t)]\mathrm{d}t, \quad x\in L_j,\ j=1,\ldots,p. \tag{2.3}$$

The proposed problem may be transferred to the following boundary value problem of mixed-type:

$$\varphi(x)+x\overline{\varphi'(x)}+\overline{\psi(x)}=f(x)+c_j, \quad x\in L_j,\ j=1,\ldots,p, \tag{2.4}$$

$$-\kappa\varphi(x)+x\overline{\varphi'(x)}+\overline{\psi(x)}=0, \quad x\in L', \tag{2.5}$$

where $\varphi(x)=\varphi^+(x)$, etc., with requirements $\varphi(z),\psi(z)$ to be bounded in the vicinities of a_j, b_j and to be zero at infinity.

Let

$$\omega(z)=\begin{cases}\kappa\varphi(z), & z\in Z^+,\\ z\,\overline{\varphi}'(z)+\overline{\psi}(z), & z\in Z^-,\end{cases} \tag{2.6}$$

where $\overline{\varphi}(z)=\overline{\varphi(\overline{z})}$, etc. (obviously $\overline{\varphi'}(z)=\overline{\varphi}'(z)$). By noting that $\overline{\omega}^+(x)=\overline{\omega^-(x)}$, by (2.5), $\omega(z)$ is holomorphic on $Z_0=Z\setminus L$, has L as its curve of discontinuity and $\omega(z)=O(1/z)$ $(z\to\infty)$. By (2.4), we get

$$\omega^+(x)+\kappa\omega^-(x)=\kappa f(x)+\kappa c_j, \quad x\in L_j,\ j=1,\ldots,p. \tag{2.7}$$

This is a Riemann boundary value problem with the requirements $\omega(a_j),\omega(b_j)$ to be finite and $\omega(\infty)=0$. That is, we should solve R_{-1} problem (2.6) in class h_{2p} (for notations, cf. [2]), where the c_j's are undetermined constants. In order to avoid to determine these con-

stants, we set

$$\Omega(z)=\omega'(z), \tag{2.8}$$

then $\Omega(z)$ is holomorphic in Z_0 with

$$\Omega(z)=O(1/|z|^2) \quad (z\to\infty), \tag{2.9}$$

Then, if we put $\Phi(z)=\varphi'(z)$, $\Psi(z)=\psi'(z)$, we have

$$\Omega(z)=\begin{cases}\kappa\Phi(z), & z\in Z^+,\\ \overline{\Phi}(z)+z\overline{\Phi}'(z)+\overline{\Psi}(z), & z\in Z^-.\end{cases} \tag{2.10}$$

By (2.7), $\Omega(z)$ must satisfy the boundary condition

$$\Omega^+(x)=-\kappa\Omega^-(x)+\kappa f'(x), \quad x\in L, \tag{2.11}$$

while $\Omega(z)$ may have singularities of order less than 1 in the vicinities of a_j, b_j and

$$f'(x)=-Y_n(x)+\mathrm{i}X_n(x), \quad x\in L, \tag{2.12}$$

by (2.3). Following the notations in [2], we should solve R_{-2} problem (2.11) in class h_0. It is easy to see the index of the problem is p.

Once $\Omega(z)$ is obtained, we then have, by (2.10),

$$\left.\begin{aligned}\Phi(z)&=\frac{1}{\kappa}\Omega(z),\\ \Psi(z)&=\overline{\Omega}(z)-\Phi(z)-z\Phi'(z),\end{aligned}\right\} \quad z\in Z^+; \tag{2.13}$$

thus, the problem is solved if $\Omega(z)$ is found out.

If we denote

$$\beta=\frac{1}{2\pi}\ln\kappa, \tag{2.14}$$

then the characteristic function of the problem is

$$X(z)=\prod_{j=1}^{p}X_j(z), \quad X_j(z)=(z-a_j)^{-\frac{1}{2}+\mathrm{i}\beta}(z-b_j)^{-\frac{1}{2}-\mathrm{i}\beta}, \tag{2.15}$$

where $X_j(z)$ has been taken as a continuous branch in Z_0, e. g., the branch such that

$$\lim_{z\to\infty} zX_j(z)=1, \quad j=1,\ldots,p. \tag{2.16}$$

By the general theory, the general solution of the problem is

$$\Omega(z)=\frac{\kappa X(z)}{2\pi\mathrm{i}}\int_L\frac{f'(t)\,\mathrm{d}t}{X^+(t)(t-z)}+P_{p-2}(z)X(z), \tag{2.17}$$

where

$$P_{p-2}(z)=C_0+C_1z+\cdots+C_{p-2}z^{p-2} \tag{2.18}$$

is a polynomial with undetermined coefficients ($P_{p-2}=0$ when $p=1$).

In order to determine the coefficients of $P_{p-2}(z)$, we draw closed smooth contours Λ_j surrounding L_j, $j=1,\ldots,p$, mutually exterior to each other, and take the clockwise sense as the positive direction. Noting that $\omega(z)$ is single-valued and continuous on Λ_j, we have

$$\int_{\Lambda_j}\Omega(z)\mathrm{d}z=\int_{\Lambda_j}\omega'(z)\mathrm{d}z=[\omega(z)]_{\Lambda_j}=0, \quad j=1,\ldots,p,$$

Denote $\Lambda=\sum_{j=1}^{p}\Lambda_j$. Then $\int_\Lambda\Omega(z)\mathrm{d}z=0$ since $\Omega(z)$ has a zero point of order at least 2 at

infinity. Hence there are only $p-1$ independent equalities among them, which could just determine $C_0, \ldots, C_{p-2}$. Shrinking Λ_j to L_j, we immediately get (omitting one superfluous equality)

$$\int_{a_j}^{b_j} [\Omega^+(x) - \Omega^-(x)] \mathrm{d}x = 0, \quad j = 1, \ldots, p-1,$$

since $\Omega(z)$ has singularities of order less than one at the ends of L_j. By (2.11), they may be rewritten as

$$\int_{a_j}^{b_j} \Omega^+(x) \mathrm{d}x = \frac{\kappa}{\kappa+1} [f(b_j) - f(a_j)]$$
$$= \frac{\mathrm{i}\kappa}{\kappa+1} (X_j + \mathrm{i}Y_j), \quad j = 1, \ldots, p-1, \tag{2.19}$$

where $X_j + \mathrm{i}Y_j$ is the principal vector of the external stresses on L_j, which is a given number:

$$X_j + \mathrm{i}Y_j = \int_{a_j}^{b_j} [X_n(t) + \mathrm{i}Y_n(t)] \mathrm{d}t. \tag{2.20}$$

Thus, the solution of the proposed problem is obtained as (2.17), where the undetermined coefficients are determined by (2.19). The unique solvability of the linear algebraic system of equations (2.20) satisfied by them may be proved by the natural equilibrium for the elastic mixed-type problem with zero boundary value conditions, which is usually utilized and will be omitted here.

§ 3 The Case of Uniform Loads

The integral expression (2.17) could not be simplified in general. However, when uniform loads are subjected to L: $X_n(t) + \mathrm{i}Y_n(t) = T + \mathrm{i}P$, $t \in L$ (P, T being constants), then an analytic expression of $\Omega(z)$ in finite form may be obtained and then so do for $\Phi(z), \Psi(z)$. In this case, $f'(t) = -(P - \mathrm{i}T)$, therefore, by (2.17),

$$\Omega(z) = -\frac{\kappa(P - \mathrm{i}T)X(z)}{2\pi \mathrm{i}} \int_L \frac{\mathrm{d}t}{X^+(t)(t-z)} + P_{p-2}(z)X(z). \tag{3.1}$$

And since $X_j + \mathrm{i}Y_j = -(P - \mathrm{i}T)(b_j - a_j)$ now, so (2.19) becomes

$$\int_{a_j}^{b_j} \Omega^+(x) \mathrm{d}x = -\frac{\kappa(P - \mathrm{i}T)}{\kappa+1}(b_j - a_j), \quad j = 1, \ldots, p-1. \tag{3.2}$$

In order to simplify $\Omega(z)$, we take Λ as before but sufficiently closing to L such that z situates in its exterior. By residue theorem, it is easy to prove

$$\frac{1}{2\pi \mathrm{i}} \int_\Lambda \frac{\mathrm{d}\zeta}{X(\zeta)(\zeta - z)} = \frac{1}{X(z)} - Q(z),$$

where $Q(z)$ is the principal part of $1/X(z)$ at $z = \infty$, which is a given polynomial of degree p. Shrinking Λ to L again, we have

$$\frac{1}{2\pi \mathrm{i}} \int_L \frac{\mathrm{d}t}{X^+(t)(t-z)} = \frac{1}{\kappa+1} \left[\frac{1}{X(z)} - Q(z) \right]. \tag{3.3}$$

Substituting it into (3. 1), we obtain

$$\Omega(z)=\frac{\kappa(P-\mathrm{i}T)}{\kappa+1}[Q(z)X(z)-1]+P_{p-2}(z)X(z);$$

or, what is the same,

$$\Omega(z)=\frac{\kappa(P-\mathrm{i}T)}{\kappa+1}\{[q_0z^p+q_1z^{p-1}+P_{p-2}(z)]X(z)-1\},\tag{3.4}$$

where $P_{p-2}(z)$ remains to be a polynomial with undetermined coefficients and $q_0z^p+q_1z^{p-1}$ is the first two terms of highest powers in the Laurant expansion of $1/X(z)$ at $z=\infty$. Now the conditions (3. 2) become

$$\int_{a_j}^{b_j}P_{p-2}(x)\,X^+(x)\mathrm{d}x=-\frac{\kappa(P-\mathrm{i}T)}{\kappa+1}\int_{a_j}^{b_j}(q_0x^p+q_1x^{p-1})\,X^+(x)\mathrm{d}x,$$

$$j=1,\dots,p-1.\tag{3.5}$$

Hence, on substituting the elementary expression of $X^+(x)$ into it, the linear algebraic system of equations for $C_0,\dots,C_{p-2}$ is obtained, the coefficients of which are integrals of certain elementary functions. Their approximate values and then those of $C_0,\dots,C_{p-2}$ may be calculated by means of formulae of mechanical quadrature. Therefore $\Omega(z)$ is completely determined.

Once $\Omega(z)$ is obtained, $\Phi(z),\Psi(z)$ may be obtained then by (2. 13). For this aim, we should calculate $\overline{\Omega}(z)$. By (3. 4),

$$\overline{\Omega}(z)=\frac{\kappa(P+\mathrm{i}T)}{\kappa+1}\{[\overline{q}_0z^p+\overline{q}_1z^{p-1}+\overline{P}_{p-2}(z)]\overline{X}(z)-1\},\tag{3.6}$$

thereby we want to simplify $\overline{X}(z)$.

By the chosen branch of the multi-valued function and (2. 15), it is easy to see, in the neighborhood of $z=\infty$,

$$\overline{X}_j(z)=\varepsilon_j(z-a_j)^{-\frac{1}{2}-\mathrm{i}\theta}(z-b_j)^{-\frac{1}{2}+\mathrm{i}\theta}=\varepsilon_j/[(z-a_j)(z-b_j)X_j(z)],$$

where ε_j is a constant, determined by the chosen branch. But it is obvious, by (2. 16),

$$\lim_{z\to\infty}zX_j(z)=1,$$

from which we could conclude $\varepsilon_j=1$ at once. Hence,

$$\overline{X}(z)=1\Big/\Big\{X(z)\prod_{j=1}^{p}(z-a_j)(z-b_j)\Big\}.\tag{3.7}$$

Thus, $\overline{\Omega}(z)$ and $\Psi(z)$ may be also expressed in terms of $X(z)$.

§ 4 Two Important Examples

In this section we consider two important examples in practice. Previously we mention a result: if the proposed problem is symmetric with respect to the imaginary axis, i. e., L as well as the loads on it is symmetric with respect to the y-axis, then, owing to the symmetric condition for the stresses, we easily know

$$\Phi(-\overline{z})=\overline{\Phi(z)},\quad\Psi(-\overline{z})=\overline{\Psi(z)},\tag{4.1}$$

and hence

$$\Omega(-\bar{z}) = \overline{\Omega(z)}. \tag{4.2}$$

Example 1. Suppose $p=1$. Assume $L=[-a,a]$, to which uniform pressure P is applied, and there are no displacements on $X\backslash L$. Find the equilibrium.

This particular example is the simplest one. By (3.4),

$$\Omega(z) = \frac{\kappa P}{\kappa+1}[(q_0 z + q_1)X(z) - 1].$$

In the neighborhood of $z=\infty$, by (2.15),

$$1/X(z) = (z+a)^{\frac{1}{2}-i\beta}(z-a)^{\frac{1}{2}+i\beta} = z - 2i\beta a + O(1/|z|),$$

and then

$$\Omega(z) = \frac{\kappa P}{\kappa+1}[(z-2i\beta a)X(z) - 1]. \tag{4.3}$$

Then, by (3.7), we know

$$\overline{\Omega}(z) = \frac{\kappa P}{\kappa+1}\Big[\frac{z+2i\beta a}{z^2-a^2}\frac{1}{X(z)} - 1\Big].$$

Noting that

$$X'(z) = -\frac{z+2i\beta a}{z^2-a^2}X(z),$$

we obtain at length, by (2.13),

$$\Phi(z) = \frac{P}{\kappa+1}[(z-2ia)X(z) - 1], \tag{4.4}$$

$$\Psi(z) = \frac{P}{\kappa+1}\Big\{\Big[2i\beta a - z + \frac{(1+4\beta^2)a^2 z}{z^2-a^2}\Big]X(z) + \frac{(z+2i\beta a)}{(z^2-a^2)X(z)} - (\kappa-1)\Big\}. \tag{4.5}$$

It is easy to verify, in case of the chosen single-valued branch,

$$X(-\bar{z}) = -\overline{X(z)},$$

and so (4.1) is actually valid.

Example 2. Suppose $p=2$, L_1, L_2 with the same length: $L_1=[-b,-a]$, $L_2=[a,b]$ $(0<a<b)$, to which uniform pressure P is applied, and there are no displacements on $X\backslash L$. Find the equilibrium.

In this case, by (3.4), we have

$$\Omega(z) = \frac{\kappa P}{\kappa+1}[(q_0 z^2 + q_1 z + C)X(z) - 1]. \tag{4.6}$$

Now,

$$X_1(z) = (z+a)^{-\frac{1}{2}-i\beta}(z+b)^{-\frac{1}{2}+i\beta} = \frac{1}{z} - \Big[(\frac{1}{2}+i\beta)a + (\frac{1}{2}-i\beta)b\Big]\frac{1}{z^2} + \cdots,$$

$$X_2(z) = (z-a)^{-\frac{1}{2}+i\beta}(z-b)^{-\frac{1}{2}-i\beta} = \frac{1}{z} + \Big[(\frac{1}{2}-i\beta)a + (\frac{1}{2}+i\beta)b\Big]\frac{1}{z^2} + \cdots,$$

and then

$$X(z) = \frac{1}{z^2} + \frac{2i\beta(b-a)}{z^3} + \cdots, \tag{4.7}$$

by which we get immediately

$$1/X(z) = z^2 - 2i\beta(b-a)z + O(1/|z|). \tag{4.8}$$

At the same time, we know

$$X(-\bar{z}) = \overline{X(z)}. \tag{4.9}$$

Thus, by (4.6), we obtain

$$\Omega(z) = \frac{\kappa P}{\kappa+1}\{[z^2 - 2i\beta(b-a)z + C]X(z) - 1\}, \tag{4.10}$$

where C, by using (2.19) and noting that $X_j=0$, $Y_j=P(b-a)$ $(j=1,2)$, may be determined by

$$\int_a^b [x^2 - 2i\beta(b-a)x + C]\, X^+(x)\mathrm{d}x = 0,$$

or,

$$C = -\int_a^b [x^2 - 2i\beta(b-a)x]\, X^+(x)\mathrm{d}x \Big/ \int_a^b X^+(x)\mathrm{d}x. \tag{4.11}$$

On account of symmetry to the y-axis in this example, it is easy to know, by (4.9),

$$X^+(-x) = \overline{X^+(x)}, \tag{4.12}$$

and therefore C is a real number by (4.11). In order to verify this and give a formula easy for calculation, we proceed as follows.

Using the expansion (4.7) of $X(z)$, we see, by the residue theorem,

$$\int_\Lambda X(z)\mathrm{d}z = 0, \quad \int_\Lambda zX(z)\mathrm{d}z = -2\pi i, \quad \int_\Lambda z^2X(z)\mathrm{d}z = 4\pi\beta(b-a).$$

Shrinking Λ to L and remembering $X^+(x) = -\kappa X^-(x)$, we know (e. g., by the first equality above),

$$\begin{aligned}\int_\Lambda X(z)\mathrm{d}z &= \int_{\Lambda_1} X(z)\mathrm{d}z + \int_{\Lambda_2} X(z)\mathrm{d}z \\ &= \int_{-b}^{-a}[X^+(x) - X^-(x)]\mathrm{d}x + \int_a^b [X^+(x) - X^-(x)]\mathrm{d}x \\ &= \frac{\kappa+1}{\kappa}\left(\int_{-b}^{-a} X^+(x)\mathrm{d}x + \int_a^b X^+(x)\mathrm{d}x\right) \\ &= \frac{\kappa+1}{\kappa}\int_a^b [X^+(x) + \overline{X^+(x)}]\mathrm{d}x = 0.\end{aligned}$$

Similarly, we may get

$$\int_a^b x[X^+(x) - \overline{X^+(x)}]\mathrm{d}x = -\frac{2\pi\kappa i}{\kappa+1},$$

$$\int_a^b x^2[X^+(x) - X^-(x)]\mathrm{d}x = \frac{4\pi\kappa(b-a)}{\kappa+1}.$$

If we denote

$$X^+(x) = R(x) + iI(x), \quad a < x < b, \tag{4.13}$$

then, by the above equalities, we get at once

$$\int_a^b R(x)\mathrm{d}x = 0, \quad \int_a^b xI(x)\mathrm{d}x = -\frac{\pi\kappa}{\kappa+1},$$

$$\int_a^b x^2R(x)\mathrm{d}x = \frac{2\pi\beta\kappa(b-a)}{\kappa+1}.$$

Substituting (4.13) into (4.11) and considering these equalities, we get

$$C = \left[2\beta(b-a)\int_a^b xR(x)\mathrm{d}x - \int_a^b x^2 I(x)\mathrm{d}x\right] \Big/ \int_a^b I(x)\mathrm{d}x, \tag{4.14}$$

which shows C is actually real.

According to the chosen branch of the multi-valued function, we may obtain

$$X^+(x) = \frac{-\mathrm{i}e^{\beta\pi}}{\sqrt{(b^2-x^2)(x^2-a^2)}}\exp\left[\mathrm{i}\beta\ln\frac{(b+x)(x-a)}{(b-x)(x+a)}\right], \quad a<x<b,$$

in which the square root has been taken to be positive. Thereby,

$$R(x) = e^{\beta\pi}\sin\beta\ln\frac{(b+x)(x-a)}{(b-x)(x+a)}\Big/\sqrt{(b^2-x^2)(x^2-a^2)},$$

$$I(x) = -e^{\beta\pi}\cos\beta\ln\frac{(b+x)(x-a)}{(b-x)(x+a)}\Big/\sqrt{(b^2-x^2)(x^2-a^2)}.$$

Hence, we get finally

$$C = -\left[2\beta(b-a)\int_a^b xR_0(x)\mathrm{d}x + \int_a^b xI_0(x)\mathrm{d}x\right] \Big/ \int_a^b I_0(x)\mathrm{d}x, \tag{4.15}$$

where

$$R_0(x) = \sin\beta\ln\frac{(b+x)(x-a)}{(b-x)(x+a)}\Big/\sqrt{(b^2-x^2)(x^2-a^2)},$$

$$I_0(x) = \cos\beta\ln\frac{(b+x)(x-a)}{(b-x)(x+a)}\Big/\sqrt{(b^2-x^2)(x^2-a^2)}. \tag{4.16}$$

It is not difficult to write out the expressions of $\Phi(z)$, $\Psi(z)$, which will be omitted here.

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PLANE ELASTIC PROBLEMS OF DIFFERENT MEDIA WITH CRACKS ON THE INTERFACE*

Abstract

The equilibrium problem for the infinite elastic plane consisting of two different media with many cracks on the interface is discussed. It is transferred to a boundary value problem for analytic functions and then further reduced to a singular integral equation, the unique solvability and an effective method of solution for which are established. A practical example in applications is illustrated, the solution of which is obtained in closed form.

Key Words. Cracks, Boundary Value Problems, Singular Integral Equations.

1. Introduction

There were many research works on plane elastic problems for the infinite plane bonded (or welded) by two different isotropic media with cracks each lying in the interior of one of the material, or touching or passing through the interface, for instance [1～3]. In literature, discussions had been made for the case in which the interface is a circle and a single crack occurs on the interface with symmetric boundary conditions. The method used certainly is also effective when the interface is an infinite straight line but is not in effect for the case in which the interface is a general closed contour with general boundary conditions.

In this paper we give a method for solving such problems in general case (even a displacement difference function may be given on the uncracked arcs of the interface), using the basic idea in [4] or [5] with some modification. Any problem of this kind may be reduced to a boundary value problem for analytic functions and further to a uniquely solvable singular integral equation on the cracks (or on the uncracked arcs of the interface). Finally, an important example in practice is illustrated for the case in which only one crack on the interface circle occurs and the plane is subjected to an extension in one direction not necessarily symmetric to the crack.

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2. Formulation of the Problems and Reduction to Boundary Value Problems

Assume the infinite elastic plane consists of two different isotropic media with a Liapunov contour L as the interface of Hölder index $>\frac{1}{2}$, on which there are p cracks $\gamma_1,\ldots,\gamma_p$. L is oriented counter-clockwisely and $\gamma_j = a_jb_j$ is oriented inductively. The interior and the exterior regions bounded by L are denoted respectively by S^+ and S^- with elastic constants κ^+, μ^+ and κ^-, μ^-. Denote $\gamma_j' = b_ja_{j+1}$ $(j=1,\ldots,p;\ a_{p+1}=a_1)$, $\gamma = \sum_{j=1}^{p}\gamma_j$ and $\gamma' = \sum_{j=1}^{p}\gamma_j'$ (Fig. 1).

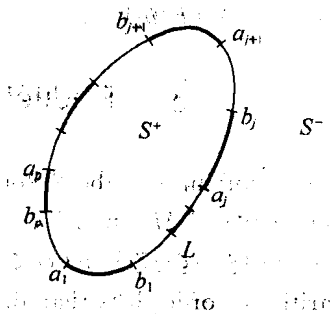

Fig. 1

We confine ourselves to the first fundamental problem: given the external loads $X_j^{\pm}(t)+iY_j^{\pm}(t)\in H$ (class of Hölder continuous functions) on both sides of γ_j and the displacement difference

$$g(\zeta)=[u^+(\zeta)+iv^+(\zeta)]-[u^-(\zeta)+iv^-(\zeta)],\ \zeta\in\gamma' \quad (g(\zeta)=g_j(\zeta),\ \zeta\in\gamma_j'),$$

with $g'(\zeta)\in H$, where $u^{\pm}(\zeta)+iv^{\pm}(\zeta)$ are the displacements at the point $\zeta\in\gamma'$ on the positive and the negative sides respectively. Besides, the stresses and the angle of rotation at infinity are also given. To find the elastic equilibrium is equivalent to find the complex Airy functions $\varphi(z)$ and $\psi(z)$ by the complex variable method. [6]

For simplicity, we assume that the resultant principal vector of the external stresses on the two sides of each crack γ_j is zero. When denoting

$$X_j^{\pm}+iY_j^{\pm}=\int_{\gamma_j}[X_j^{\pm}(t)+iY_j^{\pm}(t)]ds,\quad j=1,\ldots,p,$$

where s is the arc-length parameter at t on γ_j, we assume

$$X_j+iY_j=(X_j^{+}+iY_j^{+})+(X_j^{-}+iY_j^{-})=0,\quad j=1,\ldots,p.$$

We also assume that the stresses and the angle of rotation at infinity are zeros, otherwise these factors may be transferred to external stresses on γ_j and displacement difference on γ_j' (as shown in the example in Section 4). Thus, $\varphi(z)$ and $\psi(z)$ are sectionally holomorphic functions with L as the jump curve and $\varphi(\infty)=\psi(\infty)=0$.

As in [5], we denote

$$f^{\pm}(\tau)=f_j^{\pm}(\tau)=\pm i\int_{a_j}^{\tau}[X_j^{\pm}(t)+iY_j^{\pm}(t)]ds,\quad \tau\in\gamma_j,\ j=1,\ldots,p.$$

Then $f^{\pm}(a_j)=0$, $f^{+}(b_j)=f^{-}(b_j)$.

By the conditions given above, the problem is reduced to the following boundary value problem for $\varphi(z)$ and $\psi(z)$:

$$\varphi^{\pm}(\tau)+\tau\overline{\varphi'^{\pm}(\tau)}+\overline{\psi^{\pm}(\tau)}=f^{\pm}(\tau)+c(\tau),\quad \tau\in\gamma, \tag{2.1}$$

$$\varphi^+(t)+t\overline{\varphi'^+(t)}+\overline{\psi^+(t)}=\varphi^-(t)+t\overline{\varphi'^-(t)}+\overline{\psi^-(t)},\quad t\in\gamma',\tag{2.2}$$

$$\alpha^+\varphi^+(t)-\beta^+[t\overline{\varphi'^+(t)}+\overline{\psi^+(t)}]=\alpha^-\varphi^-(t)-\beta^-[t\overline{\varphi'^-(t)}+\overline{\psi^-(t)}]$$
$$+2g(t),\quad t\in\gamma'\ (\alpha^\pm=\kappa^\pm/\mu^\pm,\ \beta^\pm=1/\mu^\pm).\tag{2.3}$$

with $\varphi(\infty)=\psi(\infty)=0$, where $c(\tau)=c_j$, $\tau\in\gamma_j$, $j=1,\ldots,p$, are undetermined constants.

In the sequel, we shall denote

$$F(\tau)=\frac{1}{2}[f^+(\tau)-f^-(\tau)],\ G(\tau)=\frac{1}{2}[f^+(\tau)+f^-(\tau)],\quad \tau\in\gamma.\tag{2.4}$$

Hence $F(a_j)=F(b_j)=0$.

3. Reduction to Singular Integral Equations

For solving the above boundary value problem, as in [5], we introduce an unknown function $\omega(\zeta)\in H_0$ on $L=\gamma+\gamma'$ with $\omega'(\zeta)\in H^*$ (for notation, cf. [7]), that is to say, $\omega(\zeta)=\omega_{\gamma_j}(\zeta)$ $(\zeta\in\gamma_j)$ and $\omega(\zeta)=\omega_{\gamma_j'}(\zeta)$ $(\zeta\in\gamma_j')\in H$ for every j, and $\omega'(\zeta)$ may have singularities of order less than one at the tips a_j,b_j, such that

$$\varphi(z)=\frac{1}{2\pi i}\int_L\frac{\omega(\zeta)}{\zeta-z}d\zeta,\quad z\notin L,\tag{3.1}$$

$$\psi(z)=-\frac{1}{2\pi i}\int_L\frac{\overline{\omega(\zeta)}+\bar\zeta\omega'(\zeta)}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_\gamma\frac{\overline{F(\tau)}}{\tau-z}d\tau,\quad z\notin L.\tag{3.2}$$

We assume that

$$\omega_{\gamma_{j+1}}(a_j)-\omega_{\gamma_j'}(a_j)=0,\ \omega_{\gamma_j}(b_j)-\omega_{\gamma_j'}(b_j)=0,\quad j=1,\ldots,p,\tag{3.3}$$

temporarily, i. e., $\omega(\zeta)\in H$ on L. The existence of such an $\omega(\zeta)$ will be proved later.

Substituting (3.1) and (3.2) in (2.1) and taking the boundary values on both sides of γ, we get the same equation

$$K_1\omega\equiv\frac{1}{\pi i}\int_L\frac{\omega(\zeta)}{\zeta-\tau}d\zeta-\frac{1}{2\pi i}\int_L\omega(\zeta)d\log\frac{\zeta-\tau}{\bar\zeta-\bar\tau}-\frac{1}{2\pi i}\int_L\overline{\omega(\zeta)}d\frac{\zeta-\tau}{\bar\zeta-\bar\tau}$$
$$=f^*(\tau)+c(\tau),\quad \tau\in\gamma,\tag{3.4}$$

where we have set

$$f^*(\tau)=\frac{1}{2\pi i}\int_\gamma\frac{F(\tau_1)}{\bar\tau_1-\bar\tau}d\bar\tau_1+\frac{1}{2}G(\tau),\quad \tau\in\gamma.$$

Substituting them in (2.2), we find that it is identically satisfied.

Substituting them in (2.3), an equation on γ' is obtained:

$$K_2\omega\equiv A\omega(t)+\frac{B}{\pi i}\int_L\frac{\omega(\zeta)}{\zeta-t}d\zeta+\frac{\beta^+-\beta^-}{\pi i}\left\{\int_L\omega(\zeta)d\log\frac{\zeta-t}{\bar\zeta-\bar t}+\int_L\overline{\omega(\zeta)}d\frac{\zeta-t}{\bar\zeta-\bar t}\right\}$$
$$=4g(t)-\frac{\beta^+-\beta^-}{\pi i}\int_\gamma\frac{F(\tau)}{\bar\tau-\bar t}d\bar\tau,\quad t\in\gamma',\tag{3.5}$$

where

$$\begin{cases}A=\alpha^++\alpha^-+\beta^++\beta^-=\dfrac{\kappa^++1}{\mu^+}+\dfrac{\kappa^-+1}{\mu^-},\\[2mm] B=\alpha^+-\alpha^--\beta^++\beta^-=\dfrac{\kappa^+-1}{\mu^+}-\dfrac{\kappa^--1}{\mu^-}.\end{cases}\tag{3.6}$$

(3. 4)~(3. 5) constitute a singular integral equation of normal type on L with a_j, b_j $(j=1, \ldots, p)$ as nodes to be solved in class h_{2p}, that is, $\omega(\zeta)$ is required to be bounded at these points. It is easy to see that the index of this equation in h_{2p} is $-p$.

Analogous to the method described in [5], it is easily proved that this equation has a unique solution in h_{2p}. Substituting this solution in (3. 1) and (3. 2), the Airy functions $\varphi(z), \psi(z)$ may be obtained respectively.

As the functions appeared on the right sides of (3. 4) and (3. 5) are bounded at a_j, b_j since $F(a_j)=F(b_j)=0$, (3. 3) must be fulfilled, as otherwise, on the left sides of them there would occur logarithmic singularities at these points.

In practical problems, usually only the stress distribution is required so that it is enough to get $\Phi(z)=\varphi'(z)$ and $\Psi(z)=\psi'(z)$. Then it is sufficient to find $\Omega(\zeta)=\omega'(\zeta)$, $\zeta \in L$.

For this purpose, differentiating (3. 4) and (3. 5), we obtain a singular integral equation $\Omega(\zeta)$ on L consisting of

$$K_1\Omega \equiv \frac{1}{\pi i}\int_L \frac{\Omega(\zeta)}{\zeta-\tau}d\zeta+\frac{1}{2\pi i}\int_L \frac{\partial}{\partial\tau}\log\frac{\zeta-\tau}{\bar{\zeta}-\bar{\tau}}\,\Omega(\zeta)d\zeta+\frac{1}{2\pi i}\int_L \frac{\partial}{\partial\tau}\frac{\zeta-\tau}{\bar{\zeta}-\bar{\tau}}\overline{\Omega(\zeta)}d\bar{\zeta}$$
$$=f^*(\tau), \quad \tau\in\gamma, \tag{3. 7}$$

$$K_2\Omega \equiv A\Omega(t)+\frac{B}{\pi i}\int_L \frac{\Omega(\zeta)}{\zeta-t}d\zeta$$
$$-\frac{\beta^+-\beta^-}{\pi i}\left\{\int_L \frac{\partial}{\partial t}\log\frac{\zeta-t}{\bar{\zeta}-\bar{t}}\,\Omega(\zeta)d\zeta+\int_L \frac{\partial}{\partial t}\left(\frac{\zeta-t}{\bar{\zeta}-\bar{t}}\right)\overline{\Omega(\zeta)}d\bar{\zeta}\right\}$$
$$=4g'(t)-\frac{\beta^+-\beta^-}{\pi i}\int_\gamma \frac{\partial}{\partial t}\left(\frac{1}{\bar{\tau}-\bar{t}}\right)F(\tau)d\bar{\tau}, \quad t\in\gamma', \tag{3. 8}$$

which should be solved in class h_0, that is, $\Omega(\zeta)$ is permitted to have singularities of order less than 1 at a_j, b_j.

Replacing t in (3. 8) by $\tau\in\gamma$ while $g(t)$ being changed to $g(\tau)=g^+(\tau)-g^-(\tau)$, the unknown displacement difference at τ of the two sides of γ, an equation similar to (3. 7) is obtained. Although $g'(\tau)$ is unknown on τ, but

$$\int_{\gamma_j} g'(\tau)d\tau = g(b_j)-g(a_j), \quad j=1,\ldots,p$$

are given, so that, by the continuity of the displacements on γ, it follows that the following supplementary condition must be fulfilled:

$$\int_{\gamma_j}(K_2'\Omega)(\tau)d\tau=4[g(b_j)-g(a_j)]-\frac{\beta^+-\beta^-}{\pi i}\int_{\gamma_j}d\tau\int_\gamma \frac{\partial}{\partial\tau}\left(\frac{1}{\bar{\tau}_1-\bar{\tau}}\right)F(\tau_1)d\bar{\tau}_1$$
$$=4[g(b_j)-g(a_j)]-\frac{\beta^+-\beta^-}{\pi i}\int_\gamma F(\tau)\left(\frac{1}{\bar{\tau}-\bar{b}_j}-\frac{1}{\bar{\tau}-\bar{a}_j}\right)d\bar{\tau},$$
$$j=1,\ldots,p. \tag{3. 9}$$

Thus, our problem is reduced to solve equations (3. 6)~(3. 7) in h_0 with supplementary condition (3. 9).

It is worthy to note that

$$\int_L \Omega(\zeta)\mathrm{d}\zeta = \int_L \omega'(\zeta)\mathrm{d}\zeta = 0 \tag{3.10}$$

because of (3.3), which is implied in condition (3.9). Then, by the definition of $\boldsymbol{K}_2'\Omega$, it is easy to see that

$$\int_L (\boldsymbol{K}_2'\Omega)(\zeta)\mathrm{d}\zeta = 2(\alpha^- + \beta^+)\int_L \Omega(\zeta)\mathrm{d}\zeta, \tag{3.11}$$

and, by integrating (3.8) along the γ_j''s, adding together with (3.9) and summing up from $j=1$ to $j=p$,

$$\int_L (\boldsymbol{K}_2'\Omega)(\zeta)\mathrm{d}\zeta = 0. \tag{3.12}$$

Thus, by (3.11), we see that (3.10) and (3.12) are equivalent if (3.9) is fulfilled. In other words, (3.10) or (3.12) may be replaced by one of the equations in (3.9). In particular, when $p=1$, (3.10) or (3.12) may be in place of (3.9).

Moreover, if the external stresses on the two sides of γ are symmetric:

$$X_j^+(\tau) + \mathrm{i}Y_j^+(\tau) = -[X_j^-(\tau) + \mathrm{i}Y_j^-(\tau)], \quad j = 1, \ldots, p,$$

which usually occurs in practical problems, then $f^+(\tau) = f^-(\tau)$, denoted by $f(\tau)$, so $F(\tau)=0$, $f^*(\tau)=G(\tau)=f(\tau)$, $\tau\in\gamma$. In this case, (3.7), (3.8) are respectively particularly simple:

$$\boldsymbol{K}_1'\Omega = G'(\tau), \quad \tau\in\gamma, \tag{3.13}$$

$$\boldsymbol{K}_2'\Omega = 4g'(t), \quad \tau\in\gamma', \tag{3.14}$$

where

$$G'(\tau) = f'(\tau) = \mathrm{i}[X_j^+(\tau) + \mathrm{i}Y_j^+(\tau)]\frac{\mathrm{d}s}{\mathrm{d}\tau}, \quad \tau\in\gamma,$$

which belongs to class H because L is a Liapunov contour.

4. An Important Example

We shall illustrate our method by a practical example being important in application. Let L be the unit circle $|\zeta|=1$ on which there is a crack $\gamma=ab$, where $a=\mathrm{e}^{-\mathrm{i}\theta}$, $b=\mathrm{e}^{\mathrm{i}\theta}$ $(-\pi<\theta<\pi)$. We assume that an extension N in the direction with the angle of inclination λ with respect to the x-axis is subjected to the infinite plane, but there are no loads on both sides of γ and no displacement difference on $\gamma'=L\backslash\gamma$ (Fig. 2).

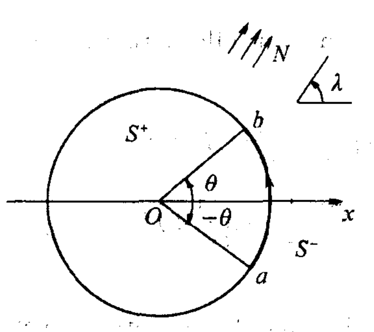

Fig. 2

It is well-known that

$$\Gamma = \frac{N}{4}, \quad \Gamma' = \frac{N}{2}\mathrm{e}^{-2\mathrm{i}\lambda}$$

(for notation, cf. [5]), and so

$$\varphi(z) = \varphi_0(z) + \frac{N}{4}z, \quad \psi(z) = \psi_0(z) - \frac{N}{2}\mathrm{e}^{-2\mathrm{i}\lambda}z,$$

where $\varphi_0(z), \psi_0(z)$ are sectionally holomorphic in the entire plane with $\varphi_0(\infty)=\psi_0(\infty)=0$. Therefore, in place of (3.1) and (3.2), we assume respectively

$$\varphi(z)=\frac{1}{2\pi i}\int_L \frac{\omega(\zeta)}{\zeta-z}d\zeta+\frac{N}{4}z, \quad z\notin L, \tag{4.1}$$

$$\psi(z)=-\frac{1}{2\pi i}\int_L \frac{\overline{\omega(\zeta)}+\bar{\zeta}\omega'(\zeta)}{\zeta-z}d\zeta-\frac{N}{2}e^{-2i\lambda}z, \quad z\notin L. \tag{4.2}$$

Substituting them in (2.1)~(2.3), we obtain the boundary condition satisfied by $\varphi_0(z), \psi_0(z)$, among which (2.2) is unchanged and

$$f^{\pm}(\tau)=f(\tau)=-\frac{N}{2}\tau+\frac{N}{2}e^{2i\lambda}\bar{\tau}, \quad \tau\in\gamma, \tag{4.3}$$

$$2g(t)=-\frac{B}{4}Nt-\frac{\beta^+-\beta^-}{2}Ne^{-2i\lambda}\bar{t}, \quad t\in\gamma', \tag{4.4}$$

in (3.1) and (3.3) respectively. Now $F(\tau)=0$, $G(\tau)=f(\tau)$ and thereby (3.13) and (3.14) become respectively

$$\boldsymbol{K}_1'\Omega=-\frac{N}{2}(1+e^{2i\lambda}\tau^{-2}), \quad \tau\in\gamma,$$

$$\boldsymbol{K}_2'\Omega=-\frac{B}{2}N+(\beta^+-\beta^-)Ne^{-2i\lambda}t^{-2}, \quad t\in\gamma'.$$

By the relations

$$\frac{\zeta-\tau}{\bar{\zeta}-\bar{\tau}}=-\zeta\tau, \ \frac{d\tau}{\bar{\zeta}-\bar{\tau}}=\frac{\zeta d\tau}{\tau(\zeta-\tau)} \quad (|\zeta|=1, \ |\tau|=1),$$

these two equations may be reduced respectively to the simpler equations

$$\frac{1}{\pi i}\int_L \frac{\Omega(\zeta)}{\zeta-\tau}d\zeta=-D-\frac{N}{2}(1+e^{2i\lambda}\tau^{-2}), \quad \tau\in\gamma, \tag{4.5}$$

$$A\Omega(t)+\frac{B}{\pi i}\int_L \frac{\Omega(\zeta)}{\zeta-t}d\zeta=(\beta^+-\beta^-)(2D-\frac{B}{2}N+Ne^{-2i\lambda}t^{-2}), \quad t\in\gamma', \tag{4.6}$$

where we have set

$$D=\frac{1}{2\pi i}\int_L \frac{\overline{\Omega(\zeta)}}{\zeta}d\zeta \tag{4.7}$$

or

$$\bar{D}=\frac{1}{2\pi i}\int_L \frac{\Omega(\zeta)}{\zeta}d\zeta. \tag{4.7$'$}$$

According to Section 3, equations (4.5) and (4.6) on L should be solved in h_0 with D satisfying (4.7) or (4.7)$'$ and with the supplementary condition

$$\int_L \Omega(\zeta)d\zeta=0. \tag{4.8}$$

The index of this equation in h_0 is 1, so that there is one arbitrary complex constant c in its general solution since its corresponding homogeneous equation has only the trivial solution. Regard D as another undetermined constant. Thus, c and D may be uniquely determined by (4.7) (or (4.7)$'$) and (4.8).

By the general theory of singular integral equations, the canonical function of this

equation in h_0 is

$$X(z)=Y(z)/\sqrt{R(z)}, \tag{4.9}$$

where

$$\sqrt{R(z)}=\sqrt{(z-e^{i\theta})(z-e^{-i\theta})}=\sqrt{z^2-2z\cos\theta+1} \tag{4.10}$$

is a single-valued branch in the complex plane cut along γ, for instance, the branch with $\sqrt{R(0)}=1$ and

$$Y(z)=\left(\frac{z-e^{-i\theta}}{z-e^{i\theta}}\right)^{\nu/2\pi i},\quad \nu=\ln\frac{\alpha^-+\beta^+}{\alpha^++\beta^-}, \tag{4.11}$$

which is a single-valued branch cut along γ', for instance, the branch with $Y(\infty)=1$, or, what is the same, $Y(0)=\exp\{\nu(1-\frac{\theta}{\pi})\}$. The standard function is

$$\begin{cases}Z(\tau)=X^+(\tau)=-X^-(\tau), & \tau\in\gamma,\\ Z(t)=(A+B)X^+(t)=(A-B)X^-(t), & t\in\gamma'.\end{cases} \tag{4.12}$$

For simplicity, we denote

$$\begin{cases}D_1=-D-\dfrac{N}{2},\quad D_2=2(\beta^+-\beta^-)D-\dfrac{B}{2}N,\\ E_1=-\dfrac{N}{2}e^{2i\lambda},\quad E_2=(\beta^+-\beta^-)Ne^{-2i\lambda}.\end{cases} \tag{4.13}$$

Thus, the general solution of (4.5)～(4.6) in h_0 is

$$\Omega(\tau_0)=\frac{Z(\tau_0)}{\pi i}\left[\int_\gamma\frac{D_1+E_1\tau^{-2}}{Z(\tau)(\tau-\tau_0)}d\tau+\int_{\gamma'}\frac{D_2+E_2t^{-2}}{Z(t)(t-\tau_0)}dt\right]-cZ(\tau_0),\quad \tau_0\in\gamma, \tag{4.14}$$

$$\begin{aligned}\Omega(t_0)=A^*(D_2+E_2t_0^{-2})-\frac{B^*Z(t_0)}{\pi i}\left[\int_\gamma\frac{D_1+E_1\tau^{-2}}{Z(\tau)(\tau-t_0)}d\tau+\int_{\gamma'}\frac{D_2+E_2t^{-2}}{Z(t)(t-t_0)}dt\right]\\ +cB^*Z(t_0),\quad t_0\in\gamma',\end{aligned} \tag{4.15}$$

where c is an arbitrary constant and

$$\begin{aligned}A^*&=\frac{A}{A^2-B^2}=\frac{1}{4}\left(\frac{1}{\alpha^-+\beta^+}+\frac{1}{\alpha^++\beta^-}\right),\\ B^*&=\frac{B}{A^2-B^2}=\frac{1}{4}\left(\frac{1}{\alpha^-+\beta^+}-\frac{1}{\alpha^++\beta^-}\right).\end{aligned} \tag{4.16}$$

By (4.12), (4.14) and (4.15) may be rewritten respectively as

$$\begin{aligned}\Omega(\tau_0)=\frac{X^+(\tau_0)}{\pi i}\left[\int_\gamma\frac{D_1+E_1\tau^{-2}}{X^+(\tau)(\tau-\tau_0)}d\tau+\frac{1}{A+B}\int_{\gamma'}\frac{D_2+E_2t^{-2}}{X^+(t)(t-\tau_0)}dt\right]\\ -cX^+(\tau_0),\quad \tau_0\in\gamma,\end{aligned} \tag{4.14'}$$

$$\begin{aligned}\Omega(t_0)&=A^*(D_2+E_2t_0^{-2})\\ &-\frac{B^*X^+(t_0)}{\pi i}\left[(A+B)\int_\gamma\frac{D_1+E_1\tau^{-2}}{X^+(\tau)(\tau-t_0)}d\tau+\int_{\gamma'}\frac{D_2+E_2t^{-2}}{X^+(t)(t-t_0)}dt\right]\\ &+cB^*(A+B)X^+(t_0),\quad t_0\in\gamma'.\end{aligned} \tag{4.15'}$$

All the integrals in (4.14)′ and (4.15)′ may be reduced to integrals along γ (or γ') only by the following results:

$$\frac{1}{\pi i}\int_L\frac{d\zeta}{X^+(\zeta)(\zeta-\zeta_0)}=\frac{1}{X^+(\zeta_0)},\quad \zeta_0\in L, \tag{4.17}$$

$$\frac{1}{\pi i}\int_L \frac{d\zeta}{X^+(\zeta)\zeta^2(\zeta-\zeta_0)}=\frac{1}{X^+(\zeta_0)\zeta_0^2}+2r(\zeta_0),\quad \zeta_0\in L, \tag{4.18}$$

where

$$r(\zeta_0)=\operatorname{res}\left\{\frac{1}{X(\zeta)\zeta^2(\zeta-\zeta_0)}\right\}_{\zeta=0},\quad \zeta_0\in L. \tag{4.19}$$

The evaluation of $r(\zeta_0)$ will be given in Section 5. Thus, (4.14)′ and (4.15)′ may be rewritten respectively as

$$\Omega(\tau_0)=\frac{D_2+E_2\tau_0^{-2}}{A+B}+\frac{X^+(\tau_0)}{A+B}\left\{2E_2r(\tau_0)\right.$$
$$\left.+\frac{1}{\pi i}\int_\gamma \frac{(A-B)D_1-D_2+[(A+B)E_1-E_2]\tau^{-2}}{X^+(\tau)(\tau-\tau_0)}d\tau+2C\right\},\quad \tau_0\in\gamma, \tag{4.20}$$

$$\Omega(t_0)=\frac{D_2+E_2t_0^{-2}}{A+B}-B^*X^+(t_0)\left\{2E_2r(t_0)\right.$$
$$\left.+\frac{1}{\pi i}\int_\gamma \frac{(A+B)D_1-D_2+[(A+B)E_1-E_2]\tau^{-2}}{X^+(\tau)(\tau-t_0)}d\tau+2C\right\},\quad t_0\in\gamma', \tag{4.21}$$

where $C=-c(A+B)/2$ is an arbitrary constant.

Substituting these equations in (4.7)′ and (4.8), we may get a system of linear equations

$$\begin{cases} k_{11}C+k_{12}D=\delta_1N, \\ k_{21}C+k_{22}D=\delta_2N+\overline{D}, \end{cases} \tag{4.22}$$

where k_{jl},δ_l $(j,l=1,2)$ are certain definite constants. When $N=0$, the elastic plane is in the natural equilibrium so that $\Phi(z)=\Psi(z)=0$. Then, by (4.1),

$$\Phi(z)=\frac{1}{2\pi i}\int_L \frac{\Omega(\zeta)}{\zeta-z}d\zeta \tag{4.23}$$

and $\Omega(\zeta)=\Phi^+(\zeta)-\Phi^-(\zeta)$ which implies $C=D=0$. This means (4.22) is uniquely solvable for any N. Hence, after C and D have been determined, we obtain, by taking derivatives in (4.1) and (4.2),

$$\Phi(z)=\frac{1}{2\pi i}\int_L \frac{\Omega(\zeta)}{\zeta-z}d\zeta+\frac{N}{4}, \tag{4.24}$$

$$\Psi(z)=\frac{1}{2\pi i}\int_L \frac{\overline{\Omega(\zeta)}}{\zeta^2(\zeta-z)}d\zeta-\frac{1}{2\pi i}\int_L \frac{\Omega(\zeta)}{\zeta(\zeta-z)^2}d\zeta-\frac{N}{2}e^{-2i\lambda}. \tag{4.25}$$

Thus, our problem is completely solved.

5. Evaluation of the Residue

Assume in the neighborhood of $\zeta=0$,

$$\frac{1}{X(\zeta)}=\lambda_0+\lambda_1\zeta+\lambda_2\zeta^2+\cdots, \tag{5.1}$$

where

$$\lambda_0=\frac{1}{X(0)}=\frac{\sqrt{R(0)}}{Y(0)}=e^{-\nu(1-\theta/\pi)} \tag{5.2}$$

by the branches of $\sqrt{R(z)}$ and $Y(z)$ taken as before. Since

$$\frac{1}{\zeta^2(\zeta-\zeta_0)}=-\frac{1}{\zeta_0\zeta^2}-\frac{1}{\zeta_0{}^2\zeta}-\frac{1}{\zeta_0}+\cdots,\quad \zeta_0\neq 0,$$

therefore,

$$r(\zeta_0)=-\frac{\lambda_0}{\zeta_0{}^2}-\frac{\lambda_1}{\zeta_0},\tag{5.3}$$

where

$$\lambda_1=X'(0)=\frac{R'(0)}{2\sqrt{R(0)}Y(0)}-\frac{\sqrt{R(0)}Y'(0)}{Y(0)}.$$

But $R'(0)=-2\cos\theta$; $Y'(0)/Y(0)=-\frac{\nu}{\pi}\sin\theta$ (independent of the branch of $Y(z)$ chosen), so

$$\lambda_1=-\,e^{-\nu(1-\theta/\pi)}\cos\theta+\frac{\nu}{\pi}\sin\theta.\tag{5.4}$$

Thus, $r(\zeta_0)$ is fully determined by (5.3) with λ_0 and λ_1 given by (5.2) and (5.3) respectively.

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A GENERAL METHOD FOR SOLVING PLANE CRACK PROBLEMS

0. Introduction

There have been many works on plane crack problems. Each of them has solved a particular problem for the case, either special in the location of the cracks and the interfaces or in the boundary conditions, for example, [1,2]. In this paper a unified method of solution for such problems is proposed, which is effective in the following more general case. Assume there are a set of arcwise smooth non-intersecting cracks in the composite media with certain interface. The cracks may touch or pass through the interface, or even lie on the interface. It is reduced to a singular integral equation which is uniquely solvable under certain natural additional requirements for its solution. A new idea for determining the order of singularity of the solution at any node of the problem is suggested. Here, by a node of the problem, we mean either any tip or corner point of the cracks, any corner point of the interface, or any point of intersection of a crack and the interface.

For definiteness, we consider the first fundamental problems only (Muskhelishvili [6]) although our method is also effective for the second fundamental problems or mixed boundary problems. For simplicity, we assume the interface is a straight line. We shall illustrate our method for two somewhat special cases which often occur in practice, but the method is universally in effect for the general case.

1. Bonded half-planes with cracks

Assume an elastic infinite plane consists of two bonded half-planes, the upper half-plane Z^+ and the lower half-plane Z^-, and there are p cracks $\gamma_1,\dots,\gamma_p$ in the plane, some of which lie in Z^+ or in Z^- (maybe touch the x-axis) and the others locate on or pass through the x-axis. Assume each crack $\gamma_j=a_jb_j$ is an arc-wise Lyapunov arc: the angle of inclination $\theta(t)$ of the tangent at t on each of its smooth subarcs is Hölder continuous. Denote $\gamma=\sum_{j=0}^{p}\gamma_j$, $X=\{\text{the } x\text{-axis}\}\backslash\gamma$ (which is the interface) and $S^\pm=Z^\pm\backslash\gamma$. X consists of several segments on the x-axis, two of which are actually half-rays extending to $+\infty$ and $-\infty$ respectively (Figure 1).

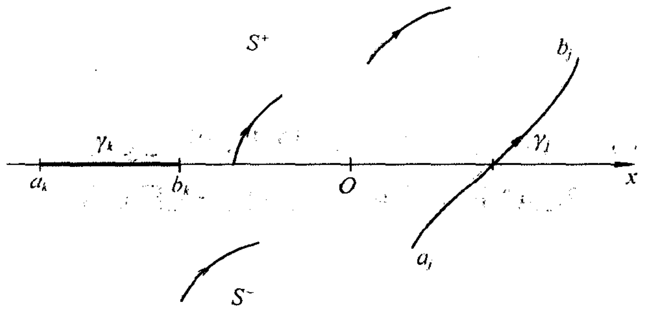

Figure 1

Let the elastic constants of $S^{\pm}$ be $\kappa^{\pm}, \mu^{\pm}$ respectively.

We shall discuss the first fundamental problem. That is, find the elastic equilibrium, given the external stresses (loads) $X_n^{\pm}(t)+iY_n^{\pm}(t)$ on γ. The principal vectors of the external stresses on $\gamma_j^{\pm}$ are

$$X_j^{\pm}+iY_j^{\pm}=\int_{\gamma_j}[X_n^{\pm}(t)+iY_n^{\pm}(t)]ds$$

respectively, where s is the arc-length parameter on γ_j. Without loss of generality, we may always assume that $X_j^+ + iY_j^+ = -(X_j^- + iY_j^-)$ and there are no stresses or rotation at infinity (cf. [3]). We always assume both $X_n^{\pm}(t)$ and $Y_n^{\pm}(t)\in H_0$ on γ (for notation, cf. [6]).

Denote

$$f_j^{\pm}(t)=\pm i\int_{a_j}^{t}[X_n^{\pm}(\tau)+iY_n^{\pm}(\tau)]ds,\quad t\in\gamma_j,\ j=1,\dots,p. \tag{1.1}$$

Thus we have

$$f_j^{\pm}(a_j)=0,\quad f_j^+(b_j)=f_j^-(b_j). \tag{1.2}$$

Moreover, denote

$$F(t)=f^+(t)-f^-(t),\ G(t)=f^+(t)+f^-(t),\quad t\in\gamma. \tag{1.3}$$

Let $\varphi(z),\psi(z)$ be the complex stress functions[6] for the problem, both of which are sectionally holomorphic in $S^+ + S^-$ with $\varphi(\infty)=\psi(\infty)=0$.

Then, the boundary conditions on $\gamma_j^{\pm}$ are

$$\varphi^{\pm}(t)+t\,\overline{\varphi'^{\pm}(t)}+\overline{\psi^{\pm}(t)}=f_j^{\pm}(t)+C_j,\quad t\in\gamma_j,\ j=1,\dots,p, \tag{1.4}$$

where C_j, $j=1,\dots,p$, are undetermined constants[3].

On the interface X, the condition of equilibrium for the external stresses is

$$\varphi^+(x)+x\,\overline{\varphi'^+(x)}+\overline{\psi^+(x)}=\varphi^-(x)+x\,\overline{\varphi'^-(x)}+\overline{\psi^-(x)},\quad x\in X, \tag{1.5}$$

and the condition of continuity of the displacements is

$$\alpha^+\varphi^+(x)-\beta^+[x\overline{\varphi'^+(x)}+\overline{\psi^+(x)}]=\alpha^-\varphi^-(x)-\beta^-[x\,\overline{\varphi'^-(x)}+\overline{\psi^-(x)}],\quad x\in X, \tag{1.6}$$

where we have put

$$\alpha^{\pm}=\kappa^{\pm}/\mu^{\pm},\quad \beta^{\pm}=1/\mu^{\pm}, \tag{1.7}$$

which are given positive constants.

Thus, our problem is transferred to the boundary value problem (1.4)~(1.6) for sectionally holomorphic functions $\varphi(z)$, $\psi(z)$ with the additional requirements

$$\varphi(\infty) = \psi(\infty) = 0. \tag{1.8}$$

The following method of solution for this problem is inspired by that of Sherman [8] for elastic problems of single medium and that of the author [4] for those of composite media without cracks.

Introduce a new unknown function $\omega(\zeta)$ on $\gamma+X$ such that

$$\varphi(z) = \frac{1}{2\pi i}\int_{\gamma+X}\frac{\omega(\zeta)}{\zeta - z}d\zeta, \quad z \notin \gamma + X, \tag{1.9}$$

$$\psi(z) = -\frac{1}{2\pi i}\int_{\gamma+X}\frac{\overline{\omega(\zeta)} + \bar{\zeta}\omega'(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_{\gamma}\frac{\overline{F_1(t)}}{t - z}dt, \quad z \notin \gamma + X, \tag{1.10}$$

in which $\omega(\zeta)\in H_0$ and $\omega'(\zeta)\in H_0^*$ are assumed, that is, $\omega(\zeta)\in H$ and $\omega'(\zeta)\in H^*$ on each crack in γ and on each segment of X (for notation, cf. [7]), which imply $\omega(x)=O(|x|^{-\mu})$ and $\omega'(x)=O(|x|^{1-\mu})$ as $x\to\infty$, $0<\mu\leqslant 1$. Of course, the existence (and uniqueness) of such a function should be proved. We also assume that, for any fixed node c,

$$\sum \omega(c) = 0, \tag{1.11}$$

where the summation extends over all the cracks and all the interface segments starting or ending at c. That means the sum of the limit values of $\omega(\zeta)$ when $\zeta\to c$ along these cracks and segments is equal to zero. Of course, this should be proved too. For the time being, we assume such $\omega(\zeta)$ exists and fulfills (1.11).

Substituting (1.9) and (1.10) in the condition (1.4), by the Plemelj formula[6] and integration by parts, we obtain the same singular integral equation on γ:

$$\begin{aligned} \boldsymbol{K}_\gamma\omega &\equiv \frac{1}{\pi i}\int_{\gamma+X}\frac{\omega(\zeta)}{\zeta-t}d\zeta - \frac{1}{2\pi i}\int_{\gamma+X}\omega(\zeta)d\log\frac{\zeta-t}{\bar\zeta-\bar t} - \frac{1}{2\pi i}\int_{\gamma+X}\overline{\omega(\zeta)}\, d\,\frac{\zeta-t}{\bar\zeta-\bar t} \\ &= f_0(t)+C(t), \quad t\in\gamma, \end{aligned} \tag{1.12}$$

where

$$f_0(t) = \frac{1}{2\pi i}\int_{\gamma}\frac{F(\tau)}{\bar\tau - \bar t}d\bar\tau + \frac{1}{2}G(t), \quad t \in \gamma, \tag{1.13}$$

and $C(t)=C_j$, $t\in\gamma_j$, $j=1,\ldots,p$, are undetermined constants. Here, the terms out of integration vanish when the process of integration by parts is applied because of (1.11). Substituting them in (1.5), we find it is identically satisfied. In the sequel, we shall denote

$$\begin{aligned} A &= \alpha^+ + \alpha^- + \beta^+ + \beta^-, \quad B = \alpha^+ - \alpha^- - \beta^+ + \beta^-, \\ C &= \alpha^+ - \alpha^-, \quad D = \beta^+ - \beta^-. \end{aligned} \tag{1.14}$$

Similarly, substituting (1.9) and (1.10) in (1.6), we get a singular integral equation on X:

$$\begin{aligned} \boldsymbol{K}_X\omega &\equiv A\omega(x) + \frac{B}{\pi i}\int_{\gamma+X}\frac{\omega(\zeta)}{\zeta-x}d\zeta + \frac{D}{\pi i}\left\{\int_\gamma\omega(\tau)d\log\frac{\tau-x}{\bar\tau-x} + \int_\gamma\overline{\omega(\tau)}d\,\frac{\tau-x}{\bar\tau-x}\right\} \\ &= -Df_1(x), \quad x\in X, \end{aligned} \tag{1.15}$$

where we have set

$$f_1(x) = \frac{1}{\pi i}\int_\gamma \frac{F(\tau)}{\bar{\tau} - x}\,d\,\bar{\tau}, \quad x \in X. \tag{1.16}$$

(1.12) and (1.15) constitute a singular integral equation of normal type on $\gamma+X$, which should be solved in the most narrow class h, i. e., the solutions are restricted to be bounded near all the nodes of $\gamma+X$. Since $F(a_j)=F(b_j)=0$, so the right-hand member of this equation belongs to class H, and therefore its solution $\omega(\zeta)$ in class h, if any, must fulfill (1.11), as otherwise there would appear logarithmic singularities on the left side of (1.12) or (1.15). $\omega(\pm\infty)=0$ are evident because all the terms except the first one on the left side of (1.15) as well as the right-hand member tend to zero as $x\to\pm\infty$.

In order to prove the obtained equation has a unique solution in class h, we first show that its corresponding homogeneous equation has only the trivial solution in h. In fact, the latter corresponds to the case where there are no stresses on γ, no stresses or rotation at infinity and $C_j=0$, $j=1,\ldots,p$. Assume $\omega(\zeta)$ is a solution of this homogeneous equation. Then, by the uniqueness theorem for elastostatic problems (which may be proved rigorously in mathematics, cf. [6]), we should have $\varphi(z)=\psi(z)=0$ since $\varphi(\infty)=\psi(\infty)=0$. Hence $\omega(\zeta)=\varphi^+(\zeta)-\varphi^-(\zeta)=0$ for any $\zeta\in\gamma+X$.

It is easily verified that the index of the obtained singular integral equation in class h is $\kappa=-p$, and so, by the Noether theorem, its adjoint equation has $2p$ linearly independent (in the sense of real coefficient domain) solutions $\sigma_1(\zeta),\ldots,\sigma_{2p}(\zeta)$, $\zeta\in\gamma+X$, in class h_0 (solutions are permitted having integrable singularities at the nodes), and it is (uniquely) solvable in this class if and only if

$$\mathrm{Re}\int_\gamma [f_0(t) + C(t)]\sigma_j(t)dt = D\,\mathrm{Re}\int_X f_1(x)\sigma_j(x)dx, \quad j = 1,\ldots,2p, \tag{1.17}$$

are satisfied (cf. [7]).

By separating the real and the imaginary parts of $C_1,\ldots,C_p$ and denoting them by $\delta_1,\ldots,\delta_{2p}$, it is seen that (1.17) is a system of real linear equations in $\delta_1,\ldots,\delta_{2p}$:

$$\sum_{k=1}^{2p} \gamma_{jk}\delta_k = \lambda_j, \quad j = 1,\ldots,2p, \tag{1.18}$$

where (γ_{jk}) is a real constant matrix relating to $\sigma_j(t)$ but independent of the boundary conditions while the λ_j's are constants relating to them. We show that (γ_{jk}) is non-singular, or, what is the same, (1.18) has only the trivial solution when $\lambda_j=0$, $j=1,\ldots,2p$. In this case, (1.18) is a system of homogeneous linear equations corresponding to the case where no external stresses on γ and no stresses or rotation at infinity. If it has a system of solutions $\delta_1^0,\ldots,\delta_{2p}^0$, we then get a set of constants $C_1^0,\ldots,C_p^0$ satisfying (1.17). Therefore, the equation (1.12),(1.15) with $f_0(t)=0$, $f_1(x)=0$, $C(t)=C^0(t)=C_j^0$, $t\in\gamma_j$, has a unique solution $\omega_0(\zeta)$. Then the functions $\varphi_0(z),\psi_0(z)$ defined respectively by (1.9), (1.10) through $\omega_0(\zeta)$ would satisfy (1.4)~(1.6) with $f_j^{\pm'}(t)=0$, $C_j=C_j^0$ and $\varphi_0(\infty)=\psi_0(\infty)=0$. By the above mentioned uniqueness theorem, we know that $\varphi_0(z)=\psi_0(z)=0$

and hence $C_j^0=0$, $j=1,\ldots,p$, or δ_j^0, $j=1,\ldots,2p$.

2. Simplification for the method of solution

It is rather difficult to determine $C_1,\ldots,C_p$ when solving the equation obtained in Section 1 since we must solve its adjoint equation first so as to obtain (1.17) or (1.18). In practice, the stress distribution of the elastic body is more important, for which it is sufficient to find $\Phi(z)=\varphi'(z)$ and $\Psi(z)=\psi'(z)$ instead of $\varphi(z)$ and $\psi(z)$ themselves. If we denote

$$\Omega(\zeta)=\omega'(\zeta),\quad \zeta\in\gamma+X, \tag{2.1}$$

then (1.9) and (1.10) respectively become

$$\Phi(z)=\frac{1}{2\pi i}\int_{\gamma+X}\frac{\Omega(\zeta)}{\zeta-z}d\zeta,\quad z\notin\gamma+X, \tag{2.2}$$

$$\begin{aligned}\Psi(z)=&-\frac{1}{2\pi i}\int_{\gamma+X}\frac{\overline{\Omega(\zeta)}}{\zeta-z}d\bar\zeta-\frac{1}{2\pi i}\int_{\gamma}\frac{\bar\zeta\,\Omega(\zeta)}{(\zeta-z)^2}d\zeta,\\&+\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{F'(t)}}{t-z}d\bar t,\quad z\notin\gamma+X.\end{aligned} \tag{2.3}$$

Differentiating (1.12), we get a singular integral equation in $\Omega(\zeta)$:

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\gamma+X}\frac{\Omega(\zeta)}{\zeta-t}d\zeta+\frac{1}{2\pi i}\int_{\gamma+X}\frac{\partial}{\partial t}\log\left(\frac{\zeta-t}{\bar\zeta-\bar t}\right)\Omega(\zeta)d\zeta\\&+\frac{1}{2\pi i}\int_{\gamma+X}\frac{\partial}{\partial t}\left(\frac{\zeta-t}{\bar\zeta-\bar t}\right)\overline{\Omega(\zeta)}\,d\bar\zeta=f_0'(t),\quad t\in\gamma,\end{aligned} \tag{2.4}$$

or

$$\begin{aligned}\check{K}_1\Omega\equiv&\frac{1}{\pi i}\int_{\gamma+X}\frac{\Omega(\zeta)}{\zeta-t}d\zeta+\frac{e^{-2i\theta(t)}}{\pi i}\int_{\gamma+X}\frac{\Omega(\zeta)}{\bar\zeta-\bar t}d\zeta\\&+\frac{1}{\pi i}\int_{\gamma+X}\frac{\partial}{\partial t}\left(\frac{\zeta-t}{\bar\zeta-\bar t}\right)\overline{\Omega(\zeta)}\,d\bar\zeta\\=&2f_0'(t),\quad t\in\gamma,\end{aligned} \tag{2.4$'$}$$

where

$$\begin{aligned}2f_0'(t)&=\frac{1}{\pi i}\int_{\gamma}\frac{\partial}{\partial\bar t}\frac{F(\tau)}{\bar\tau-\bar t}\frac{d\bar t}{dt}d\bar\tau+G'(t)\\&=\frac{e^{-2i\theta(t)}}{\pi i}\int_{\gamma}\frac{F'(\tau)}{\bar\tau-\bar t}d\tau+G'(t),\end{aligned} \tag{2.5}$$

(noting that $d\bar t/dt=e^{-2i\theta(t)}$), in which we have used integration by parts since $F(a_j)=F(b_j)=0$. By assumption, $f_0'(t)\in H_0$ on γ, that is, $f_0'(t)\in H$ on each smooth subarc of γ.

Similarly, differentiating (1.15), we obtain another singular integral equation in $\Omega(\zeta)$:

$$\begin{aligned}A\Omega(x)+\frac{B}{\pi i}\int_{\gamma+X}\frac{\Omega(\zeta)}{\zeta-x}d\zeta-\frac{D}{\pi i}\left\{\int_{\gamma}\frac{\partial}{\partial x}\log\frac{\tau-x}{\bar\tau-x}\,\Omega(\tau)d\tau+\int_{\gamma}\frac{\partial}{\partial x}\left(\frac{\zeta-x}{\bar\zeta-x}\right)\overline{\Omega(\zeta)}d\bar\zeta\right\}\\=-Df_1'(x)\end{aligned}$$

or

$$\begin{aligned}\check{K}_2\Omega \equiv & A\Omega(x) + \frac{B}{\pi i}\int_X \frac{\Omega(\xi)}{\xi - x}d\xi + \frac{C}{\pi i}\int_\gamma \frac{\Omega(\tau)}{\tau - x}d\tau \\ & - \frac{D}{\pi i}\left\{\int_\gamma \frac{\overline{\Omega(\tau)}}{\bar{\tau} - x}d\tau + \int_\gamma \frac{\zeta - \bar{\zeta}}{(\bar{\zeta} - x)^2}\overline{\Omega(\zeta)}d\bar{\zeta}\right\} \\ = & -Df_1'(x), \quad x \in X,\end{aligned} \tag{2.6'}$$

where

$$f_1'(x) = \frac{1}{\pi i}\int_\gamma \frac{F'(\tau)}{\tau - x}d\tau \tag{2.7}$$

also exists and belongs to H on X.

(2.4)′ and (2.6)′ constitute a singular integral equation for $\Omega(\zeta)$ on $\gamma + X$ without undetermined constants. Now $\Omega(\zeta) = \omega'(\zeta)$ may have integrable discontinuities at the nodes. Moreover, if we replace x in (2.6)′ by $t \in \gamma_j$, then it is easily verified that $(\check{K}_2\Omega)(t) = 4g'(t)$, where $g(t) = g^+(t) - g^-(t)$ is the displacement difference at t between the two sides of γ_j. Hence, the following condition must be satisfied for each γ_j:

$$\int_{\gamma_j}(\check{K}_2\Omega)(t)dt = 0. \tag{2.8}$$

For any crack γ_j not lying on nor passing through the x-axis, it is easily seen that (2.8) is reduced to

$$\int_{\gamma_j}\Omega(t)dt = 0; \tag{2.9}$$

for a crack $\gamma_j = a_jb_j$ passing through the interface at a point c_j (as shown in Figure 1), it is reduced to

$$(\alpha^- + \beta^-)\int_{a_jc_j}\Omega(t)dt + (\alpha^+ + \beta^+)\int_{c_jb_j}\Omega(t)dt = 0. \tag{2.10}$$

Thus, the rather complicated form (2.8) remains only for cracks lying on the interface.

Therefore, our problem is reduced to solve a singular integral equation along $\gamma + X$ for $\Omega(\zeta)$ in class h_0 with supplementary requirements as shown above. Its unique solvability is guaranteed since $\omega(\zeta)$ is determined as the unique solution of (1.12) and (1.15).

3. The order of singularity

It is important to determine the order of singularity at each node for the stress functions, which is fully determined by the order of $\Omega(\zeta)$ at the same node. It is well known that, for any node which is a free tip (not on the interface) of any crack, the order is 1/2. For the other nodes, the situation is much more complex. Near such a node c, usually it is assumed

$$\Omega(\zeta) = \frac{\Omega_0(\zeta)}{(\zeta - c)^{\alpha + i\beta}} +, \quad 0 < \alpha < 1, \tag{3.1}$$

where $\Omega_0(\zeta)$ is bounded at $\zeta=c$ (the unwritten terms are of lower order of singularity, the same as below). In many cases, the limit of $\Omega_0(\zeta)$ does not exist as $\zeta\to c$ along any crack or any segment of the interface. In fact, $\Omega_0(\zeta)$ oscillates near the node in general.

It seems more reasonable and exact to assume

$$\Omega(\zeta)=\frac{C_1}{(\zeta-c)^{\alpha+i\beta}}+\frac{C_2}{(\zeta-c)^{\alpha-i\beta}}+\cdots\quad (C_1,C_2 \text{ not both zero}) \tag{3.2}$$

near $\zeta=c$ along a definite crack or segment mentioned above. Thus, the behavior of $\Omega(\zeta)$ near c becomes much clearer. If $\beta=0$, then $\Omega(\zeta)=\Omega_0/(\zeta-c)^{\alpha}+\cdots$ has a real order at c, Ω_0 being a constant, so that there is no oscillation at c, which is in harmony with the known results.

We illustrate this idea by two examples.

Example 1. Assume the elastic body occupies the entire plane with two rectilinear cracks $\gamma_j=Ob_j$, $j=1,2$, having a common tip $z=0$ and oriented from O to b_j. It may be bonded by several different media with interfaces not passing through the cracks (Figure 2). The angle of inclination of γ_j is θ_j, $0\leqslant\theta_j<2\pi$, and the length of γ_j is r_j, i. e., $b_j=r_je^{i\theta_j}$.

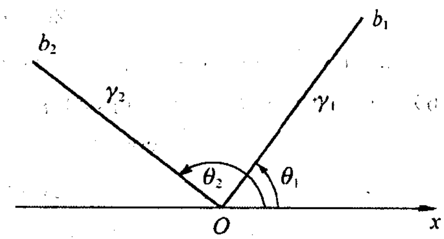

Figure 2

Denote $\theta=\theta_2-\theta_1$. Let $t=re^{i\theta_j}$, $\zeta=\rho e^{i\theta_j}$ and $\Omega(\zeta)=\Omega_j(\rho)$ on γ_j. Then, (2.4) becomes: for $t\in\gamma_1$,

$$\begin{aligned}
&\frac{1}{\pi i}\int_0^{r_1}\frac{\Omega_1(\rho)}{\rho-r}d\rho+\frac{1}{\pi i}\int_0^{r_2}\frac{\Omega_2(\rho)e^{i\theta_2}}{\rho e^{i\theta_2}-re^{i\theta_1}}d\rho\\
&\quad+\frac{1}{2\pi i}\int_0^{r_2}e^{-i\theta_1}\frac{\partial}{\partial r}\log\left(\frac{\rho e^{i\theta_2}-re^{i\theta_1}}{\rho e^{-i\theta_2}-re^{-i\theta_1}}\right)\Omega_2(\rho)e^{i\theta_2}d\rho\\
&\quad+\frac{1}{2\pi i}\int_0^{r_2}e^{-i\theta_1}\frac{\partial}{\partial r}\left(\frac{\rho e^{i\theta_2}-re^{i\theta_1}}{\rho e^{-i\theta_2}-re^{-i\theta_1}}\right)\overline{\Omega_2(\rho)}e^{-i\theta_2}d\rho\\
&=f_0'(t),\quad t=re^{i\theta_1},\ 0<r<r_1,
\end{aligned}$$

or,

$$\begin{aligned}
&\frac{1}{\pi i}\int_0^{r_1}\frac{\Omega_1(\rho)}{\rho-r}d\rho-\frac{1}{2\pi i}\int_0^{r_2}\frac{\Omega_2(\rho)}{\rho-re^{-i\theta}}d\rho+\frac{e^{2i\theta}}{2\pi i}\int_0^{r_2}\frac{\Omega_2(\rho)}{\rho-re^{i\theta}}d\rho\\
&\quad+\frac{\sin\theta\, e^{i\theta}}{\pi}\left\{\int_0^{r_2}\frac{\overline{\Omega_2(\rho)}}{\rho-re^{i\theta}}d\rho+r\frac{d}{dr}\int_0^{r_2}\frac{\overline{\Omega_2(\rho)}}{\rho-re^{i\theta}}d\rho\right\}\\
&=f_0'(t),\quad 0<r<r_1,
\end{aligned}\tag{3.3}$$

while equation (2.6) has no connection with the singularity at O. Assume

$$\begin{aligned}
&\Omega_1(\rho)=A_1/\rho^{\gamma}+\overline{B}_1/\rho^{\bar\gamma}+\cdots,\\
&\Omega_2(\rho)=A_2/\rho^{\gamma}+\overline{B}_2/\rho^{\bar\gamma}+\cdots\\
&(\gamma=\alpha+i\beta;\ A_1,B_1,A_2,B_2 \text{ not all zero}).
\end{aligned}\tag{3.4}$$

Then, by the property of Cauchy principal value integrals near the end O, we should have

$$\frac{\cot \gamma\pi}{\mathrm{i}r^{\gamma}}A_1 + \frac{\cot \bar{\gamma}\pi}{\mathrm{i}r^{\bar{\gamma}}}\overline{B}_1 - \frac{\mathrm{e}^{\mathrm{i}\gamma\pi}}{2\mathrm{i}r^{\gamma}\mathrm{e}^{-\mathrm{i}\theta\gamma}\sin \gamma\pi}A_2 - \frac{\mathrm{e}^{\mathrm{i}\bar{\gamma}\pi}}{2\mathrm{i}r^{\bar{\gamma}}\mathrm{e}^{-\mathrm{i}\theta\bar{\gamma}}\sin \bar{\gamma}\pi}\overline{B}_2$$
$$+ \mathrm{e}^{2\mathrm{i}\theta}\left(\frac{\mathrm{e}^{\mathrm{i}\gamma\pi}}{2\mathrm{i}r^{\gamma}\mathrm{e}^{\mathrm{i}\theta\gamma}\sin \gamma\pi}A_2 + \frac{\mathrm{e}^{\mathrm{i}\bar{\gamma}\pi}}{2\mathrm{i}r^{\bar{\gamma}}\mathrm{e}^{\mathrm{i}\theta\bar{\gamma}}\sin \bar{\gamma}\pi}\overline{B}_2\right)$$
$$+ \sin\theta\, \mathrm{e}^{\mathrm{i}\theta}\left(1 + r\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{\mathrm{e}^{\mathrm{i}\bar{\gamma}\pi}}{r^{\bar{\gamma}}\mathrm{e}^{\mathrm{i}\theta\bar{\gamma}}\sin \bar{\gamma}\pi}\overline{A}_2 + \frac{\mathrm{e}^{\mathrm{i}\gamma\pi}}{r^{\gamma}\mathrm{e}^{\mathrm{i}\theta\gamma}\sin \gamma\pi}B_2\right) = \cdots.$$

Therefore,

$$\begin{aligned}&2\cos \gamma\pi\, A_1 - \mathrm{e}^{\mathrm{i}\gamma(\pi+\theta)}A_2 + \mathrm{e}^{\mathrm{i}[\gamma(\pi-\theta)+2\theta]}A_2 + 2\mathrm{i}\sin\theta\,(1-\gamma)\mathrm{e}^{\mathrm{i}[\gamma(\pi-\theta)+\theta]}B_2 = 0,\\ &2\cos \bar{\gamma}\pi\, \overline{B_1} - \mathrm{e}^{\mathrm{i}\bar{\gamma}(\pi+\theta)}\overline{B}_2 + \mathrm{e}^{\mathrm{i}[\bar{\gamma}(\pi-\theta)+2\theta]}\overline{B}_2 + 2\mathrm{i}\sin\theta\,(1-\bar{\gamma})\mathrm{e}^{\mathrm{i}[\bar{\gamma}(\pi-\theta)+\theta]}\overline{A}_2 = 0\end{aligned} \tag{3.5}$$

or

$$\begin{aligned}&2\cos \gamma\pi\, B_1 - \mathrm{e}^{-\mathrm{i}\gamma(\pi+\theta)}B_2 + \mathrm{e}^{-\mathrm{i}[\gamma(\pi-\theta)+2\theta]}B_2\\ &\quad - 2\mathrm{i}\sin\theta\,(1-\gamma)\mathrm{e}^{-\mathrm{i}[\gamma(\pi-\theta)+\theta]}A_2 = 0.\end{aligned} \tag{3.6}$$

Similarly for $t = r\mathrm{e}^{\mathrm{i}\theta_2} \in \gamma_2$, we have, by interchanging the subscripts 1, 2 in (3.5), (3.6) and at the same time replacing θ by $-\theta$,

$$\begin{aligned}&2\cos \gamma\pi\, A_2 - \mathrm{e}^{\mathrm{i}\gamma(\pi-\theta)}A_1 + \mathrm{e}^{\mathrm{i}[\gamma(\pi+\theta)-2\theta]}A_1\\ &\quad - 2\mathrm{i}\sin\theta\,(1-\gamma)\mathrm{e}^{\mathrm{i}[\gamma(\pi+\theta)-\theta]}B_1 = 0,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&2\cos \gamma\pi\, B_2 - \mathrm{e}^{-\mathrm{i}\gamma(\pi-\theta)}B_1 + \mathrm{e}^{-\mathrm{i}[\gamma(\pi+\theta)-2\theta]}B_1\\ &\quad + 2\mathrm{i}\sin\theta\,(1-\gamma)\mathrm{e}^{-\mathrm{i}[\gamma(\pi+\theta)-\theta]}A_1 = 0.\end{aligned} \tag{3.8}$$

Equations (3.5), (3.7), (3.6) and (3.8) form a homogeneous linear system in A_1, B_1, A_2, B_2, which has nontrivial solution if and only if its coefficient determinant is zero:

$$|a_{jk}| = 0, \quad j,k = 1,2,3,4, \tag{3.9}$$

where

$$\begin{aligned}
&a_{11} = 2\cos \gamma\pi, && a_{12} = \mathrm{e}^{\mathrm{i}[\gamma(\pi-\theta)+2\theta]} - \mathrm{e}^{\mathrm{i}\gamma(\pi+\theta)},\\
&a_{13} = 0, && a_{14} = 2\mathrm{i}\sin\theta\,(1-\gamma)\mathrm{e}^{\mathrm{i}[\gamma(\pi-\theta)+\theta]};\\
&a_{21} = \mathrm{e}^{\mathrm{i}[\gamma(\pi+\theta)-2\theta]} - \mathrm{e}^{\mathrm{i}\gamma(\pi-\theta)}, && a_{22} = 2\cos \gamma\pi,\\
&a_{23} = -2\mathrm{i}\sin\theta\,(1-\gamma)\mathrm{e}^{\mathrm{i}[\gamma(\pi+\theta)-\theta]}, && a_{24} = 0;\\
&a_{31} = 0, && a_{32} = -2\mathrm{i}\sin\theta\,(1-\gamma)\mathrm{e}^{-\mathrm{i}[\gamma(\pi-\theta)+\theta]},\\
&a_{33} = 2\cos \gamma\pi, && a_{34} = \mathrm{e}^{-\mathrm{i}[\gamma(\pi-\theta)+2\theta]} - \mathrm{e}^{-\mathrm{i}\gamma(\pi+\theta)};\\
&a_{41} = 2\mathrm{i}\sin\theta\,(1-\gamma)\mathrm{e}^{-\mathrm{i}[\gamma(\pi+\theta)+\theta]}, && a_{42} = 0,\\
&a_{43} = \mathrm{e}^{-\mathrm{i}[\gamma(\pi+\theta)-2\theta]} - \mathrm{e}^{-\mathrm{i}\gamma(\pi-\theta)}, && a_{44} = 2\cos \gamma\pi.
\end{aligned} \tag{3.10}$$

Remark. If the root $\gamma = \alpha + \mathrm{i}\beta$, $0 < \alpha < 1$, of (3.9) has been found, then A_1, B_1, A_2 and B_2 are proportional to the algebraic complements of the elements in the first row of $|a_{jk}|$.

Example 2. Let $Z^{\pm}$ be two isotropic media with elastic constants $\kappa^{\pm}$ and $\mu^{\pm}$ respectively, containing a rectilinear crack γ from O to $b = a\mathrm{e}^{\mathrm{i}\theta}$, $a > 0$, with inclination θ, $0 < \theta < \pi$ (Figure 3). Denote

$$\Omega(x) = \begin{cases}\Omega_-(x) & -\infty < x < 0,\\ \Omega_+(x), & 0 < x < +\infty,\end{cases}$$

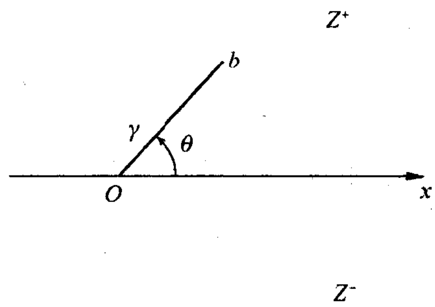

Figure 3

$$\Omega(\tau) = \Omega_0(\rho), \quad 0 \leqslant \rho \leqslant a, \ \tau = \rho e^{i\theta}.$$

Then equation (2.4) becomes (let $t = re^{i\theta}$)

$$\begin{aligned}
&\frac{1}{\pi i}\int_0^a \frac{\Omega_0(\rho)}{\rho - r}d\rho + \frac{1}{\pi i}\int_{-\infty}^0 \frac{\Omega_-(\zeta)}{\zeta - re^{i\theta}}d\zeta + \frac{1}{\pi i}\int_0^{+\infty}\frac{\Omega_+(\zeta)}{\zeta - re^{i\theta}}d\zeta \\
&\quad + \frac{e^{-i\theta}}{2\pi i}\int_{-\infty}^0 \frac{\partial}{\partial r}\log\frac{\zeta - re^{i\theta}}{\zeta - re^{-i\theta}}\,\Omega_-(\zeta)d\zeta \\
&\quad + \frac{e^{-i\theta}}{2\pi i}\int_0^{+\infty} \frac{\partial}{\partial r}\log\frac{\zeta - re^{i\theta}}{\zeta - re^{-i\theta}}\,\Omega_+(\zeta)d\zeta \\
&\quad + \frac{e^{-i\theta}}{2\pi i}\int_{-\infty}^0 \frac{\partial}{\partial r}\frac{\zeta - re^{i\theta}}{\zeta - re^{-i\theta}}\,\overline{\Omega_-(\zeta)}d\zeta \\
&\quad + \frac{e^{-i\theta}}{2\pi i}\int_0^{+\infty} \frac{\partial}{\partial r}\frac{\zeta - re^{i\theta}}{\zeta - re^{-i\theta}}\,\overline{\Omega_+(\zeta)}d\zeta \\
&= 2f_0'(t), \quad t \in \gamma,
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{\pi i}\int_0^a \frac{\Omega_0(\rho)}{\rho - r}d\rho + \frac{1}{2\pi i}\int_{-\infty}^0 \frac{\Omega_-(\zeta)}{\zeta - re^{i\theta}}d\zeta + \frac{e^{-2i\theta}}{2\pi i}\int_{-\infty}^0 \frac{\Omega_-(\zeta)}{\zeta - re^{-i\theta}}d\zeta \\
&\quad + \frac{1}{2\pi i}\int_0^{+\infty}\frac{\Omega_+(\zeta)}{\zeta - re^{i\theta}}d\zeta + \frac{e^{-2i\theta}}{2\pi i}\int_0^{+\infty}\frac{\Omega_+(\zeta)}{\zeta - re^{-i\theta}}d\zeta \\
&\quad - \frac{1 - e^{-2i\theta}}{2\pi i}\left\{\int_{-\infty}^0 \frac{\overline{\Omega_-(\zeta)}}{\zeta - re^{-i\theta}}d\zeta + r\frac{d}{dr}\int_{-\infty}^0 \frac{\overline{\Omega_-(\zeta)}}{\zeta - re^{-i\theta}}d\zeta\right\} \\
&\quad - \frac{1 - e^{-2i\theta}}{2\pi i}\left\{\int_0^{+\infty} \frac{\overline{\Omega_+(\zeta)}}{\zeta - re^{-i\theta}}d\zeta + r\frac{d}{dr}\int_0^{+\infty} \frac{\overline{\Omega_+(\zeta)}}{\zeta - re^{-i\theta}}d\zeta\right\} \\
&= 2f_0'(t), \quad t \in \gamma.
\end{aligned} \tag{3.11}$$

Let ($\gamma = \alpha + i\beta$)

$$\begin{aligned}
\Omega_0(\rho) &= A_0/\rho^{\gamma} + \overline{B}_0/\rho^{\overline{\gamma}} + \cdots, \quad 0 < \rho < a, \\
\Omega_-(x) &= A_-/x^{\gamma} + \overline{B}_-/x^{\overline{\gamma}} + \cdots, \quad -\infty < x < 0, \\
\Omega_+(x) &= A_+/x^{\gamma} + \overline{B}_+/x^{\overline{\gamma}} + \cdots, \quad 0 < x < +\infty.
\end{aligned} \tag{3.12}$$

Then, substituting in (3.11), we obtain

$$\frac{\cot\gamma\pi\, A_0}{i\rho^{\gamma}} + \frac{\cot\overline{\gamma}\pi\,\overline{B}_0}{i\rho^{\overline{\gamma}}}$$

$$+\left(\frac{1}{2\mathrm{i}\sin\gamma\pi\,\rho^{\gamma}\mathrm{e}^{\mathrm{i}\gamma\theta}}+\frac{\mathrm{e}^{-2\mathrm{i}\theta}}{2\mathrm{i}\sin\gamma\pi\,\rho^{\gamma}\mathrm{e}^{-\mathrm{i}\gamma\theta}}\right)(\mathrm{e}^{\mathrm{i}\gamma\pi}A_{+}-\mathrm{e}^{-\mathrm{i}\gamma\pi}A_{-})$$
$$+\left(\frac{1}{2\mathrm{i}\sin\bar{\gamma}\pi\,\rho^{\bar{\gamma}}\mathrm{e}^{\mathrm{i}\bar{\gamma}\theta}}+\frac{\mathrm{e}^{-2\mathrm{i}\theta}}{2\mathrm{i}\sin\bar{\gamma}\pi\,\rho^{\bar{\gamma}}\mathrm{e}^{-\mathrm{i}\bar{\gamma}\theta}}\right)(\mathrm{e}^{\mathrm{i}\bar{\gamma}\pi}\bar{B}_{+}-\mathrm{e}^{-\mathrm{i}\bar{\gamma}\pi}\bar{B}_{-})$$
$$+\frac{(1-\mathrm{e}^{-2\mathrm{i}\theta})(1-\gamma)}{2\mathrm{i}\sin\gamma\pi\,\rho^{\gamma}\mathrm{e}^{-\mathrm{i}\gamma\theta}}(\mathrm{e}^{-\mathrm{i}\gamma\pi}B_{-}-\mathrm{e}^{\mathrm{i}\gamma\pi}B_{+})$$
$$+\frac{(1-\mathrm{e}^{-2\mathrm{i}\theta})(1-\bar{\gamma})}{2\mathrm{i}\sin\bar{\gamma}\pi\,\rho^{\bar{\gamma}}\mathrm{e}^{-\mathrm{i}\bar{\gamma}\theta}}(\mathrm{e}^{-\mathrm{i}\bar{\gamma}\pi}\bar{A}_{-}-\mathrm{e}^{\mathrm{i}\bar{\gamma}\pi}\bar{A}_{+})=0,$$

which gives rise to

$$2\cos\gamma\pi\,A_0-\mathrm{e}^{-\mathrm{i}\gamma(\pi+\theta)}A_{-}-\mathrm{e}^{-\mathrm{i}[\gamma(\pi-\theta)+2\theta]}A_{-}+\mathrm{e}^{\mathrm{i}\gamma(\pi-\theta)}A_{+}+\mathrm{e}^{\mathrm{i}[\gamma(\pi+\theta)-2\theta]}A_{+}$$
$$+(1-\mathrm{e}^{-2\mathrm{i}\theta})(1-\gamma)[\mathrm{e}^{-\mathrm{i}\gamma(\pi-\theta)}B_{-}-\mathrm{e}^{\mathrm{i}\gamma(\pi+\theta)}B_{+}]=0, \tag{3.13}$$
$$2\cos\gamma\pi\,B_0-\mathrm{e}^{\mathrm{i}\gamma(\pi+\theta)}B_{-}-\mathrm{e}^{\mathrm{i}[\gamma(\pi-\theta)+2\theta]}B_{-}+\mathrm{e}^{-\mathrm{i}\gamma(\pi-\theta)}B_{+}+\mathrm{e}^{-\mathrm{i}[\gamma(\pi+\theta)-2\theta]}B_{+}$$
$$+(1-\mathrm{e}^{-2\mathrm{i}\theta})(1-\gamma)[\mathrm{e}^{\mathrm{i}\gamma(\pi-\theta)}A_{-}-\mathrm{e}^{-\mathrm{i}\gamma(\pi+\theta)}A_{+}]=0. \tag{3.14}$$

The equation (2.6) now becomes either of the equations

$$A\Omega_{\pm}(x)+\frac{B}{\pi\mathrm{i}}\left(\int_{-\infty}^{0}\frac{\Omega_{-}(\zeta)}{\zeta-x}\mathrm{d}\zeta+\int_{0}^{+\infty}\frac{\Omega_{+}(\zeta)}{\zeta-x}\mathrm{d}\zeta\right)$$
$$+\frac{C}{\pi\mathrm{i}}\int_0^a\frac{\Omega(\rho)\mathrm{d}\rho}{\rho-x\mathrm{e}^{-\mathrm{i}\theta}}-\frac{D\mathrm{e}^{2\mathrm{i}\theta}}{\pi\mathrm{i}}\int_0^a\frac{\Omega(\rho)\mathrm{d}\rho}{\rho-x\mathrm{e}^{\mathrm{i}\theta}}$$
$$+\frac{D(1-\mathrm{e}^{2\mathrm{i}\theta})}{\pi\mathrm{i}}\left(1+x\frac{\mathrm{d}}{\mathrm{d}x}\right)\int_0^a\frac{\overline{\Omega(\rho)}}{\rho-x\mathrm{e}^{\mathrm{i}\theta}}\mathrm{d}\rho$$
$$=-Df_1'(x)$$

according as $0<x<+\infty$ or $-\infty<x<0$. It follows then that for $x>0$,

$$\frac{AA_{+}}{x^{\gamma}}-\frac{B\mathrm{e}^{-\mathrm{i}\gamma\pi}A_{-}}{\mathrm{i}\sin\gamma\pi\,x^{\gamma}}+\frac{B\cot\gamma\pi\,A_{+}}{\mathrm{i}x^{\gamma}}+\frac{C\mathrm{e}^{\mathrm{i}\gamma\pi}A_0}{\mathrm{i}\sin\gamma\pi\,x^{\gamma}\mathrm{e}^{-\mathrm{i}\gamma\theta}}$$
$$-\frac{D\mathrm{e}^{2\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\gamma\pi}A_0}{\mathrm{i}\sin\gamma\pi\,x^{\gamma}\mathrm{e}^{\mathrm{i}\gamma\theta}}+\frac{D(1-\mathrm{e}^{2\mathrm{i}\theta})(1-\bar{\gamma})\mathrm{e}^{\mathrm{i}\bar{\gamma}\pi}\bar{A}_0}{\mathrm{i}\sin\bar{\gamma}\pi\,x^{\bar{\gamma}}\mathrm{e}^{\mathrm{i}\bar{\gamma}\theta}}$$
$$+\frac{A\bar{B}_{+}}{x^{\bar{\gamma}}}-\frac{B\mathrm{e}^{-\mathrm{i}\bar{\gamma}\pi}\bar{B}_{-}}{\mathrm{i}\sin\bar{\gamma}\pi\,x^{\bar{\gamma}}}+\frac{B\cot\bar{\gamma}\pi\,\bar{B}_{+}}{\mathrm{i}x^{\bar{\gamma}}}$$
$$+\frac{C\mathrm{e}^{\mathrm{i}\bar{\gamma}\pi}\bar{B}_0}{\mathrm{i}\sin\bar{\gamma}\pi\,x^{\bar{\gamma}}\mathrm{e}^{-\mathrm{i}\bar{\gamma}\theta}}-\frac{D\mathrm{e}^{2\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\bar{\gamma}\pi}\bar{B}_0}{\mathrm{i}\sin\bar{\gamma}\pi\,x^{\bar{\gamma}}\mathrm{e}^{\mathrm{i}\bar{\gamma}\theta}}$$
$$+\frac{D(1-\mathrm{e}^{2\mathrm{i}\theta})(1-\gamma)\mathrm{e}^{\mathrm{i}\gamma\pi}B_0}{\mathrm{i}\sin\gamma\pi\,x^{\gamma}\mathrm{e}^{\mathrm{i}\gamma\theta}}=0,$$

from which it follows

$$\mathrm{i}A\sin\gamma\pi\,A_{+}-B\mathrm{e}^{-\mathrm{i}\gamma\pi}A_{-}+B\cos\gamma\pi\,A_{+}+C\mathrm{e}^{\mathrm{i}\gamma(\pi+\theta)}A_0-D\mathrm{e}^{\mathrm{i}[\gamma(\pi-\theta)+2\theta]}A_0$$
$$+D(1-\mathrm{e}^{2\mathrm{i}\theta})(1-\gamma)\mathrm{e}^{\mathrm{i}\gamma(\pi-\theta)}B_0=0, \tag{3.15}$$
$$-\mathrm{i}A\sin\gamma\pi\,B_{+}-B\mathrm{e}^{\mathrm{i}\gamma\pi}B_{-}+B\cos\gamma\pi\,B_{+}+C\mathrm{e}^{-\mathrm{i}\gamma(\pi+\theta)}B_0-D\mathrm{e}^{-\mathrm{i}[\gamma(\pi-\theta)+2\theta]}B_0$$
$$+D(1-\mathrm{e}^{-2\mathrm{i}\theta})(1-\gamma)\mathrm{e}^{-\mathrm{i}\gamma(\pi-\theta)}A_0=0; \tag{3.16}$$

similarly, for $x<0$,

$$\frac{AA_{-}}{x^{\gamma}}+\frac{B\mathrm{e}^{\mathrm{i}\gamma\pi}A_{+}}{\mathrm{i}\sin\gamma\pi\,x^{\gamma}}-\frac{B\cot\gamma\pi\,A_{-}}{\mathrm{i}x^{\gamma}}+\frac{C\mathrm{e}^{\mathrm{i}\gamma\pi}A_0}{\mathrm{i}\sin\gamma\pi\,x^{\gamma}\mathrm{e}^{-\mathrm{i}\gamma\theta}}$$

$$
\begin{aligned}
&-\frac{De^{2i\theta}e^{i\gamma\pi}A_0}{i\sin\gamma\pi\, x^{\gamma}e^{i\gamma\theta}}+\frac{D(1-e^{2i\theta})(1-\bar{\gamma})e^{i\bar{\gamma}\pi}\,\overline{A_0}}{i\sin\,\bar{\gamma}\pi\, x^{\bar{\gamma}}e^{i\bar{\gamma}\theta}}\\
&+\frac{A\,\overline{B}_-}{x^{\bar{\gamma}}}+\frac{Be^{i\bar{\gamma}\pi}\overline{B}_+}{i\sin\,\bar{\gamma}\pi\, x^{\bar{\gamma}}}-\frac{B\cot\,\bar{\gamma}\pi\,\overline{B}_-}{ix^{\bar{\gamma}}}\\
&+\frac{Ce^{i\bar{\gamma}\pi}\,\overline{B_0}}{i\sin\bar{\gamma}\pi\, x^{\bar{\gamma}}e^{-i\bar{\gamma}\theta}}-\frac{De^{2i\theta}e^{i\bar{\gamma}\pi}\,\overline{B_0}}{i\sin\,\bar{\gamma}\pi\, x^{\bar{\gamma}}e^{i\bar{\gamma}\theta}}\\
&+\frac{D(1-e^{2i\theta})(1-\gamma)e^{i\gamma\pi}B_0}{i\sin\,\gamma\pi\, x^{\gamma}e^{i\gamma\theta}}=0
\end{aligned}
$$

by which it follows

$$
\begin{aligned}
&iA\sin\gamma\pi\, A_-+Be^{i\gamma\pi}A_+-B\cos\gamma\pi\, A_-+Ce^{i\gamma(\pi+\theta)}A_0-De^{i[\gamma(\pi-\theta)+2\theta]}A_0\\
&\qquad+D(1-e^{2i\theta})(1-\gamma)e^{i\gamma(\pi-\theta)}B_0=0, \qquad (3.17)\\
&-iA\sin\gamma\pi\, B_-+Be^{-i\gamma\pi}B_+-B\cos\gamma\pi\, B_-+Ce^{-i\gamma(\pi+\theta)}B_0-De^{-i[\gamma(\pi-\theta)+2\theta]}B_0\\
&\qquad+D(1-e^{-2i\theta})(1-\gamma)e^{-i\gamma(\pi-\theta)}A_0=0. \qquad (3.18)
\end{aligned}
$$

Therefore, we obtain a system of linear equations (3.13)~(3.18) in $A_0, B_0, A_\pm$ and $B_\pm$. Denote $A_0=A_1$, $B_0=A_2$, $A_+=A_3$, $B_+=A_4$, $A_-=A_5$ and $B_-=A_6$. Then, this system is

$$\sum_{k=1}^{6}a_{jk}A_k=0,\quad j=1,2,3,4,5,6, \qquad (3.19)$$

where

$$
\begin{array}{ll}
a_{11}=2\cos\gamma\pi, & a_{12}=0,\\
a_{13}=e^{i\gamma(\pi-\theta)}+e^{i[\gamma(\pi+\theta)-2\theta]}, & a_{14}=-(1-e^{-2i\theta})(1-\gamma)e^{i\gamma(\pi+\theta)},\\
a_{15}=-e^{-i\gamma(\pi+\theta)}-e^{-i[\gamma(\pi-\theta)+2\theta]}, & a_{16}=(1-e^{-2i\theta})(1-\gamma)e^{-i\gamma(\pi-\theta)};\\
a_{21}=0, & a_{22}=2\cos\gamma\pi,\\
a_{23}=-(1-e^{2i\theta})(1-\gamma)e^{-i\gamma(\pi+\theta)}, & a_{24}=e^{-i\gamma(\pi-\theta)}+e^{-i[\gamma(\pi+\theta)-2\theta]},\\
a_{25}=(1-e^{2i\theta})(1-\gamma)e^{i\gamma(\pi-\theta)}, & a_{26}=-e^{i\gamma(\pi+\theta)}-e^{i[\gamma(\pi-\theta)+2\theta]};\\
a_{31}=Ce^{i\gamma(\pi+\theta)}-De^{i[\gamma(\pi-\theta)+2\theta]}, & a_{32}=D(1-e^{2i\theta})(1-\gamma)e^{i\gamma(\pi-\theta)},\\
a_{33}=iA\sin\gamma\pi+B\cos\gamma\pi, & a_{34}=0,\\
a_{35}=-Be^{-i\gamma\pi}, & a_{36}=0;\\
a_{41}=D(1-e^{-2i\theta})(1-\gamma)e^{-\gamma(\pi-\theta)}, & a_{42}=Ce^{-i\gamma(\pi+\theta)}-De^{-i[\gamma(\pi-\theta)+2\theta]},\\
a_{43}=0, & a_{44}=-iA\sin\gamma\pi+B\cos\gamma\pi,\\
a_{45}=0, & a_{46}=-Be^{i\gamma\pi};\\
a_{51}=Ce^{i\gamma(\pi+\theta)}-De^{i[\gamma(\pi-\theta)+2\theta]}, & a_{52}=D(1-e^{2i\theta})(1-\gamma)e^{i\gamma(\pi-\theta)},\\
a_{53}=Be^{i\gamma\pi}, & a_{54}=0,\\
a_{55}=iA\sin\gamma\pi-B\cos\gamma\pi, & a_{56}=0;\\
a_{61}=D(1-e^{-2i\theta})(1-\gamma)e^{-i\gamma(\pi-\theta)}, & a_{62}=Ce^{-i\gamma(\pi+\theta)}-De^{-i[\gamma(\pi-\theta)+2\theta]},\\
a_{63}=0, & a_{64}=Be^{-i\gamma\pi},\\
a_{65}=0, & a_{66}=-iA\sin\gamma\pi-B\cos\gamma\pi.
\end{array}
\qquad (3.20)
$$

The root $\gamma=\alpha+i\beta$, $0<\alpha<1$, of the equation $|a_{jk}|=0$ determines the order of singularity at O to be sought.

The remark at the end of Example 1 is also effective for this example.

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